

Control System Engineering

Section D : Frequency Domain Analysis

Topic Covered : Relationship between frequency response and time response for 2nd order system, polar, Nyquist, Bode plots, stability, Gain-margin and Phase Margin, relative stability, frequency response specifications.

The Root Locus Method

No matter what we pick K to be, the closed-loop system must always have n poles, where n is the number of poles of $G(s)$.

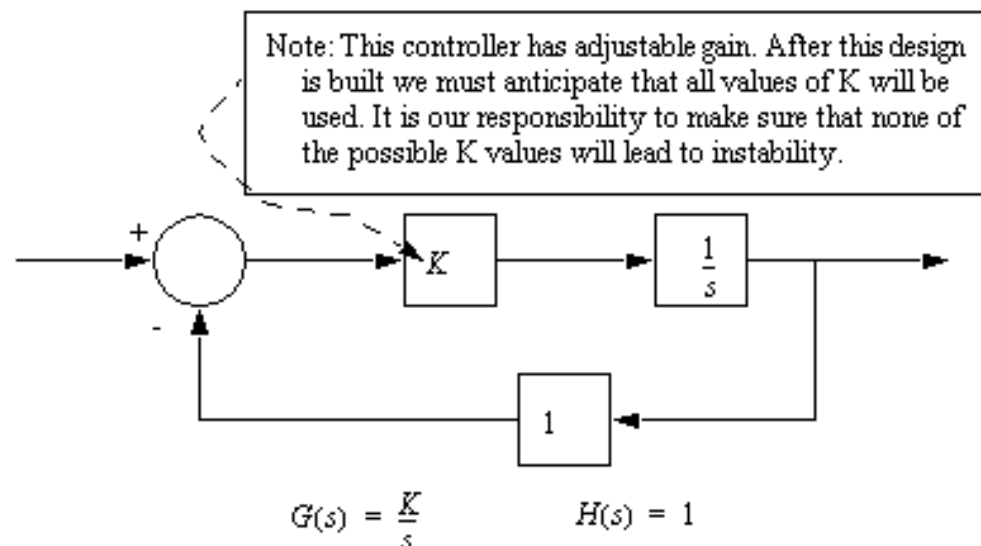
The root locus must have n branches, each branch starts at a pole of $G(s)$ and goes to a zero of $G(s)$.

If $G(s)$ has more poles than zeros (as is often the case), $m < n$ and we say that $G(s)$ has zeros at infinity. In this case, the limit of $G(s)$ as $s \rightarrow \infty$ is zero.

The number of zeros at infinity is $n-m$, the number of poles minus the number of zeros, and is the number of branches of the root locus that go to infinity (asymptotes).

Since the root locus is actually the locations of all possible closed loop poles, from the root locus we can select a gain such that our closed-loop system will perform the way we want. If any of the selected poles are on the right half plane, the closed-loop system will be unstable. The poles that are closest to the imaginary axis have the greatest influence on the closed-loop response, so even though the system has three or four poles, it may still act like a second or even first order system depending on the location(s) of the dominant pole(s).

Example

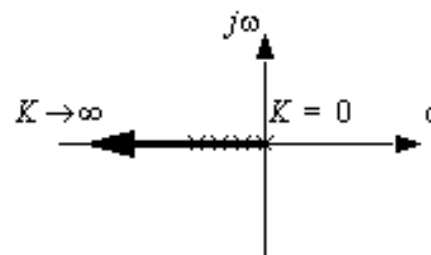


First, we must develop a transfer function for the entire control system.

$$G_g(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{\left(\frac{K}{s}\right)}{1 + \left(\frac{K}{s}\right)(1)} = \frac{K}{s + K}$$

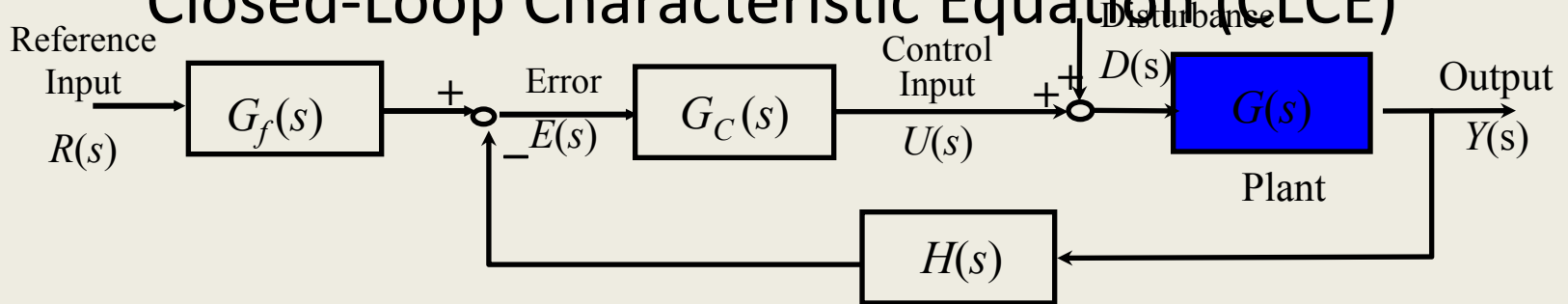
Next, we use the characteristic equation of the denominator to find the roots as the value of K varies. These can then be plotted on a complex plane. Note: the value of gain ' K ' is normally found from 0 to +infinity.

$s + K = 0$	K	root
	0	
	1	
	2	
	3	
	etc..	



Note: because all of the roots for all values of K are real negative this system will always be stable, and it will always tend to have a damped response. The larger the value of K , the more stable the system becomes.

Closed-Loop Characteristic Equation (CLCE)



The closed-loop transfer function $G_{YR}(s)$ is:

$$G_{YR}(s) = \frac{G(s)G_c(s)G_f(s)}{1 + G(s)G_c(s)H(s)}$$

The closed-loop characteristic equation (CLCE) is:

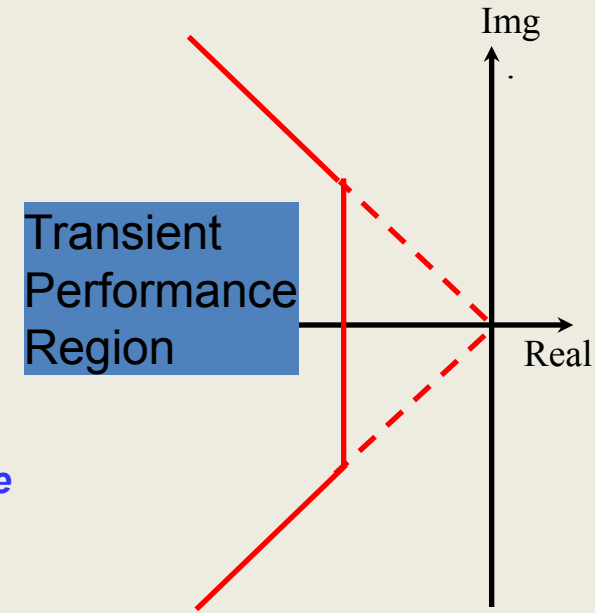
$$1 + G(s)G_c(s)H(s) = 0$$

For simplicity, assume a simple proportional feedback controller:

$$G_c(s) = K_p \Rightarrow 1 + K_p GH = 0$$

The transient performance specifications define a region on the complex plane where the closed-loop poles should be located.

Q: How should we choose K_p such that the CL poles are within the desired performance boundary?



Motivation

Ex: The closed-loop characteristic equation for the DC motor positioning system under proportional control is:

$$1 + K_p K_s G(s) = 0 \quad \Rightarrow \quad 1 + K_p \cdot 0.03 \cdot \frac{16}{s(0.0174s + 1)} = 0$$

Q: **How to choose K_p such that the resulting closed-loop poles are in the desired performance region?**

– How do we find the roots of the equation:

$$1 + K_p \cdot 0.03 \cdot \frac{16}{s(0.0174s + 1)} = 0$$

as a function of the design parameter K_p ?

– Graphically display the locations of the closed-loop poles for all $K_p > 0$ on the complex plane, from which we know the range of values for K_p that CL poles are in the performance region.

Root Locus – Definition

Root Locus is the method of graphically displaying the roots of a polynomial equation having the following form on the complex plane when the parameter K varies from 0 to ∞ :

$$1 + K \cdot G(s) = 0 \quad \text{or} \quad 1 + K \cdot \frac{N(s)}{D(s)} = 0$$

where $N(s)$ and $D(s)$ are known polynomials in factorized form:

$$N(s) = (s - z_1)(s - z_2) \cdots (s - z_{N_z})$$

$$D(s) = (s - p_1)(s - p_2) \cdots (s - p_{N_p})$$

Conventionally, the N_z roots of the polynomial $N(s)$, z_1, z_2, \dots, z_{N_z} , are called the finite open-loop zeros. The N_p roots of the polynomial $D(s)$, p_1, p_2, \dots, p_{N_p} , are called the finite open-loop poles.

Note: By transforming the closed-loop characteristic equation of a feedback controlled system with a single positive design parameter K into the above standard form, one can use the Root Locus technique to determine the range of K that have CL poles in the performance region.

Methods of Obtaining Root Locus

- Given a value of K , numerically solve the $1 + K G(s) = 0$ equation to obtain all roots. Repeat this procedure for a set of K values that span from 0 to ∞ and plot the corresponding roots on the complex plane.
- In MATLAB, use the commands *rlocus* and *rlocfind*. A very efficient root locus design tool is the command *rltool*. You can use on-line help to find the usage for these commands.

$$1 + K_P \cdot 0.03 \cdot \frac{16}{s(0.0174s + 1)} = 0 \quad \Rightarrow \quad 1 + K_P \cdot \frac{0.48}{0.0174s^2 + s} = 0$$

```
>> op_num=[0.48];
```

No open-loop zeros

```
>> op_den=[0.0174 1 0];
```

Two open-loop poles

```
>> rlocus(op_num,op_den);
```

```
>> [K, poles]=rlocfind(op_num,op_den);
```

- Apply the following root locus sketching rules to obtain an *approximated* root locus plot.

Root Locus Sketching Rules

$$1 + K \cdot \frac{N(s)}{D(s)} = 0 \quad \Rightarrow \quad 1 + K \cdot \frac{(s - z_1)(s - z_2) \cdots (s - z_{N_z})}{(s - p_1)(s - p_2) \cdots (s - p_{N_p})} = 0$$

Rule 1: The number of branches of the root locus is equal to the number of closed-loop poles (or roots of the characteristic equation). In other words, the number of branches is equal to the number of open-loop poles or open-loop zeros, whichever is greater. $D(s) + KN(s) = 0$

Rule 2: Root locus starts at open-loop poles (when $K=0$) and ends at open-loop zeros (when $K=\infty$). If the number of open-loop poles is greater than the number of open-loop zeros, some branches starting from finite open-loop poles will terminate at zeros at infinity (i.e., go to infinity). If the reverse is true, some branches will start at poles at infinity and terminate at the finite open-loop zeros.

$$D(s) + KN(s) = 0$$

$K = 0?$

$K = \infty?$

Rule 3: Root locus is symmetric about the real axis, which reflects the fact that closed-loop poles appear in complex conjugate pairs.

Rule 4: Along the real axis, the root locus includes all segments that are to the left of an *odd* number of finite real open-loop poles and zeros.

Check the phases

$$\angle K \frac{N(s)}{D(s)} = \angle -1 = \pi [\text{rad}] = 180^\circ$$

Root Locus Sketching Rules

Rule 5: If number of poles N_P exceeds the number of zeros N_Z , then as $K \rightarrow \infty$, $(N_P - N_Z)$ branches will become asymptotic to straight lines. These straight lines intersect the real axis with angles θ_k at σ_0 .

$$\sigma_0 = \frac{\sum p_i - \sum z_i}{N_P - N_Z} = \frac{\text{Sum of open-loop poles} - \text{Sum of open-loop zeros}}{\# \text{ of open-loop poles} - \# \text{ of open-loop zeros}}$$

$$\theta_k = (2k + 1) \frac{\pi}{N_P - N_Z} \text{ [rad]} = (2k + 1) \frac{180^\circ}{N_P - N_Z} \text{ [deg]}, \quad k = 0, 1, 2, \dots$$

If N_Z exceeds N_P , then as $K \rightarrow 0$, $(N_Z - N_P)$ branches behave as above.

Rule 6: Breakaway and/or break-in (arrival) points should be the solutions to the following equations:

$$\frac{d}{ds} \left(\frac{N(s)}{D(s)} \right) = 0 \quad \text{or} \quad \frac{d}{ds} \left(\frac{D(s)}{N(s)} \right) = 0$$

Root Locus Sketching Rules

Rule 7: The departure angle for a pole p_i (the arrival angle for a zero z_i) can be calculated by slightly modifying the following equation:

angle criterion →
$$\angle(s - z_1) + \angle(s - z_2) + \cdots + \angle(s - z_{N_z}) - \angle(s - p_1) - \angle(s - p_2) - \cdots - \angle(s - p_{N_p}) = 180^\circ$$

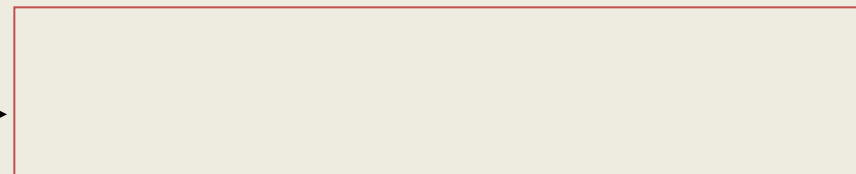
The departure angle q_j from the pole p_j can be calculated by replacing the term $\angle(s - p_j)$ with q_j and replacing all the s 's with p_j in the other terms.

Rule 8: If the root locus passes through the imaginary axis (the stability boundary), the crossing point $j\omega$ and the corresponding gain K can be found as follows:

$$K = \frac{|s - p_1| |s - p_2| \cdots |s - p_{N_p}|}{|s - z_1| |s - z_2| \cdots |s - z_{N_z}|}$$

- Replace s in the left side of the closed-loop characteristic equation with $j\omega$ to obtain the real and imaginary parts of the resulting complex number.
- Set the real and imaginary parts to zero, and solve for ω and K . This will tell you at what values of K and at what points on the $j\omega$ axis the roots will cross.

magnitude criterion →



Steps to Sketch Root Locus

Step 1: Transform the closed-loop characteristic equation into the standard form for sketching root locus:

$$1 + K \cdot \frac{N(s)}{D(s)} = 0 \quad \text{or} \quad 1 + K \cdot \frac{(s - z_1)(s - z_2) \cdots (s - z_{N_z})}{(s - p_1)(s - p_2) \cdots (s - p_{N_p})} = 0$$

Step 2: Find the open-loop zeros, z_j , and the open-loop poles, p_j . Mark the open-loop poles and zeros on the complex plane. Use \times to represent open-loop poles and \circ to represent the open-loop zeros.

Step 3: Determine the real axis segments that are on the root locus by applying Rule 4.

Step 4: Determine the number of asymptotes and the corresponding intersection σ_0 and angles θ_k by applying Rules 2 and 5.

Step 5: (If necessary) Determine the break-away and break-in points using Rule 6.

Step 6: (If necessary) Determine the departure and arrival angles using Rule 7.

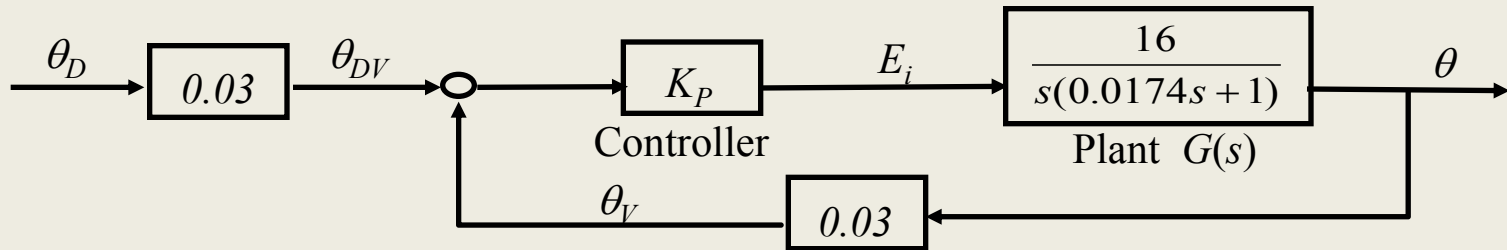
Step 7: (If necessary) Determine the imaginary axis crossings using Rule 8.

Step 8: Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus.

Example 1

DC Motor Position Control

In the previous example on the printer paper advance position control, the proportional control block diagram is:



Sketch the root locus of the closed-loop poles as the proportional gain K_p varies from 0 to ∞ .

Find closed-loop characteristic equation:

$$1 + K_p G(s) H(s) = 0$$

$$1 + K_p \frac{0.48 N(s)}{\underbrace{s(0.0174s + 1)}_{D(s)}} = 0$$

Example 1

Step 1: Transform the closed-loop characteristic equation into the standard form for sketching root locus:

$$1 + \frac{27.58K_p}{K} \frac{1}{\underbrace{s(s + 57.47)}_{D(s)}} = 0$$

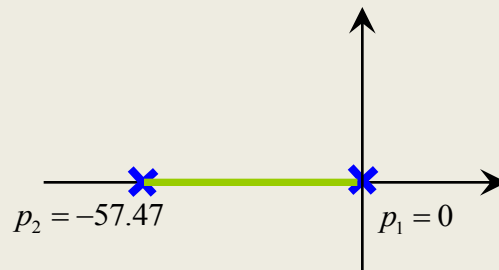
Step 2: Find the open-loop zeros, z_i , and the open-loop poles, p_i :

No open-loop zeros

open-loop poles

$$p_1 = 0, p_2 = -57.47$$

Step 3: Determine the real axis segments that are to be included in the root locus by applying Rule 4.



Example 1

Step 4: Determine the number of asymptotes and the corresponding intersection σ_0 and angles θ_k by applying Rules 2 and 5.

$$\sigma_0 = \frac{\sum p_i - \sum z_i}{N_P - N_Z} = \frac{-57.47}{2} = -28.74$$
$$\theta_k = (2k+1) \frac{\pi}{N_P - N_Z} \text{ [rad]} = \begin{cases} \frac{\pi}{2} \\ \frac{3\pi}{2} \end{cases}$$

Step 5: (If necessary) Determine the break-away and break-in points using Rule 6.

$$\frac{d}{ds} \left(\frac{N(s)}{D(s)} \right) = 0 \text{ or } \frac{d}{ds} \left(\frac{D(s)}{N(s)} \right) = 0,$$

$$\frac{d}{ds} \left(\frac{s(0.0174s+1)}{0.48} \right) = 0, 0.0348s+1=0, s=-28.74$$

Step 6: (If necessary) Determine the departure and arrival angles using Rule 7.

$$-\angle(p_2 - p_1) - \theta_{p_2} = 180^\circ, \nu_{p_2} - \nu^\circ$$

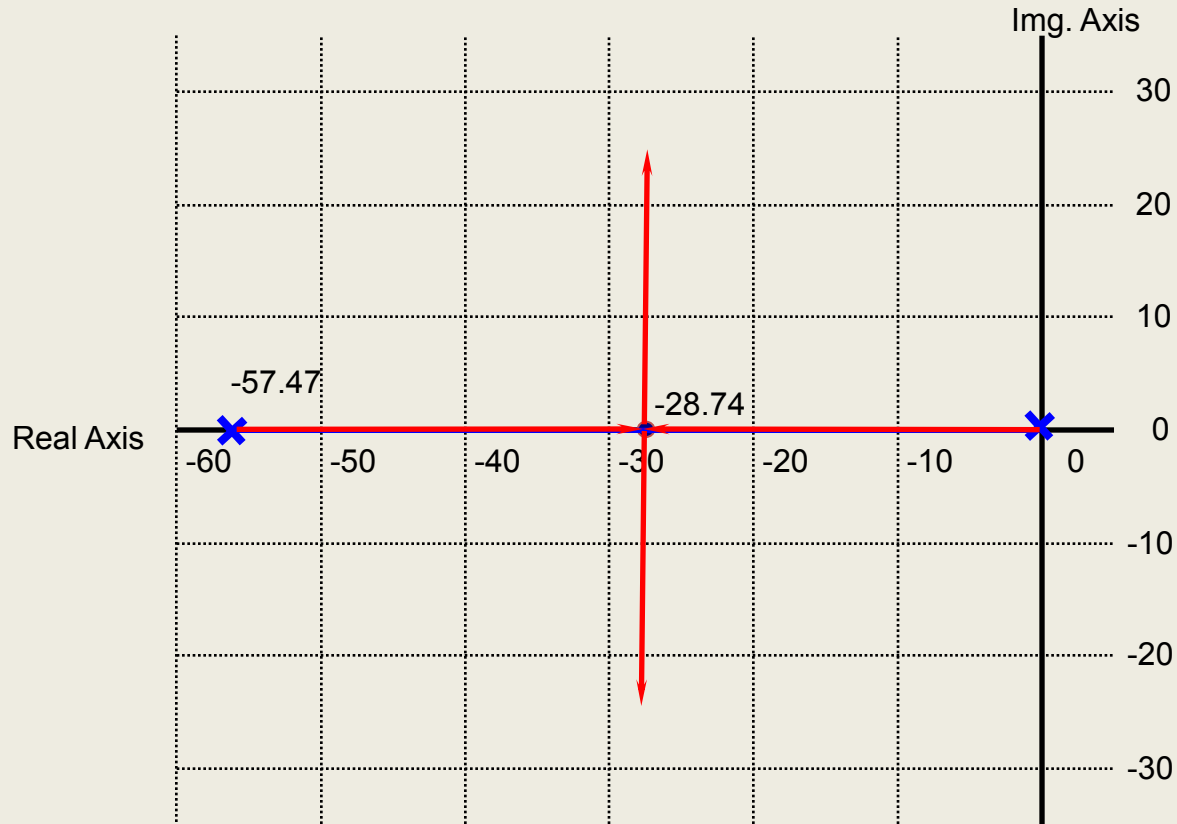
$$-\theta_{p_1} - \angle(p_1 - p_2) = 180^\circ, \nu_{p_1} - 180^\circ$$

Step 7: (If necessary) Determine the imaginary axis crossings using Rule 8.

Could s be pure imaginary in this example?

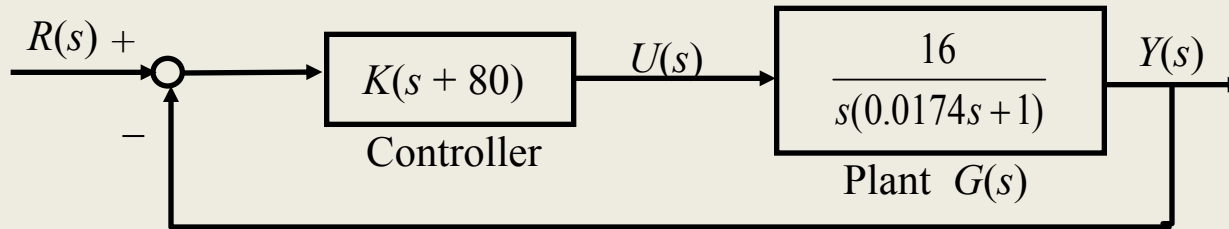
Example 1

Step 8: Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus.



Example 2

A positioning feedback control system is proposed. The corresponding block diagram is:



Sketch the root locus of the closed-loop poles as the controller gain K varies from 0 to ∞ .

Find closed-loop characteristic equation:

$$1 + G_c(s)G(s)H(s) = 0$$

$$1 + K(s + 80) \frac{16}{s(0.0174s + 1)} = 0$$

Example 2

Step 1: Formulate the (closed-loop) characteristic equation into the standard form for sketching root locus:

$$1 + K \frac{\overbrace{16(s+80)}^{N(s)}}{\underbrace{s(0.0174s+1)}_{D(s)}} = 1 + \boxed{920K} \frac{\overbrace{(s+80)}^{N(s)}}{\underbrace{s(s+57.47)}_{D(s)}} = 0$$

Step 2: Find the open-loop zeros, z_i , and the open-loop poles, p_i :

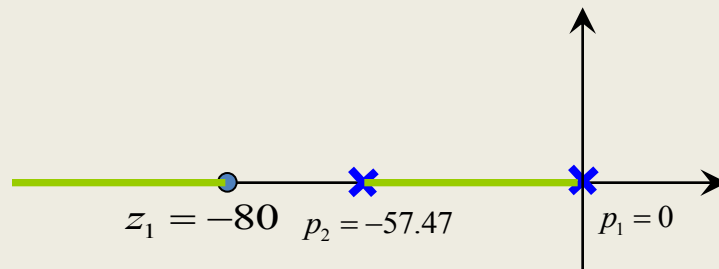
open-loop zeros

$$z_1 = -80$$

open-loop poles

$$p_1 = 0, p_2 = -57.47$$

Step 3: Determine the real axis segments that are to be included in the root locus by applying Rule 4.



Example 2

Step 4: Determine the number of asymptotes and the corresponding intersection σ_0 and angles θ_k by applying Rules 2 and 5.

Step 5: (If necessary) Determine the break-away and break-in points using Rule 6.

$$\frac{d}{ds} \left(\frac{N(s)}{D(s)} \right) = 0 \quad \text{or} \quad \frac{d}{ds} \left(\frac{D(s)}{N(s)} \right) = 0,$$

$$\frac{d}{ds} \left(\frac{(s+80)}{s(s+57.47)} \right) = \frac{s(s+57.47) - (s+80)(2s+57.47)}{[s(s+57.47)]^2} = 0,$$

$$s^2 + 160s + 4600 = 0$$

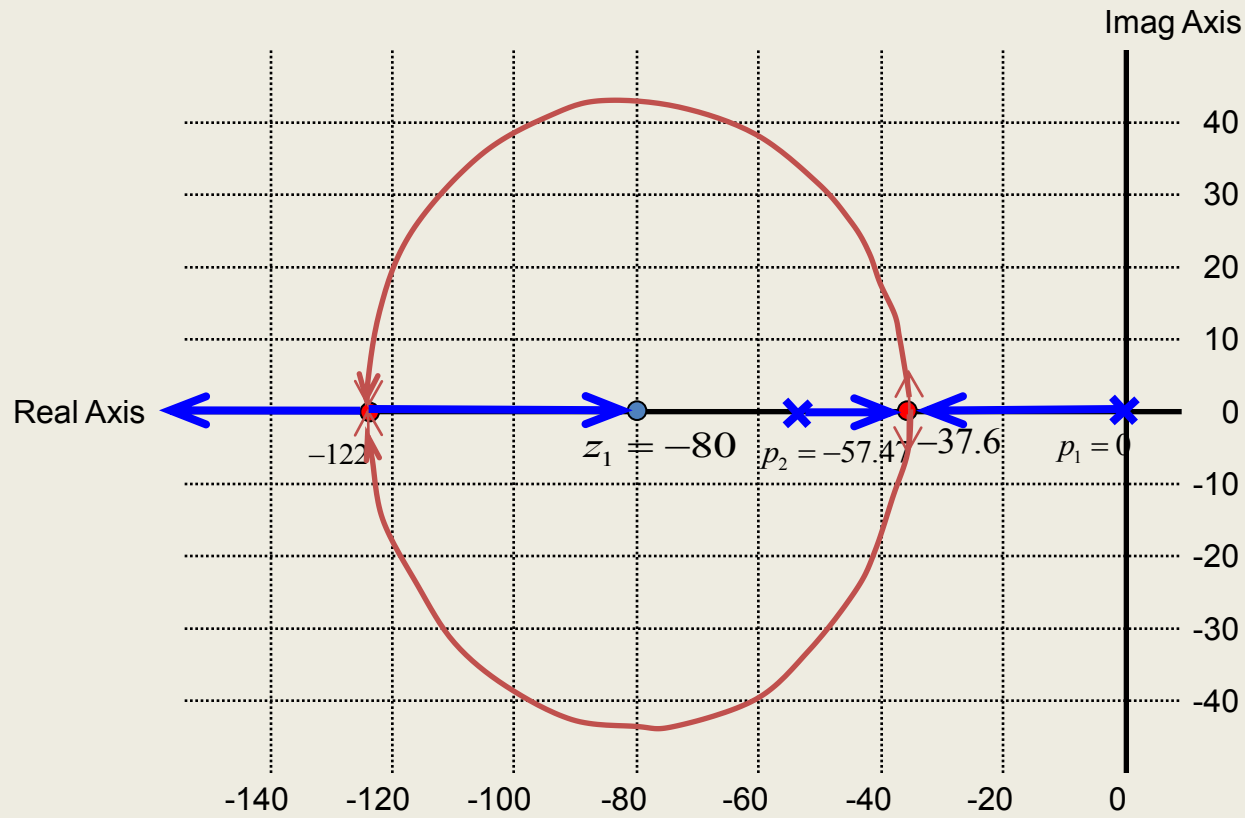
$$s_1 = -122, s_2 = -37.6$$

Example 2

Step 6: (If necessary) Determine the departure and arrival angles using Rule 7.

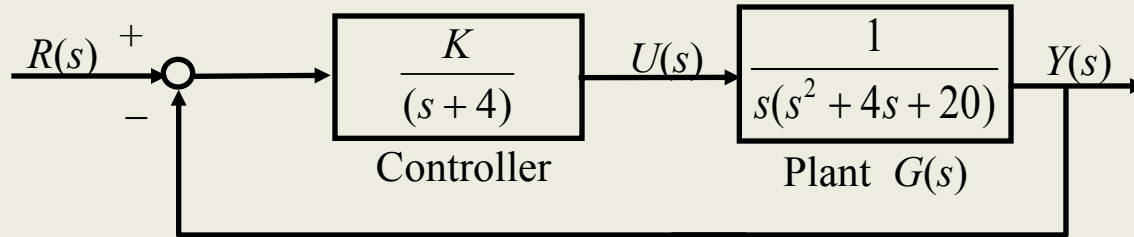
Step 7: (If necessary) Determine the imaginary axis crossings using Rule 8.

Step 8: Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus.



Example 3

A feedback control system is proposed. The corresponding block diagram is:



Sketch the root locus of the closed-loop poles as the controller gain K varies from 0 to ∞ .

Find closed-loop characteristic equation:

$$1 + G_c(s)G(s)H(s) = 0$$

$$1 + \frac{K}{s+4} \frac{1}{s(s^2+4s+20)} = 0$$

Example 3

Step 1: Transform the closed-loop characteristic equation into the standard form for sketching root locus:

$$1 + K \frac{1 \cdot N(s)}{\underbrace{s(s^2 + 4s + 20)(s + 4)}_{D(s)}} = 0$$

Step 2: Find the open-loop zeros, z_i , and the open-loop poles, p_i :

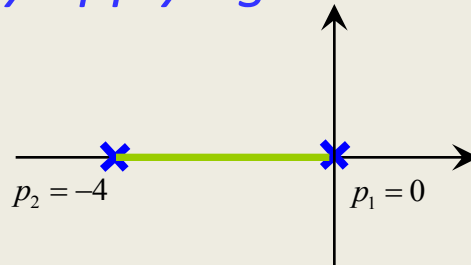
open-loop zeros

No open-loop zeros

open-loop poles

$p_1 = 0, p_2 = -4, p_{3,4} = -2 \pm 4j$

Step 3: Determine the real axis segments that are to be included in the root locus by applying Rule 4.



Example 3

Step 4: Determine the number of asymptotes and the corresponding intersection σ_0 and angles θ_k by applying Rules 2 and 5.

$$\sigma_0 = \frac{\sum p_i - \sum z_i}{N_P - N_Z} = \frac{0 + (-4) + (-2 + 4j) + (-2 - 4j)}{4 - 0} = -2$$

$$\theta_k = (2k + 1) \frac{\pi}{N_P - N_Z} \text{ [rad]} = \begin{cases} \frac{\pi}{4} \\ \frac{3\pi}{4} \\ \frac{5\pi}{4} \\ \frac{7\pi}{4} \end{cases}$$

Step 5: (If necessary) Determine the break-away and break-in points using Rule 6.

$$\frac{d}{ds} \left(\frac{N(s)}{D(s)} \right) = 0 \text{ or } \frac{d}{ds} \left(\frac{D(s)}{N(s)} \right) = 0,$$

$$\frac{d}{ds} \left(\frac{D(s)}{N(s)} \right) = \frac{d}{ds} \left(\frac{s(s^2 + 4s + 20)(s + 4)}{1} \right) = \frac{d}{ds} (s^4 + 8s^3 + 36s^2 + 80s)$$

$$= 4s^3 + 24s^2 + 72s + 80 = 0$$

$$s_1 = -2, s_{2,3} = -2 \pm 2.45j$$

Example 3

Step 6: (If necessary) Determine the departure and arrival angles using Rule 7.

$$\sum_{i=1}^{N_z} \angle(s - z_i) - \sum_{i=1}^{N_p} \angle(s - p_i) = 180^\circ$$

$$p_1 = 0 : \quad \theta_{p_1} = 180^\circ \quad p_3 = -2 + 4j : \quad \theta_{p_3} = -90^\circ$$

$$p_2 = -4 : \quad \theta_{p_2} = 0^\circ \quad p_4 = -2 - 4j : \quad \theta_{p_4} = 90^\circ$$

Step 7: (If necessary) Determine the imaginary axis crossings using Rule 8.

$$1 + K \frac{1}{s(s^2 + 4s + 20)(s + 4)} = 0 \Rightarrow$$

$$s(s^2 + 4s + 20)(s + 4) + K = 0$$

$$\Leftrightarrow s^4 + 8s^3 + 36s^2 + 80s + K = 0$$

CLCE

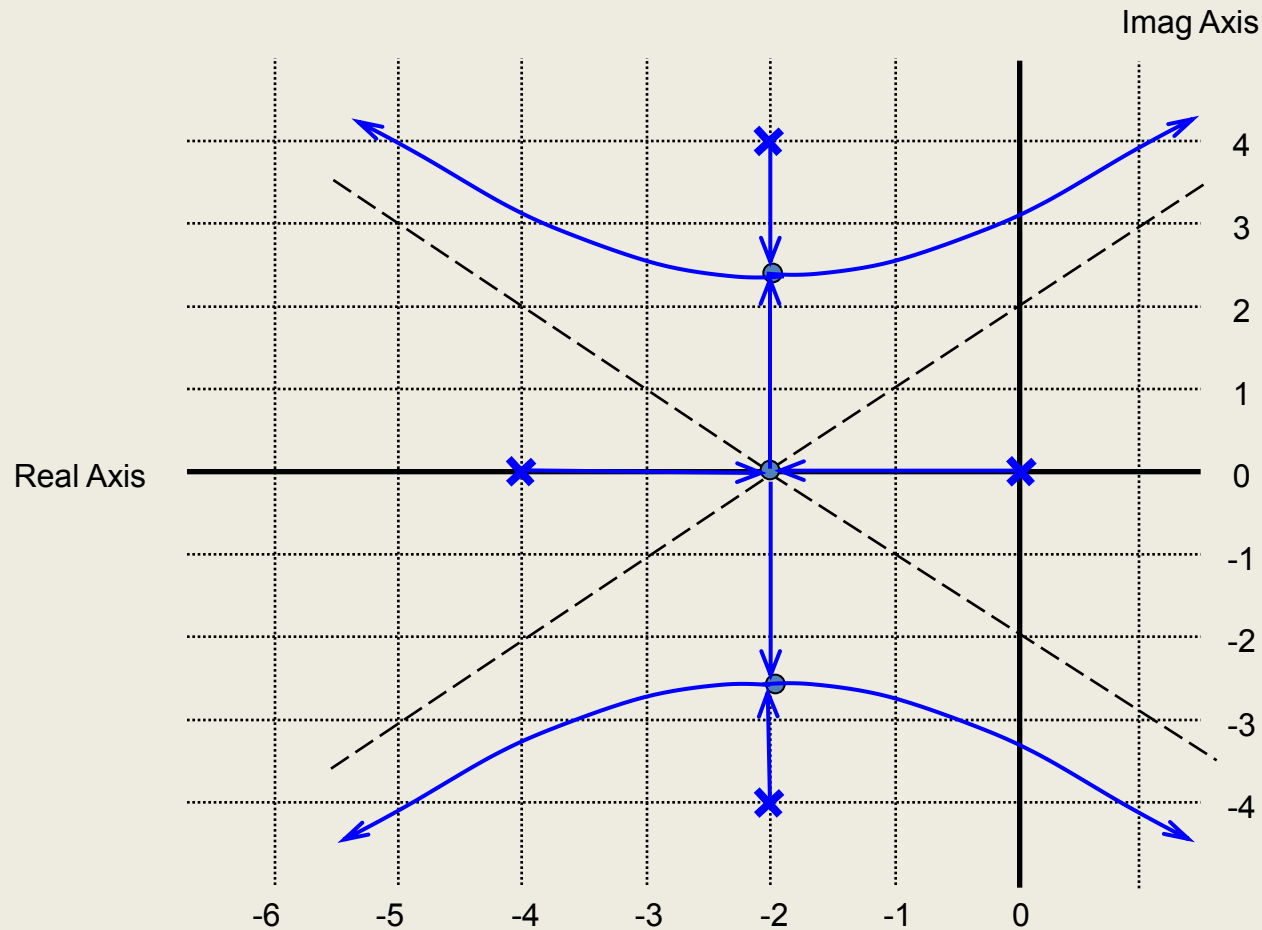
$$s = j\omega$$

$$\Rightarrow (\omega^4 - 36\omega^2 + K) + (-8\omega^3 + 80\omega)j = 0$$

$$\Rightarrow \begin{cases} \omega^4 - 36\omega^2 + K = 0 \\ -8\omega^3 + 80\omega = 0 \end{cases} \Rightarrow \begin{cases} K_1 = 0 \\ \omega_1 = 0 \end{cases}, \begin{cases} K_2 = 260 \\ \omega_2 = \sqrt{10} = 3.16 \end{cases}$$

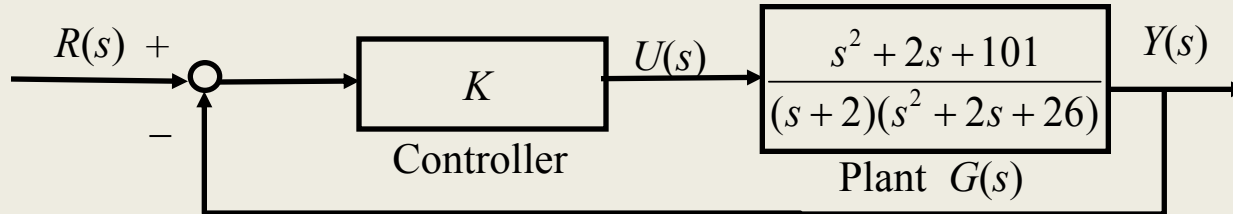
Example 3

Step 8: Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus.



Example 4

A feedback control system is proposed. The corresponding block diagram is:



Sketch the root locus of the closed-loop poles as the controller gain K varies from 0 to ∞ .

Find closed-loop characteristic equation:

$$1 + K \frac{s^2 + 2s + 101}{(s + 2)(s^2 + 2s + 26)} = 0$$

Example 4

Step 1: Formulate the (closed-loop) characteristic equation into the standard form for sketching root locus:

$$1 + K \frac{\overbrace{s^2 + 2s + 101}^{N(s)}}{\underbrace{(s + 2)(s^2 + 2s + 26)}_{D(s)}} = 0$$

Step 2: Find the open-loop zeros, z_i , and the open-loop poles, p_i :

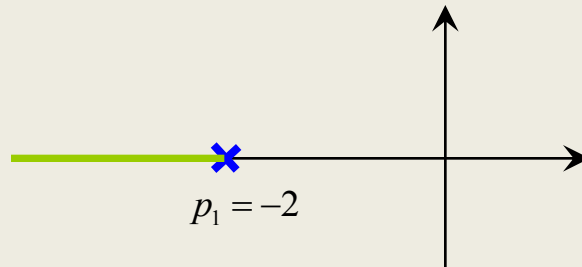
open-loop zeros

$$s^2 + 2s + 101 = (s + 1)^2 + 100 = 0, z_{1,2} = -1 \pm 10j$$

open-loop poles

$$(s + 2)((s + 1)^2 + 25) = 0, p_1 = -2, p_{2,3} = -1 \pm 5j$$

Step 3: Determine the real axis segments that are to be included in the root locus by applying Rule 4.



Example 4

Step 4: Determine the number of asymptotes and the corresponding intersection σ_0 and angles θ_k by applying Rules 2 and 5.

$$N_p - N_z = 1 \quad \text{One asymptote} \quad \theta_k = (2k+1) \times 180^\circ - 180^\circ$$

Step 5: (If necessary) Determine the break-away and break-in points using Rule 6.

Step 6: (If necessary) Determine the departure and arrival angles using Rule 7.

$$\begin{array}{llll} z_1 = -1 + 10j & \theta_{z_1} = 90^\circ - \tan^{-1}(10) - 90^\circ - 90^\circ = 180^\circ & p_1 = -2 & \theta_{p_1} = 180^\circ \\ & \theta_{z_1} = 354^\circ = -6^\circ & p_2 = -1 + 5j & \theta_{p_2} = 11^\circ \\ z_2 = -1 - 10j & \theta_{z_2} = 6^\circ & p_3 = -1 - 5j & \theta_{p_3} = -11^\circ \end{array}$$

Step 7: (If necessary) Determine the imaginary axis crossings using Rule 8.

$$(s+2)(s^2+2s+26) + K(s^2+2s+101) = 0$$

$$\Leftrightarrow s^3 + (4+K)s^2 + (30+2K)s + (52+101K) = 0$$

$$\Rightarrow \overset{s=j\omega}{\Rightarrow} [(52+101K) - (4+K)\omega^2] + [(30+2K) - \omega^2]j = 0$$

$$\begin{cases} (52+101K) - (4+K)\omega^2 = 0 \\ [(30+2K) - \omega^2]j = 0 \end{cases} \Rightarrow \begin{cases} \omega_1 = 0 \\ K_1 = -\frac{52}{101} \end{cases}, \begin{cases} \omega_2 = 9.5 \\ K_2 = 30.4 \end{cases}, \begin{cases} \omega_3 = 5.7 \\ K_3 = 1.1 \end{cases}$$

Example 4

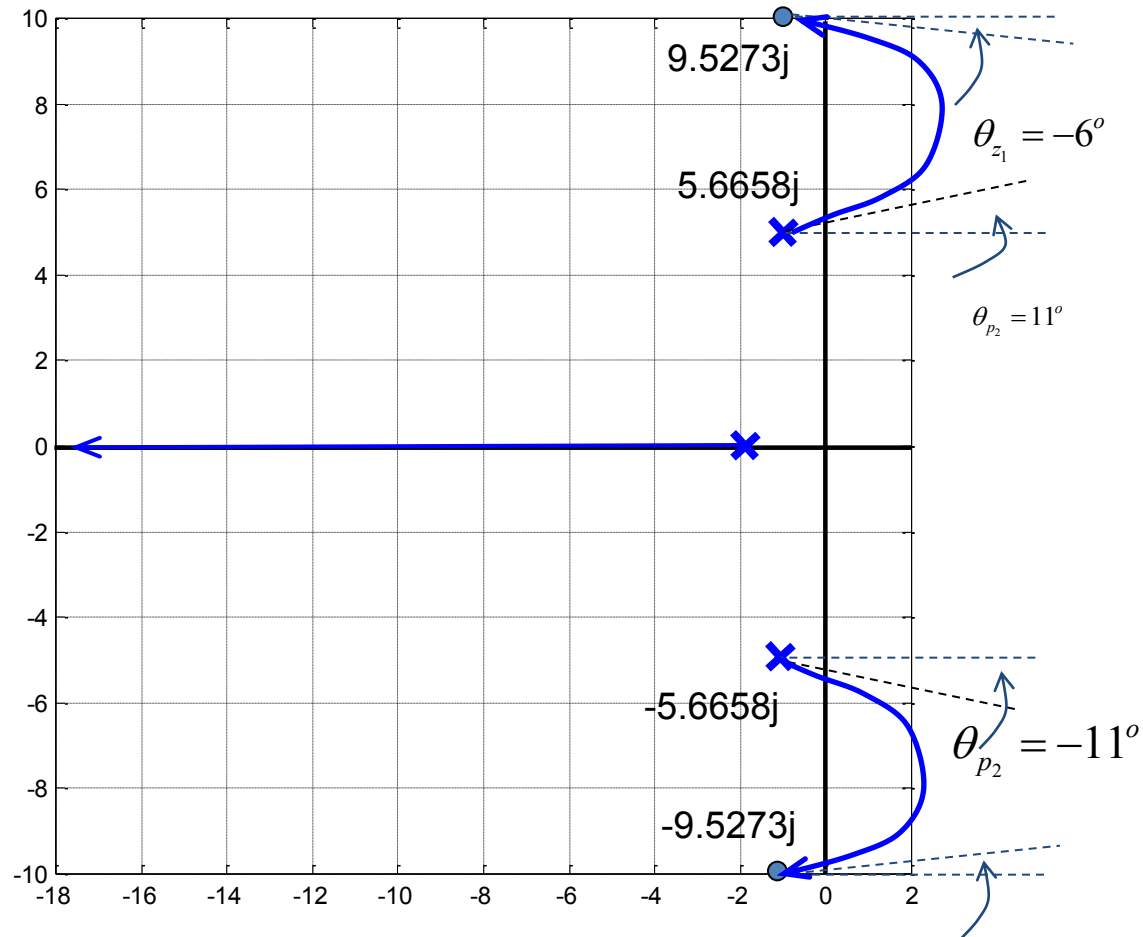
Step 8: Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus.

Stability condition

$$0 < K < 1.1$$

or

$$K > 30.4$$



Root Locus as an Analysis/Design Tool

Mechanical system response depends on the location of the system characteristic values, i.e., poles of the system transfer function. Since root locus tells us how the system poles vary w.r.t. a parameter K , we can use root locus to analyze the effect of parameter variation on system performance.

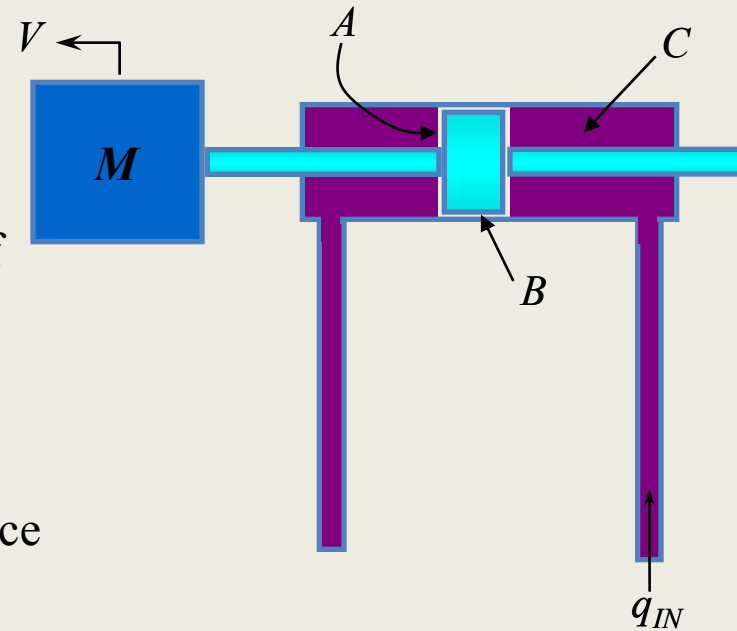
Ex: (*Motion Control of Hydraulic Cylinders*)

Recall the example of the flow control of a hydraulic cylinder that takes into account the capacitance effect of the pressure chamber. The plant transfer function is:

$$G(s) = \frac{V(s)}{Q_{IN}(s)} = \frac{A}{MCs^2 + BCs + A^2}$$

where M is the mass of the load; C is the flow capacitance of the pressure chamber; A is the effective area of the piston and B is the viscous friction coefficient.

Q: How would the plant parameters affect the system response ?



Root Locus as an Analysis/Design Tool

- Effect of load (M) on system performance:

System characteristic equation:

$$MCs^2 + BCs + A^2 = 0$$

Transform characteristic equation into standard form for root locus analysis by identifying the parameter that is to be varied. In this case, the load mass M is the varying parameter:

Standard form $1 + K \frac{N(s)}{D(s)} = 0$

$$1 + M \frac{C}{BC} \frac{s^2 \frac{N(s)}{A^2}}{\underbrace{s + \frac{A^2}{BC}}_{D(s)}} = 0$$

Varying parameter

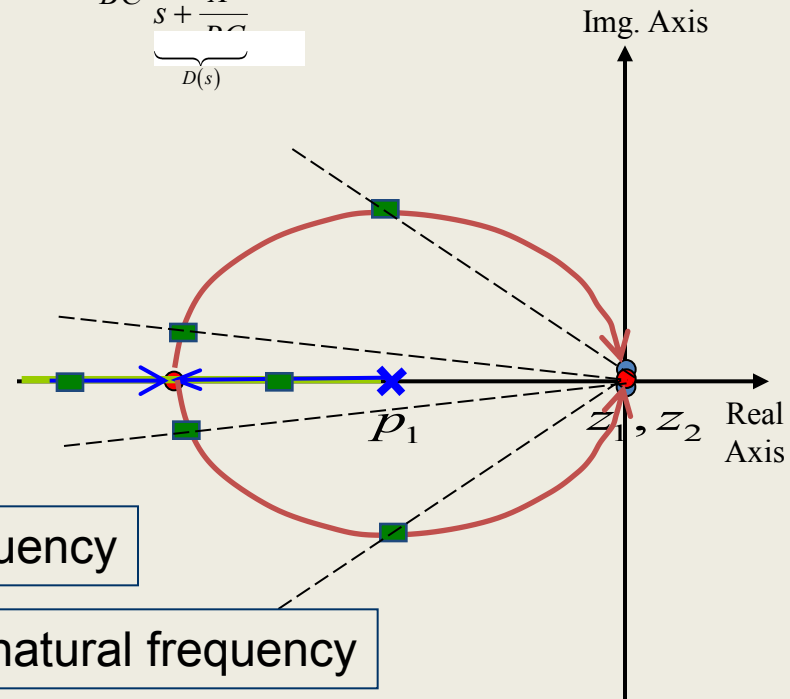
open-loop zeros

$$z_1 = z_2 = 0$$

open-loop poles

$$p_1 = -\frac{A^2}{BC}$$

$$\frac{d}{ds} \left(\frac{N(s)}{D(s)} \right) = 0 \Rightarrow s = -\frac{2A^2}{BC}, s = 0$$



Small M : less overshoot and high natural frequency

As M increases: larger overshoot and lower natural frequency

Think about the settling time

Root Locus as an Analysis/Design Tool

- Effect of flow capacitance (C) on system performance:

System characteristic equation: $MCs^2 + BCs + A^2 = 0$

Transform characteristic equation into standard form for root locus analysis by identifying the parameter that is to be varied. In this case, the flow capacitance C is the varying parameter:

Standard form $1 + K \frac{N(s)}{D(s)} = 0$

Varying parameter

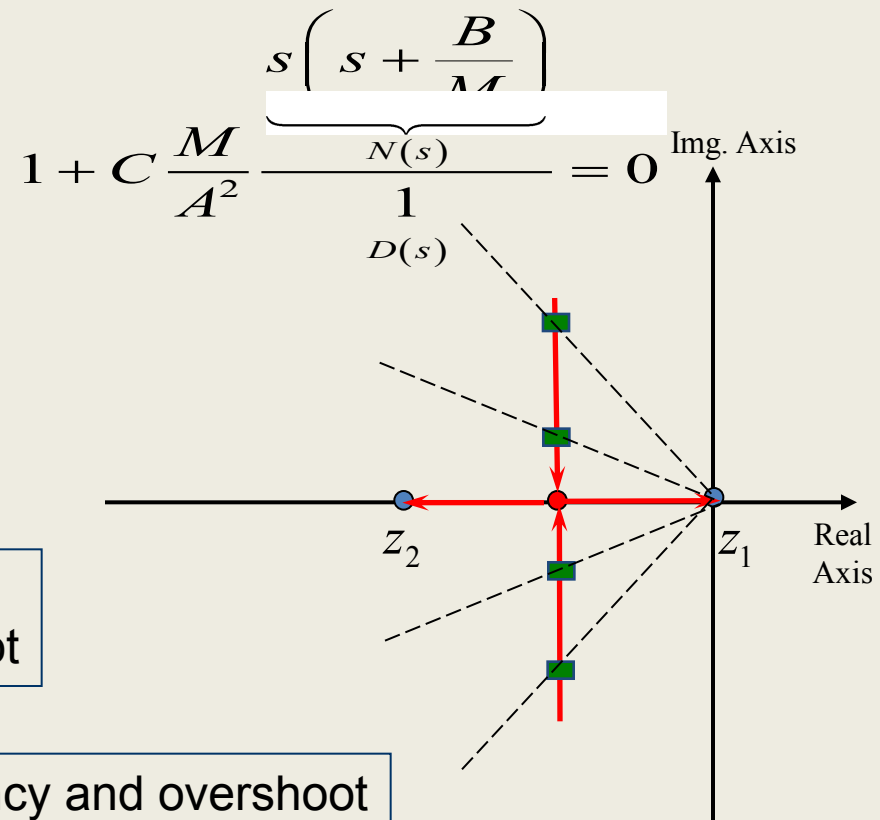
open-loop zeros $z_1 = 0, z_2 = -\frac{B}{M}$

open-loop poles NO open-loop poles

$$\frac{d}{ds} \left(\frac{N(s)}{D(s)} \right) = 0 \Rightarrow s = -\frac{B}{2M}$$

Smaller C (or less compressible fluid):
Larger oscillating frequency and overshoot

Larger C : smaller oscillating frequency and overshoot



Root Locus as an Analysis/Design Tool

- Effect of friction (B) on system performance:

System characteristic equation: $MCs^2 + BCs + A^2 = 0$

Transform characteristic equation into standard form for root locus analysis by identifying the parameter that is to be varied. In this case, the viscous friction coefficient B is the varying parameter:

Standard form $1 + K \frac{N(s)}{D(s)} = 0$ $1 + B \frac{C}{MC} \frac{s}{s^2 + \frac{A^2}{MC}} = 0$

Varying parameter

open-loop zeros

$$z_1 = 0$$

open-loop poles

$$p_{1,2} = \pm j\sqrt{\frac{A^2}{MC}}$$

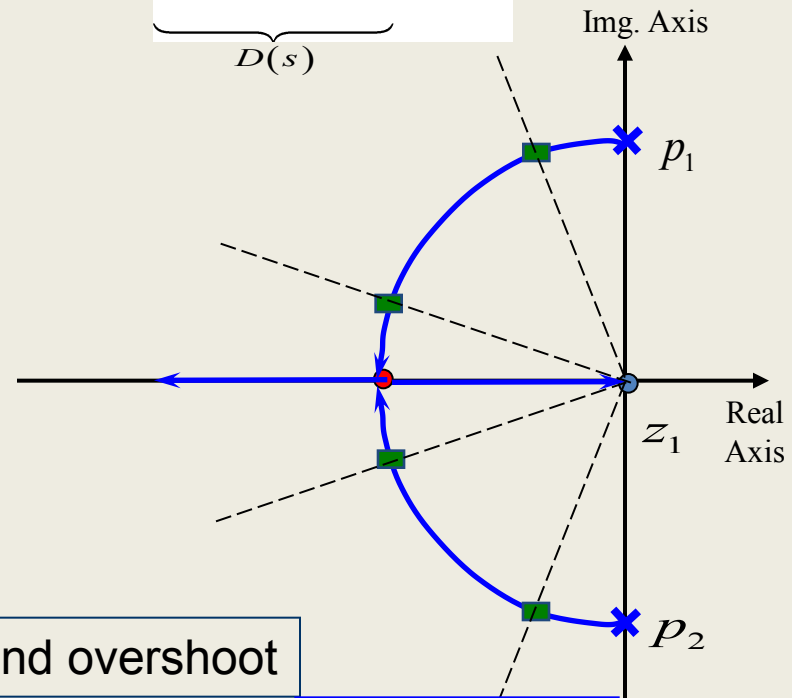
$$\frac{d}{ds} \left(\frac{D(s)}{N(s)} \right) = 0 \Rightarrow s = -\sqrt{\frac{A^2}{MC}}$$

Smaller B :

Larger oscillating frequency and overshoot

Larger B : smaller oscillating frequency and overshoot

settling time?



Nyquist Plot or Polar Plot

- Nyquist Plots were invented by Nyquist - who worked at Bell Laboratories, the premiere technical organization in the U.S. at the time.
- Nyquist Plots are a way of showing frequency responses of linear systems.
- There are several ways of displaying frequency response data, including Bode' plots and Nyquist plots.
- Bode' plots use frequency as the horizontal axis and use two separate plots to display amplitude and phase of the frequency response.
- Nyquist plots display both amplitude and phase angle on a single plot, using frequency as a parameter in the plot.
- Nyquist plots have properties that allow you to see whether a system is stable or unstable.

Nyquist Plot

- A Nyquist plot is a polar plot of the frequency response function of a linear system.
- That means a Nyquist plot is a plot of the transfer function, $G(s)$ with $s = j\omega$. That means you want to plot $G(j\omega)$.
- $G(j\omega)$ is a complex number for any angular frequency, ω , so the plot is a plot of complex numbers.
- The complex number, $G(j\omega)$, depends upon frequency, so frequency will be a parameter if you plot the imaginary part of $G(j\omega)$ against the real part of $G(j\omega)$.

Sketch the Polar plot of Frequency Response

To sketch the polar plot of $G(j\omega)$ for the entire range of frequency ω , i.e., from 0 to infinity, there are four key points that usually need to be known:

- 1) The start of plot where $\omega = 0$,
- 2) The end of plot where $\omega = \infty$,
- 3) Where the plot crosses the real axis, i.e., $\text{Im}(G(j\omega)) = 0$, and
- 4) Where the plot crosses the imaginary axis, i.e., $\text{Re}(G(j\omega)) = 0$.

Problem-1: Polar Plot of Integrator

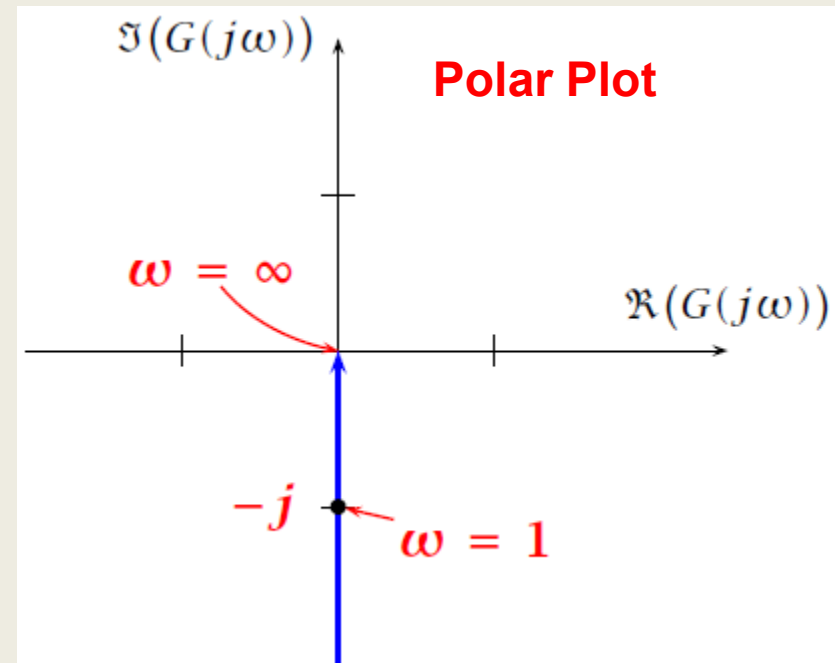
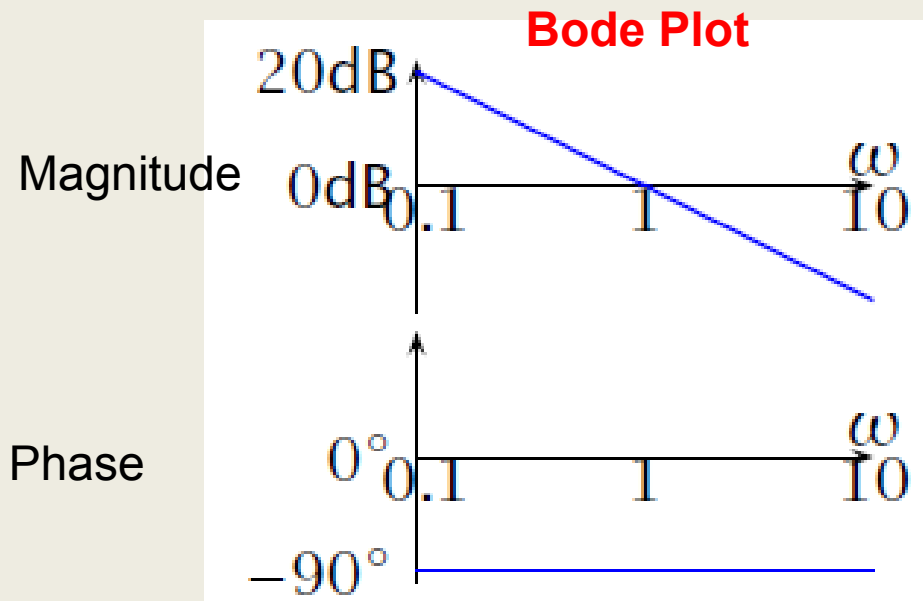
Consider a first order system, $G(s) = \frac{1}{s}$

Representing $G(s)$ in the **frequency response form** $G(j\omega)$ by replacing $s = j\omega$:

$$G(j\omega) = \frac{1}{j\omega} = \frac{-j}{\omega}$$

The **magnitude** of $G(j\omega)$, i.e., $|G(j\omega)|$, is obtained as; $|G(j\omega)| = 1/\omega$

The **phase** of $G(j\omega)$, denoted by, ϕ , is obtained as; $\angle G(j\omega) = -90^\circ$



Problem-2: Polar Plot of First Order System

Consider a first order system $G(s) = \frac{1}{1 + sT}$ where T is the time constant.

Representing $G(s)$ in the **frequency response form** $G(j\omega)$ by replacing **$s = j\omega$** :

$$G(j\omega) = \frac{1}{1 + j\omega T}$$

The **magnitude** of $G(j\omega)$, i.e., $|G(j\omega)|$, is obtained as;

$$|G(j\omega)| = \frac{1}{\sqrt{1 + \omega^2 T^2}}$$

The **phase** of $G(j\omega)$, denoted by, ϕ , is obtained as;

$$\angle G(j\omega) = -\arctan(\omega T)$$

The start of plot where $\omega = 0$

$$|G(j\omega)| = \frac{1}{\sqrt{1+0}} = 1, \quad \phi = \tan^{-1}\left(\frac{0}{1}\right) = 0$$

The end of plot where $\omega = \infty$

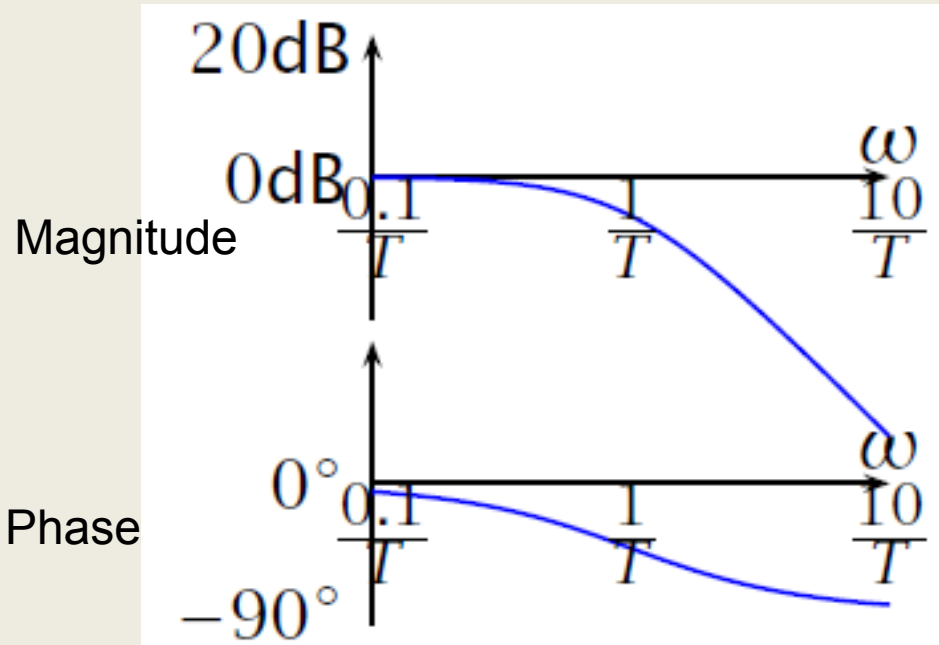
$$|G(j\omega)| = \frac{1}{\sqrt{1+\infty}} = 0, \quad \phi = \tan^{-1}\left(\frac{-\infty}{1}\right) = -90^\circ$$

The mid part of plot where $\omega = 1/T$

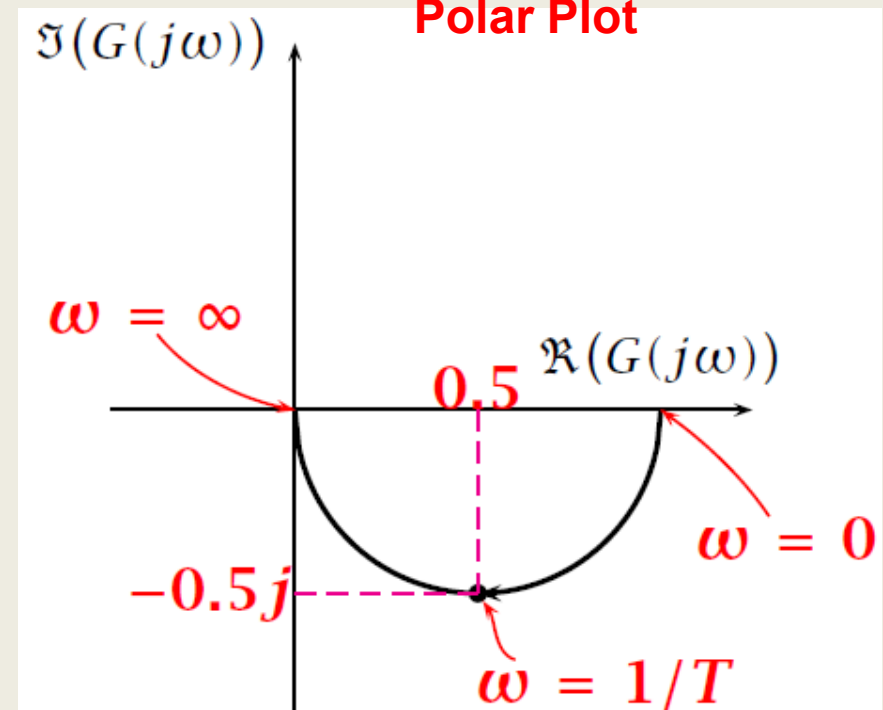
$$|G(j\omega)| = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}, \quad \phi = \tan^{-1}\left(\frac{-1}{1}\right) = -45^\circ$$

	$ G(j\omega) $	$\angle G(j\omega)$
$\omega = 0$	1	0
$\omega = \frac{1}{T}$	$\frac{1}{\sqrt{2}}$	-45°
$\omega \rightarrow \infty$	0	-90°

Bode Plot



Polar Plot



Problem-3: Polar Plot of Second Order System

Consider a second order system $G(s) = \frac{1}{(1 + sT_1)(1 + sT_2)}$ where T is the time constant.

Representing $G(s)$ in the **frequency response form** $G(j\omega)$ by replacing **$s = j\omega$** :

$$G(j\omega) = \frac{1}{(1 + j\omega T_1)(1 + j\omega T_2)}$$

The **magnitude** of $G(j\omega)$, i.e., $|G(j\omega)|$, is obtained as;

$$|G(j\omega)| = \frac{1}{\sqrt{1 + \omega^2 T_1^2} \sqrt{1 + \omega^2 T_2^2}}$$

The **phase** of $G(j\omega)$, denoted by, ϕ , is obtained as;

$$\angle G(j\omega) = -\arctan(\omega T_1) - \arctan(\omega T_2)$$

The start of plot where $\omega = 0$

$$|G(j\omega)| = \frac{1}{\sqrt{1+0}\sqrt{1+0}} = 1$$

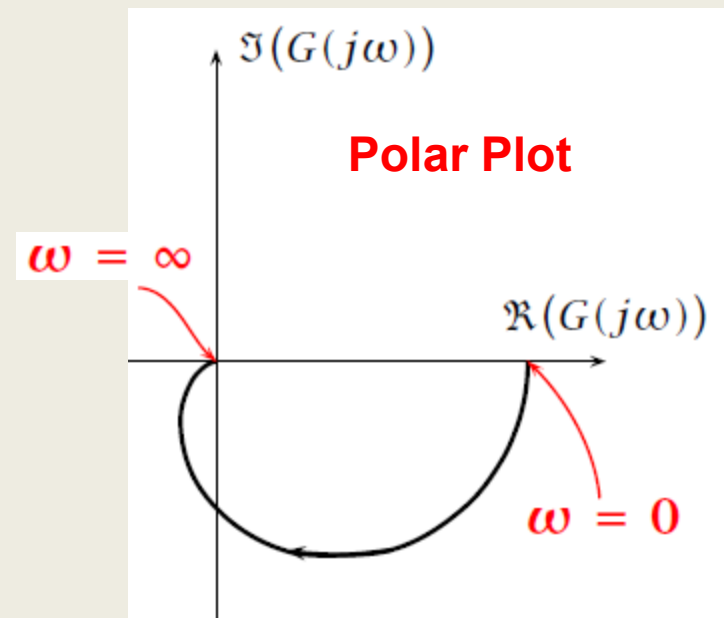
$$\angle G(j\omega) = -\tan^{-1}(0) - \tan^{-1}(0) = 0^\circ$$

The end of plot where $\omega = \infty$

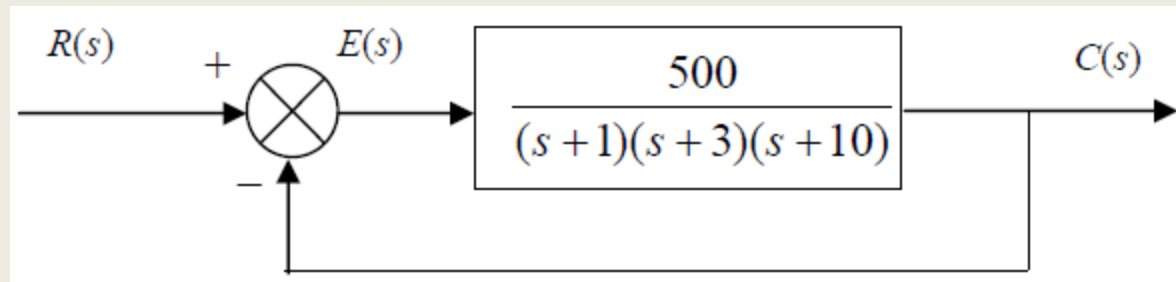
$$|G(j\omega)| = \frac{1}{\sqrt{\infty}\sqrt{\infty}} = 0$$

$$\angle G(j\omega) = -\tan^{-1}(\infty) - \tan^{-1}(\infty) = -90^\circ - 90^\circ = -180^\circ$$

	$ G(j\omega) $	$\angle G(j\omega)$
$\omega = 0$	1	0
$\omega \rightarrow \infty$	0	-180°



Problem-4: Sketch the Nyquist diagram for the system shown in the following figure, and then determine the system stability using the Nyquist criterion.



The open-loop transfer function: $G(s)H(s) = \frac{500}{(s+1)(s+3)(s+10)}$.

Replacing s with $j\omega$ yields the frequency response of $G(s)H(s)$, i.e.,

$$G(j\omega)H(j\omega) = \frac{500}{(j\omega+1)(j\omega+3)(j\omega+10)} = \frac{500}{(-14\omega^2 + 30) + j(43\omega - \omega^3)}$$

$$= 500 \frac{(-14\omega^2 + 30) - j(43\omega - \omega^3)}{(-14\omega^2 + 30)^2 + (43\omega - \omega^3)^2} \longrightarrow \text{(a)}$$

Magnitude response:

$$|G(j\omega)H(j\omega)| = \sqrt{\text{Re}^2 + \text{Im}^2} = \frac{500}{\sqrt{(-14\omega^2 + 30)^2 + (43\omega - \omega^3)^2}}$$

Phase response:

$$\angle G(j\omega)H(j\omega) = \tan^{-1}\left(\frac{\text{Im}}{\text{Re}}\right) = \tan^{-1}\left(\frac{-(43\omega - \omega^3)}{-14\omega^2 + 30}\right)$$

Now that we have expressions for the magnitude and phase of the frequency response, we can sketch the polar plot using the **4 key points**.

Point 1: The start of plot where $\omega = 0$

$$|G(0)H(0)| = \frac{500}{\sqrt{(30)^2}} = 16.67$$

$$\angle G(0)H(0) = \tan^{-1}\frac{0}{30} = 0^\circ$$

Point 2: The end of plot where $\omega = \infty$

$$|G(\infty)H(\infty)| = \frac{500}{\sqrt{\infty}} = 0$$

$$\angle G(\infty)H(\infty) = \tan^{-1} \frac{\infty^3}{30} = -3 \times 90^\circ = -270^\circ$$

Point 3: Where the plot crosses the real axis, i.e., $\text{Im}(G(j\omega)) = 0$

Take the imaginary part of equation (a), and put equal to zero, to get the value of frequency ω at the interception of real axis.

$$\frac{-(43\omega - \omega^3)}{(-14\omega^2 + 30)^2 + (43\omega - \omega^3)^2} = 0 \Rightarrow \begin{cases} 43\omega - \omega^3 = 0 & \Rightarrow \omega = 0 \quad \text{and} \quad \omega = 6.56 \text{ rad/s} \\ \omega = \infty \end{cases}$$

Point 4: Where the plot crosses the imaginary axis, $\text{Re}(G(j\omega)) = 0$

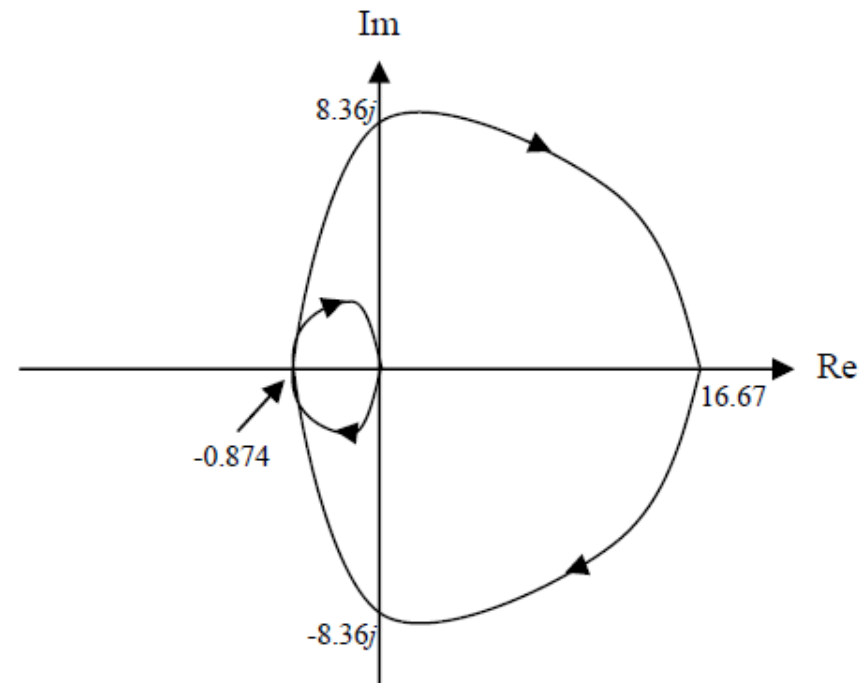
Take the real part of equation (a), and put equal to zero, to get the value of frequency ω at the interception of imaginary axis.

$$\frac{-14\omega^2 + 30}{(-14\omega^2 + 30)^2 + (43\omega - \omega^3)^2} = 0 \Rightarrow \begin{cases} -14\omega^2 + 30 = 0 & \Rightarrow \omega = 1.46 \text{ rad/s} \\ \omega = \infty \end{cases}$$

Key Points of the polar plot:

	$ GH $	$\angle GH$
$\omega = 0$	16.67	0
$\omega = \infty$	0	-270°
Cross Re:		
$\omega = 0$		
$\omega = \infty$	See above	See above
$\omega = 6.56 \text{ rad/s}$	0.874	-180°
Cross Re:		
$\omega = 0$		
$\omega = \infty$	See above	See above
$\omega = 1.46 \text{ rad/s}$	8.36	-90°

Nyquist diagram



Problem-5: Sketch the polar plot for the following transfer function.

$$G(s) = \frac{10}{s(s+1)(s+5)}$$

Representing $G(s)$ in the **frequency response form** $G(j\omega)$ by replacing $s = j\omega$, and then Multiply both numerator and denominator by the conjugate of denominator.

$$\begin{aligned} G(j\omega) &= \frac{10}{j\omega(j\omega+1)(j\omega+5)} \\ &= \frac{10}{(-\omega^2 + j\omega)(j\omega+5)} \\ &= \frac{10}{(-j\omega^3 - 5\omega^2 - \omega^2 + 5j\omega)} \\ &= \frac{10}{-6\omega^2 + j(5\omega - \omega^3)} * \frac{-6\omega^2 - j(5\omega - \omega^3)}{-6\omega^2 - j(5\omega - \omega^3)} \end{aligned}$$

$$G(j\omega) = \frac{-60\omega^2 - j10\omega(5 - \omega^2)}{36\omega^4 - (5\omega - \omega^3)^2} \longrightarrow \text{(a)}$$

Point 1: The start of plot where $\omega = 0$

At frequency $\omega = 0$, we only observe the most significant terms that take the effect.

$$\text{Magnitude at } \omega = 0: \quad |G(j\omega)|_{\omega=0} = \frac{10}{5j\omega} \Big|_{\omega=0} = \frac{2}{j\omega} \Big|_{\omega=0} = \infty$$

$$\text{Phase at } \omega = 0: \quad \angle G(j\omega) \Big|_{\omega=0} = \lim_{\omega \rightarrow 0} \angle \frac{2}{j\omega} = -90^\circ$$

Point 2: The end of plot where $\omega = \infty$

At frequency $\omega = \infty$, we shall look at the most significant term that takes effect when the frequency approaches infinity.

$$\text{Magnitude at } \omega = \infty: \quad |G(j\omega)|_{\omega \rightarrow \infty} = \lim_{\omega \rightarrow \infty} \left| \frac{10}{(j\omega^3)} \right| = \lim_{\omega \rightarrow \infty} \frac{10}{\omega^3} = 0$$

$$\text{Phase at } \omega = \infty: \quad \angle G(j\omega) \Big|_{\omega \rightarrow \infty} = \angle \lim_{\omega \rightarrow \infty} \left[\frac{10}{(j\omega)^3} \right] = -270^\circ$$

Point 3: Where the plot crosses the real axis, i.e., $\text{Im}(G(j\omega)) = 0$

Take the imaginary part of equation (a), and put equal to zero, to get the value of frequency ω at the interception of real axis.

$$\begin{aligned}\Rightarrow -\frac{10\omega(5-\omega^2)}{36\omega^4+(5\omega-\omega^3)^2} &= 0 \\ \Rightarrow 10(5-\omega^2) &= 0 \\ \Rightarrow \omega^2 &= 5 \\ \Rightarrow \omega &= \sqrt{5}\end{aligned}$$

Therefore, the intersection point between the polar plot and the real axis, when $\omega = \sqrt{5}$, is located at;

$$G(j\omega)|_{\omega=\sqrt{5}} = -\frac{1}{3}$$

Point 4: Where the plot crosses the imaginary axis, $\text{Re}(G(j\omega)) = 0$

Take the real part of equation (a), and put equal to zero, to get the value of frequency ω at the interception of imaginary axis.

$$\Rightarrow -\frac{60\omega^2}{36\omega^4 + (5\omega - \omega^3)^2} = 0$$

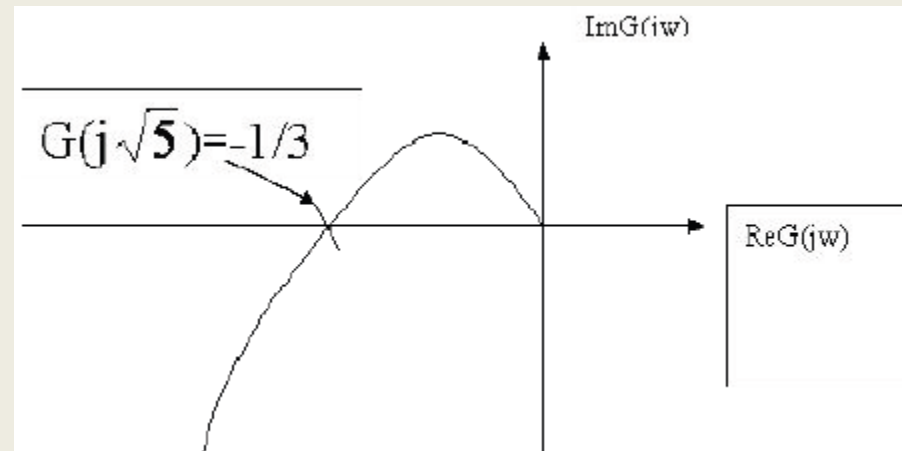
$$\Rightarrow \omega = \infty$$

Therefore, the intersection point between the polar plot and the imaginary axis is when $\omega = \infty$ is located at;

$$G(j\infty) = 0$$

Polar Plot

	$ G(j\omega) $	$\angle G(j\omega)$
$\omega = 0$	∞	-90°
$\omega = \infty$	0	-270°
Cross Re: $\omega = \sqrt{5}$	$-\frac{1}{3}$	
Cross Img: $\omega = \infty$	0	



Problem-6: Sketch the polar plot for the following transfer function.

$$GH(s) = \frac{1}{s^4(s+p)} \quad p > 0$$

Representing $G(s)H(s)$ in the **frequency response form** $G(j\omega)H(j\omega)$ by replacing **$s = j\omega$** :

$$GH(j\omega) = \frac{1}{j^4\omega^4(j\omega+p)}$$

The **magnitude** of $GH(j\omega)$ i.e., $|GH(j\omega)|$, is obtained as;

$$|GH(j\omega)| = \frac{1}{\omega^4\sqrt{\omega^2+p^2}}$$

The **phase** of $GH(j\omega)$ denoted by, φ , is obtained as;

$$\varphi = \angle GH(0) = -\tan^{-1}\left(\frac{\omega}{p}\right)$$

Point 1: The start of plot where $\omega = 0$

$$|GH(0)| = \frac{1}{0\sqrt{0+p^2}} = \infty$$

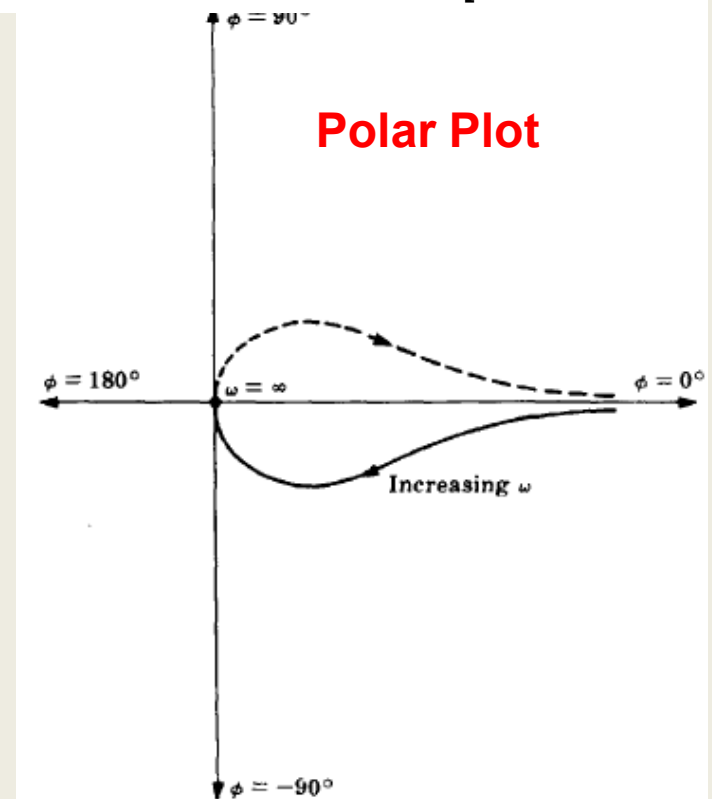
$$\varphi = \angle GH(0) = -\tan^{-1} \frac{0}{p} = 0^\circ$$

Point 2: The end of plot where $\omega = \infty$

$$|GH(\infty)| = \frac{1}{\infty\sqrt{\infty+p^2}} = 0$$

$$\varphi = \angle GH(\infty) = -\tan^{-1} \frac{\infty}{p} = -90^\circ$$

	$ GH(j\omega) $	$\angle GH(j\omega)$
$\omega = 0$	∞	0°
$\omega = \infty$	0	-90°

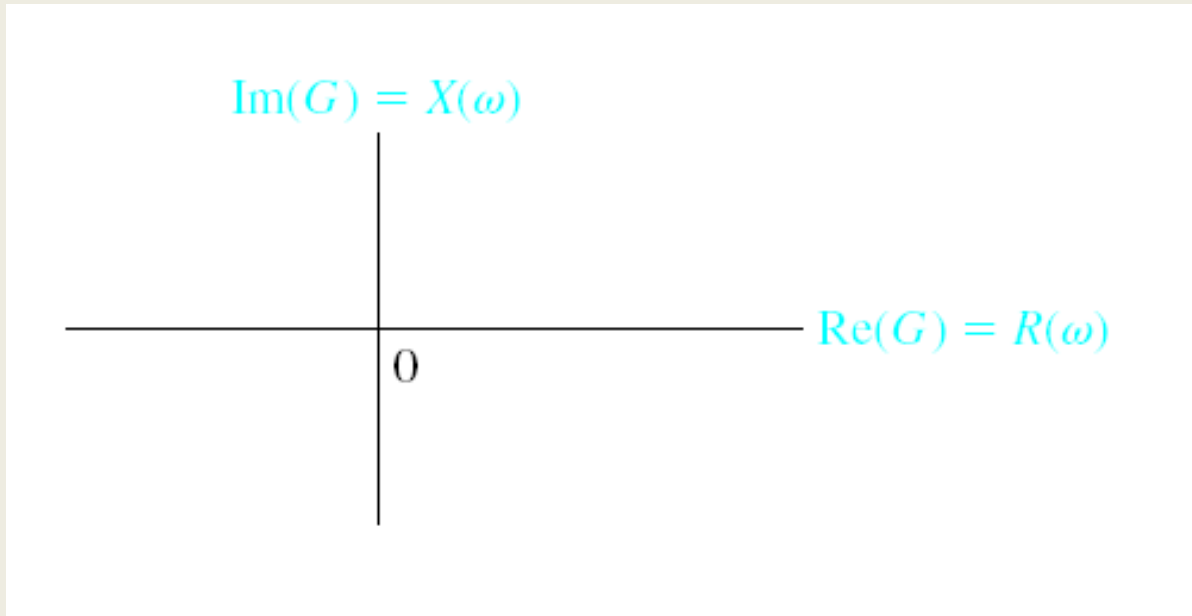


Introduction

The frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal. The sinusoid is a unique input signal, and the resulting output signal for a linear system, as well as signals throughout the system, is sinusoidal in the steady-state; it differs from the input waveform only in amplitude and phase.

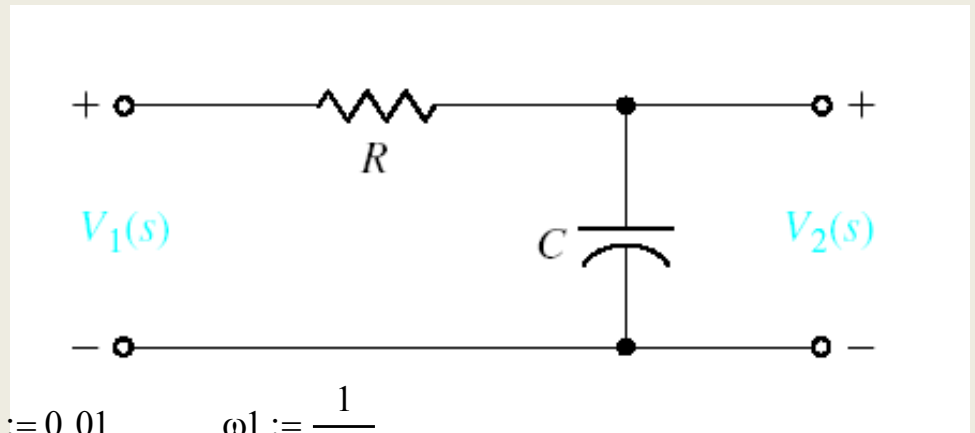
Frequency Response Plots

Polar Plots



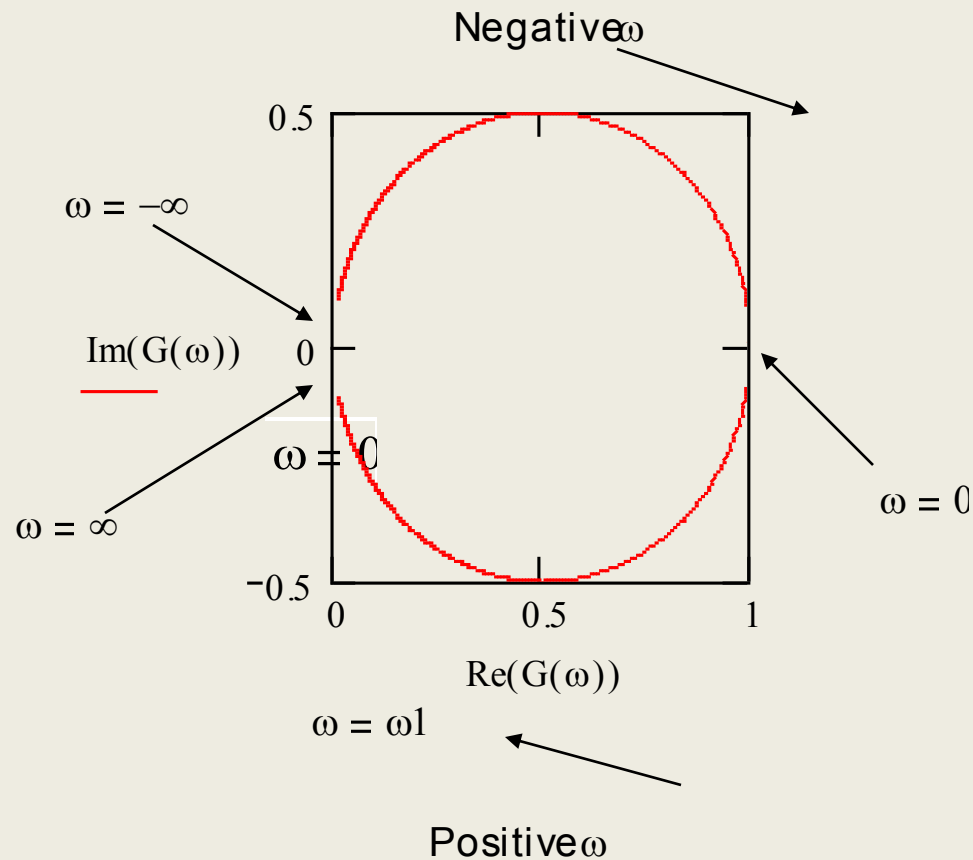
Frequency Response Plots

Polar Plots



$\omega := -1000, -999..1000$ $j := \sqrt{-1}$ $R := 1$ $C := 0.01$ $\omega_1 := \frac{1}{R \cdot C}$

$$G(\omega) := \frac{1}{\left(j \cdot \frac{\omega}{\omega_1}\right) + 1}$$



Frequency Response Plots

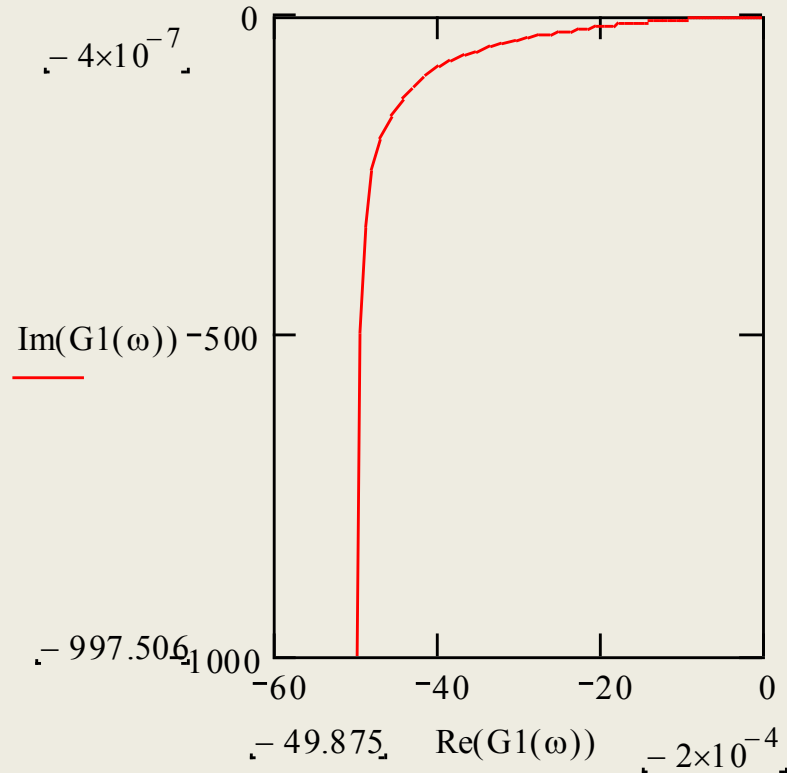
Polar Plots

$$\omega := 0, .1..1000$$

$$\tau := 0.5$$

$$K := 100$$

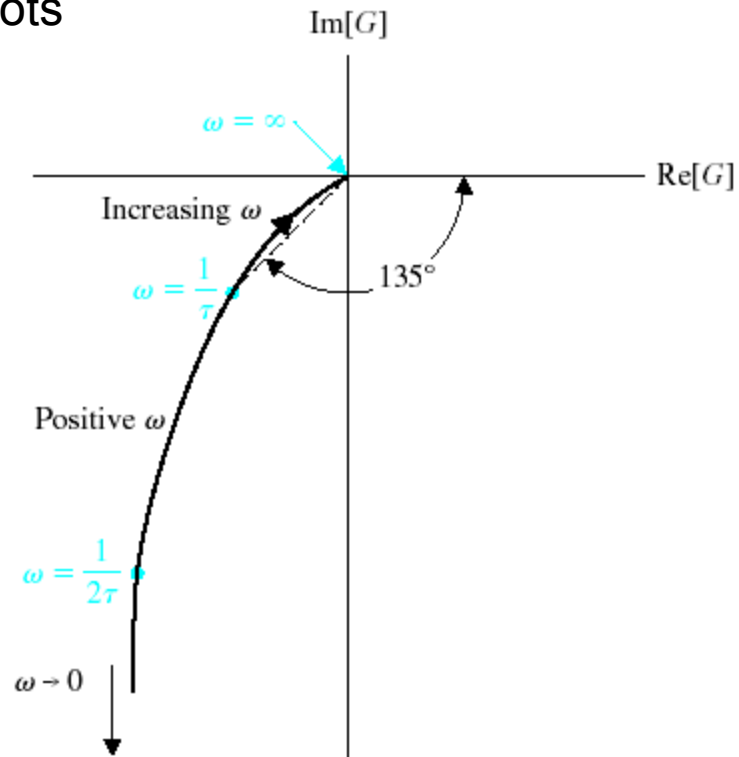
$$G1(\omega) := \frac{\frac{K}{\tau}}{j \cdot \omega \cdot \left(j \cdot \omega + \frac{1}{\tau} \right)}$$



Polar plot for $G(j\omega) = K/j\omega(j\omega\tau + 1)$. Note that $\omega = \infty$ at the origin.

Frequency Response Plots

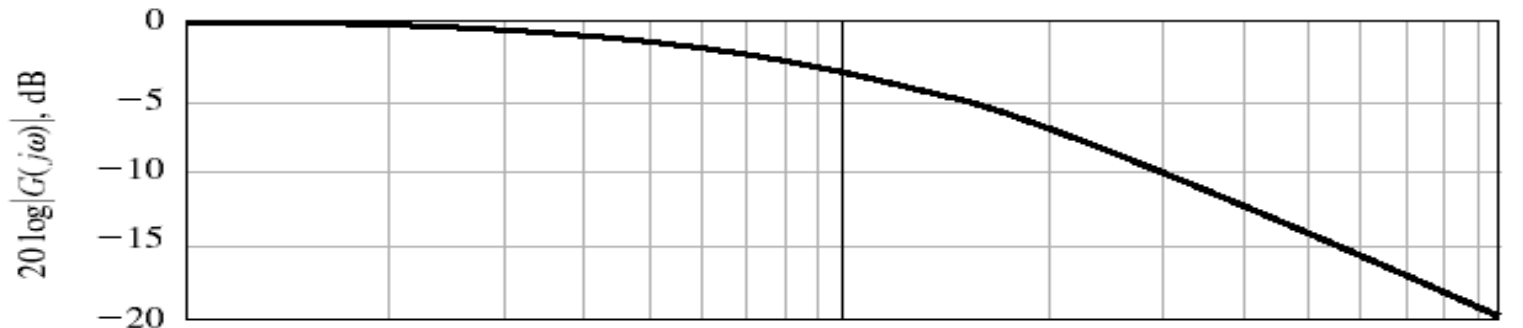
Polar Plots



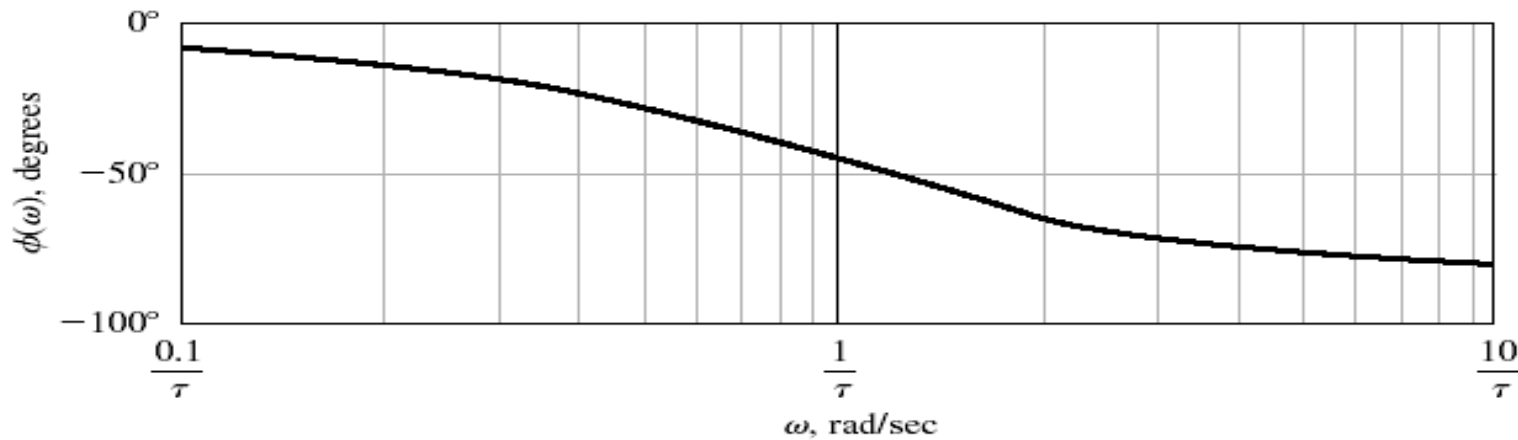
Polar plot for $G(j\omega) = K/j\omega(j\omega\tau + 1)$. Note that $\omega = \infty$ at the origin.

Frequency Response Plots

Bode Plots – Real Poles



(a)



(b)

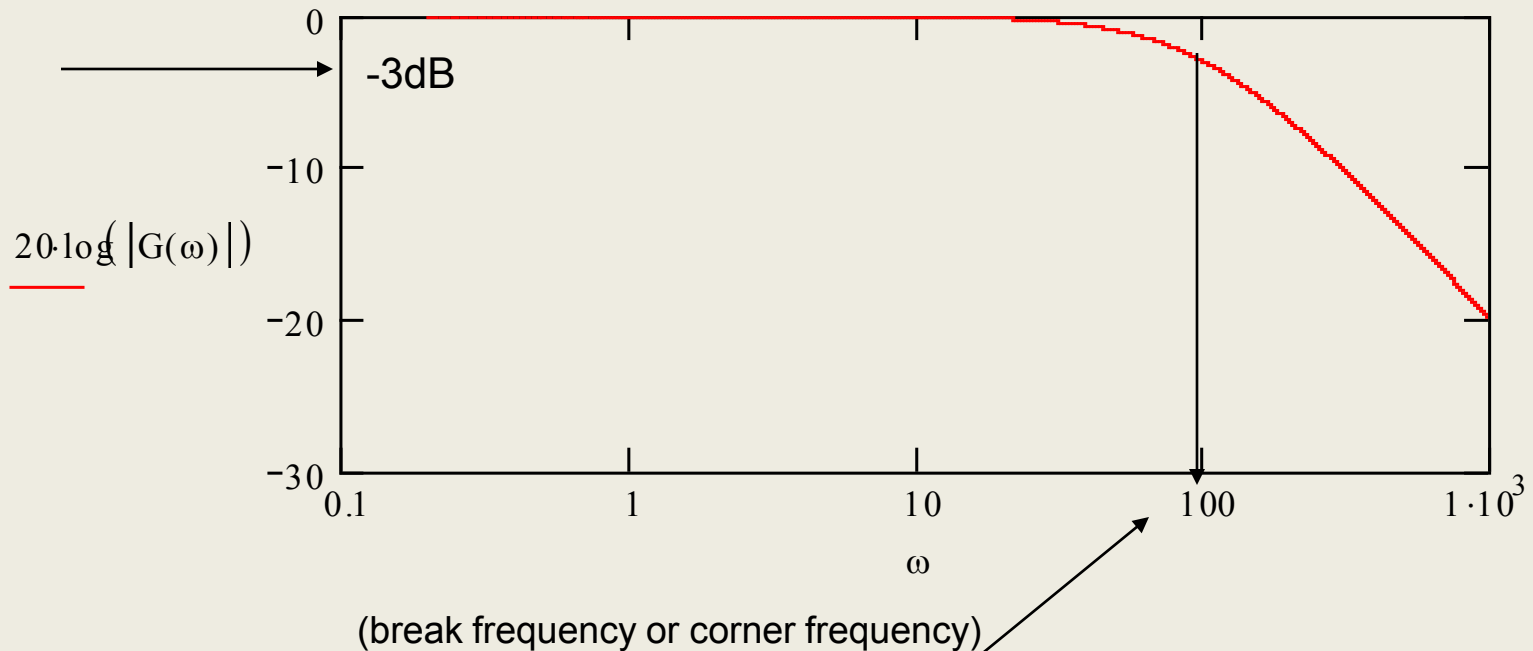
Bode diagram for $G(j\omega) = 1/(j\omega\tau + 1)$: (a) magnitude plot and (b) phase plot.

Frequency Response Plots

Bode Plots – Real Poles

$$\omega := \frac{0.1}{\tau}, \frac{0.11}{\tau} \dots 1000 \quad j := \sqrt{-1} \quad R := 1 \quad C := 0.01 \quad \tau := R \cdot C$$

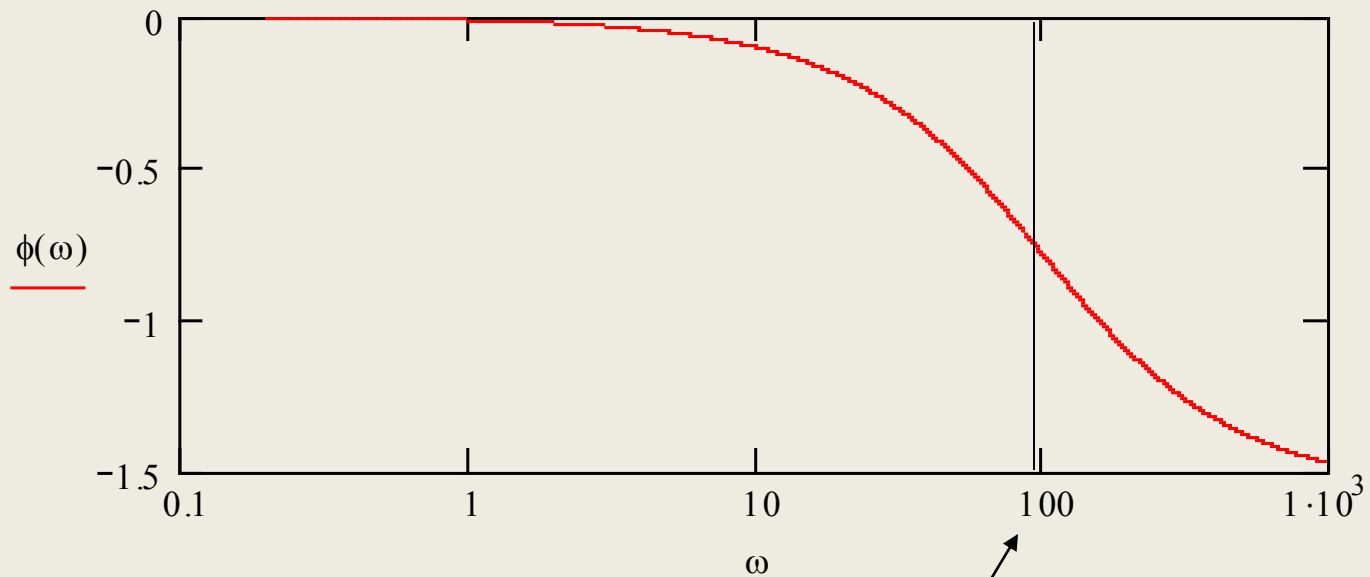
$$G(\omega) := \frac{1}{j \cdot \omega \cdot \tau + 1} \quad \omega_1 := \frac{1}{\tau} \quad \omega_1 = 100 \text{ (break frequency or corner frequency)}$$



Frequency Response Plots

Bode Plots – Real Poles

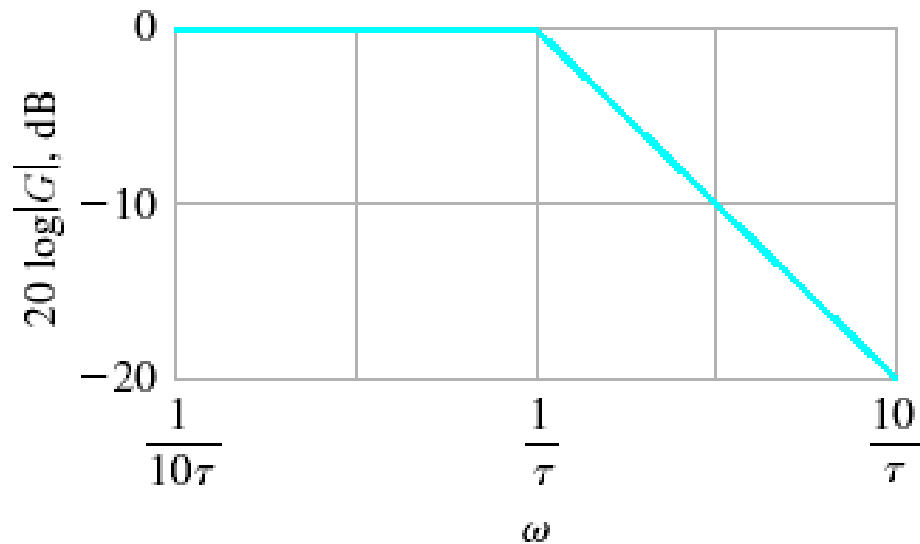
$$\phi(\omega) := -\text{atan}(\omega \cdot \tau)$$



(break frequency or corner frequency)

Frequency Response Plots

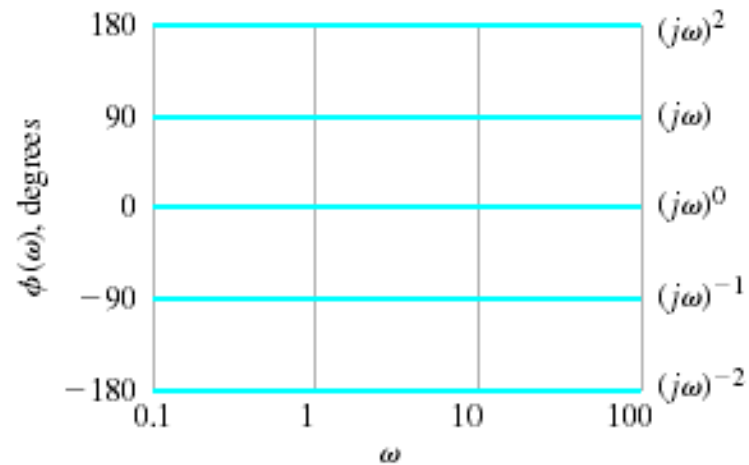
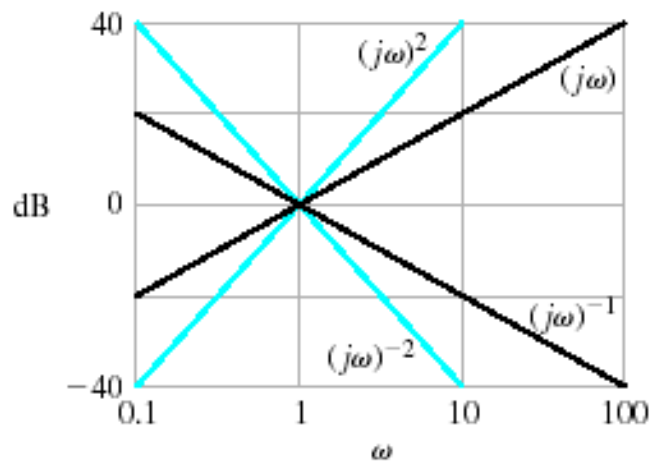
Bode Plots – Real Poles (Graphical Construction)



Asymptotic curve for $(j\omega\tau + 1)^{-1}$.

Frequency Response Plots

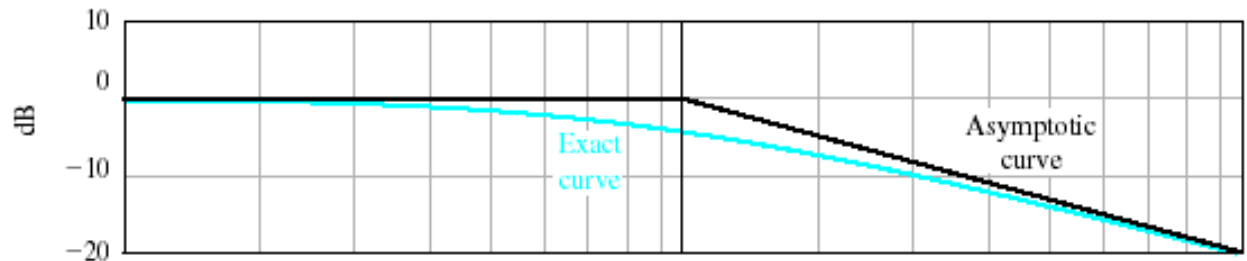
Bode Plots – Real Poles



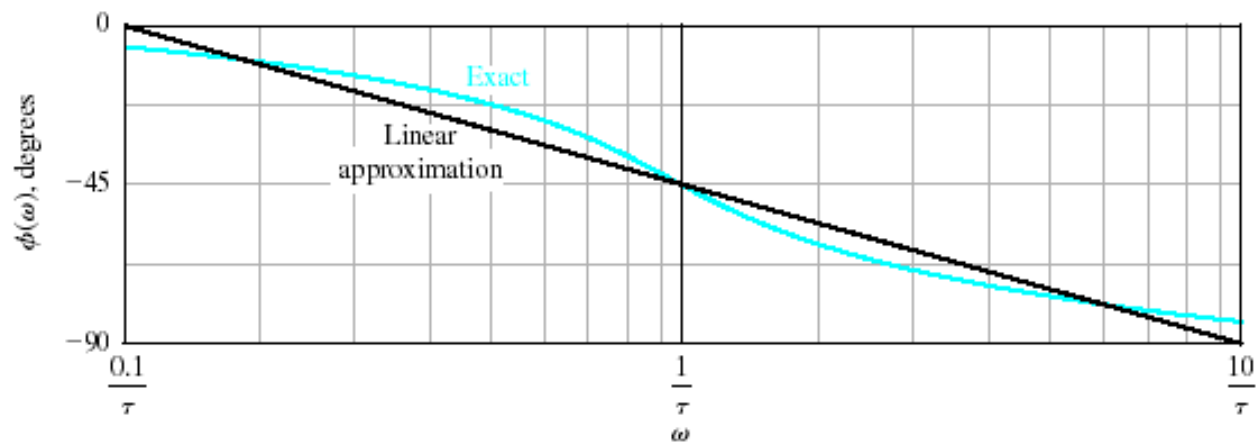
Bode diagram for $(j\omega)^{\pm N}$.

Frequency Response Plots

Bode Plots – Real Poles



(a)



(b)

Bode diagram for $(1 + j\omega\tau)^{-1}$.

Frequency Response Plots

Bode Plots – Real Poles

Magnitude:

$$\text{db}(G, \omega) := 20 \cdot \log(|G(j \cdot \omega)|)$$

Phase shift:

$$\text{ps}(G, \omega) := \frac{180}{\pi} \cdot \arg(G(j \cdot \omega)) - 360 \cdot (\text{if}(\arg(G(j \cdot \omega)) \geq 0, 1, 0))$$

Assume

$$K := 2 \quad G(s) := \frac{K}{s \cdot (1 + s) \cdot \left(1 + \frac{s}{3}\right)}$$

Next, choose a frequency range for the plots (use powers of 10 for convenient plotting):

$$\text{lowest frequency (in Hz):} \quad \omega_{\text{start}} := .01 \quad \text{number of points:} \quad N := 50$$

$$\text{highest frequency (in Hz):} \quad \omega_{\text{end}} := 100$$

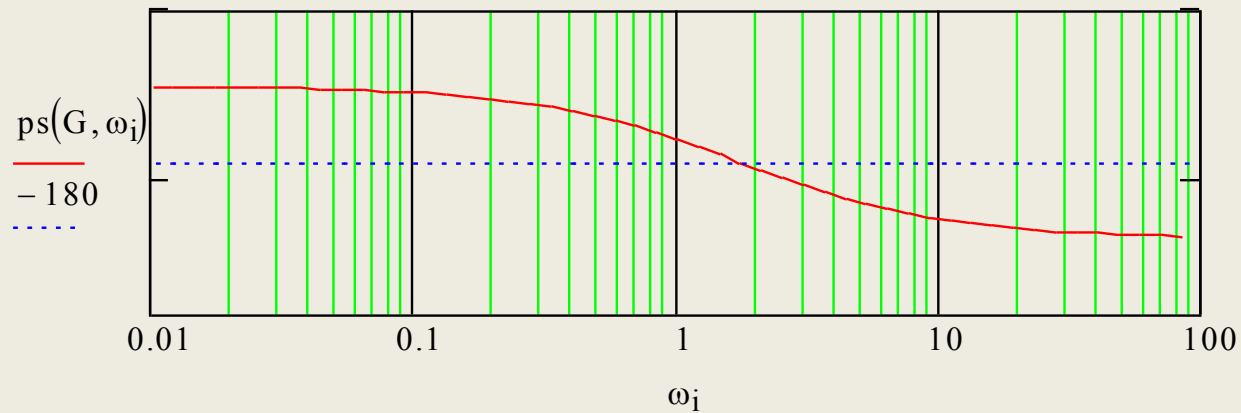
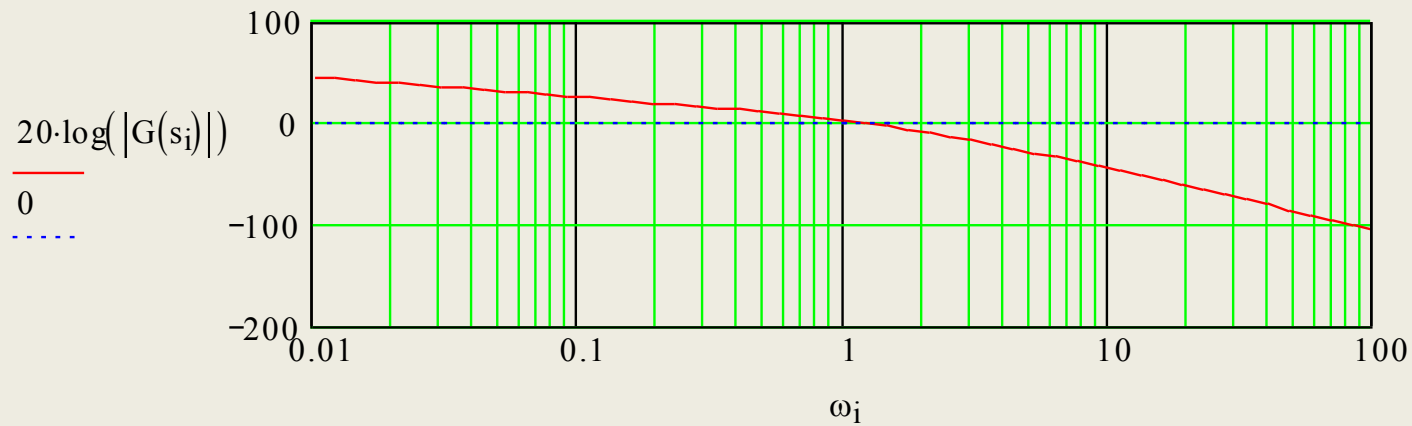
$$\text{step size:} \quad r := \log\left(\frac{\omega_{\text{start}}}{\omega_{\text{end}}}\right) \cdot \frac{1}{N}$$

$$\text{range for plot:} \quad i := 0..N \quad \text{range variable:} \quad \omega_i := \omega_{\text{end}} \cdot 10^{i \cdot r} \quad s_i := j \cdot \omega_i$$

Frequency Response Plots

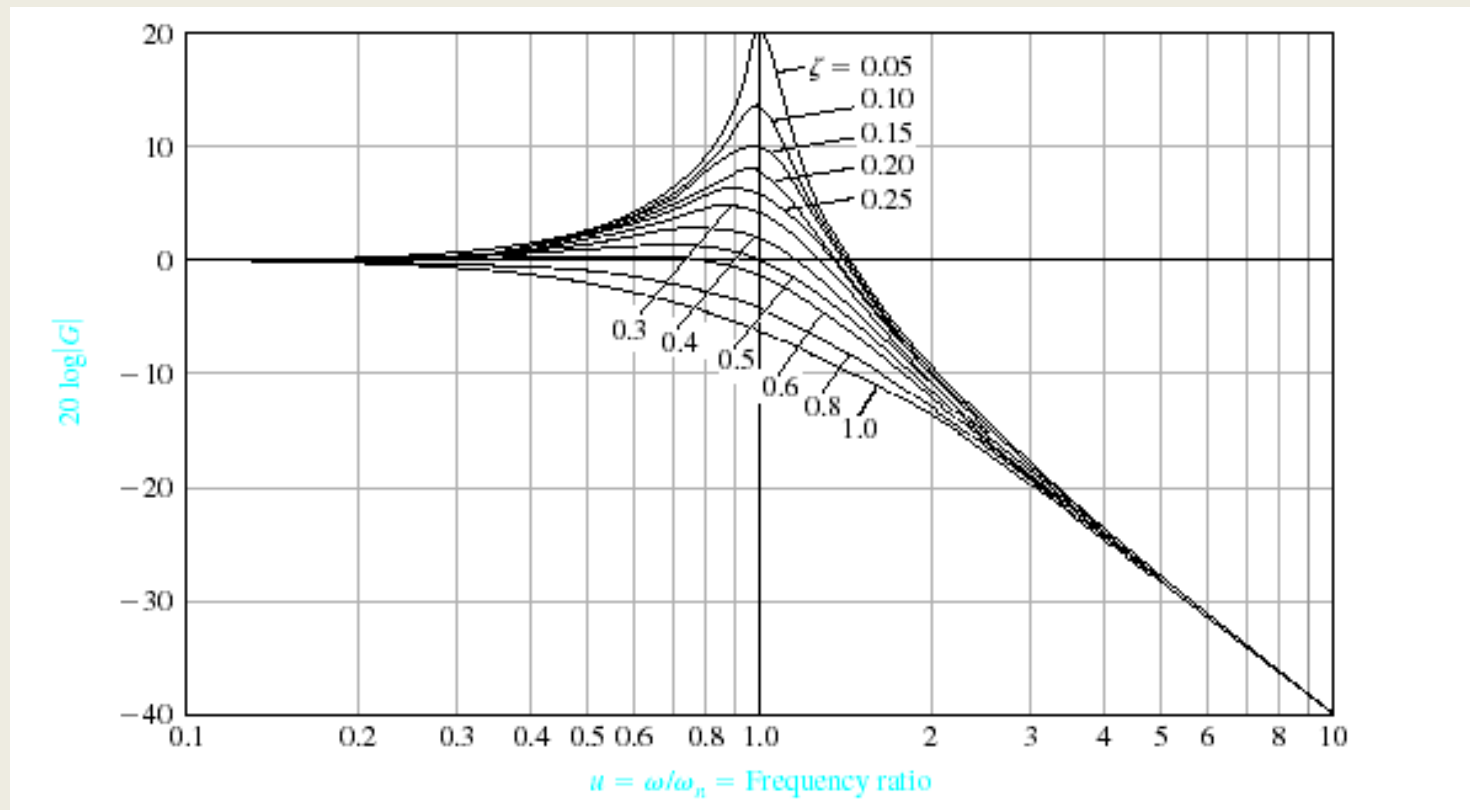
Bode Plots – Real Poles

- range for plot: $i := 0..N$ range variable: $\omega_i := \omega_{\text{end}} \cdot 10^{i \cdot r}$ $s_i := j \cdot \omega_i$



Frequency Response Plots

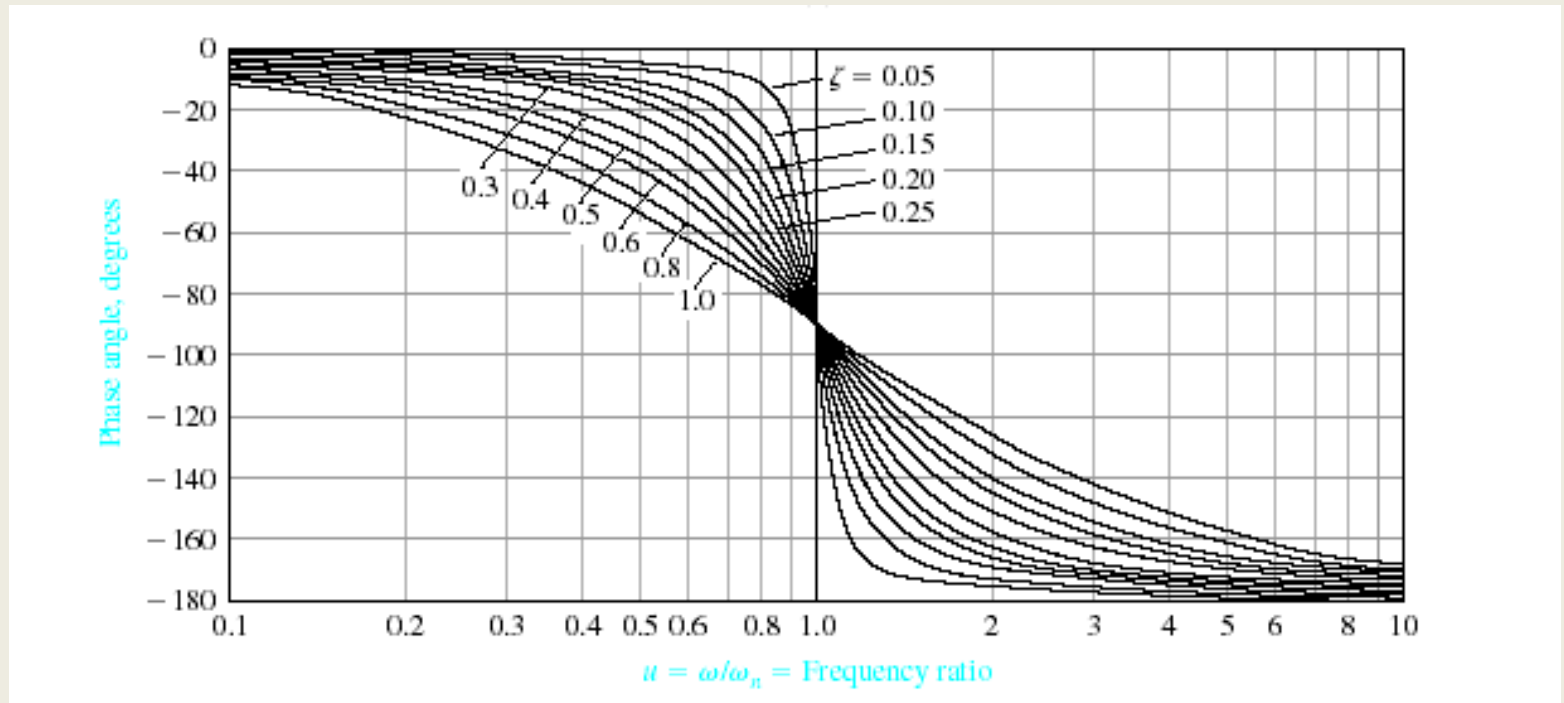
Bode Plots – Complex Poles



$$\text{Bode diagram for } G(j\omega) = [1 + (2\zeta/\omega_n)j\omega + (j\omega/\omega_n)^2]^{-1}.$$

Frequency Response Plots

Bode Plots – Complex Poles



Bode diagram for $G(j\omega) = [1 + (2\zeta/\omega_n)j\omega + (j\omega/\omega_n)^2]^{-1}$.

Frequency Response Plots

Bode Plots – Complex Poles

$$\omega_r = \omega_n \cdot \sqrt{1 - 2 \cdot \zeta^2} \quad \zeta < 0.707$$

$$M_{p\omega} = |G(\omega_r)| = \frac{1}{\left(2 \cdot \zeta \cdot \sqrt{1 - \zeta^2}\right)} \quad \zeta < 0.707$$

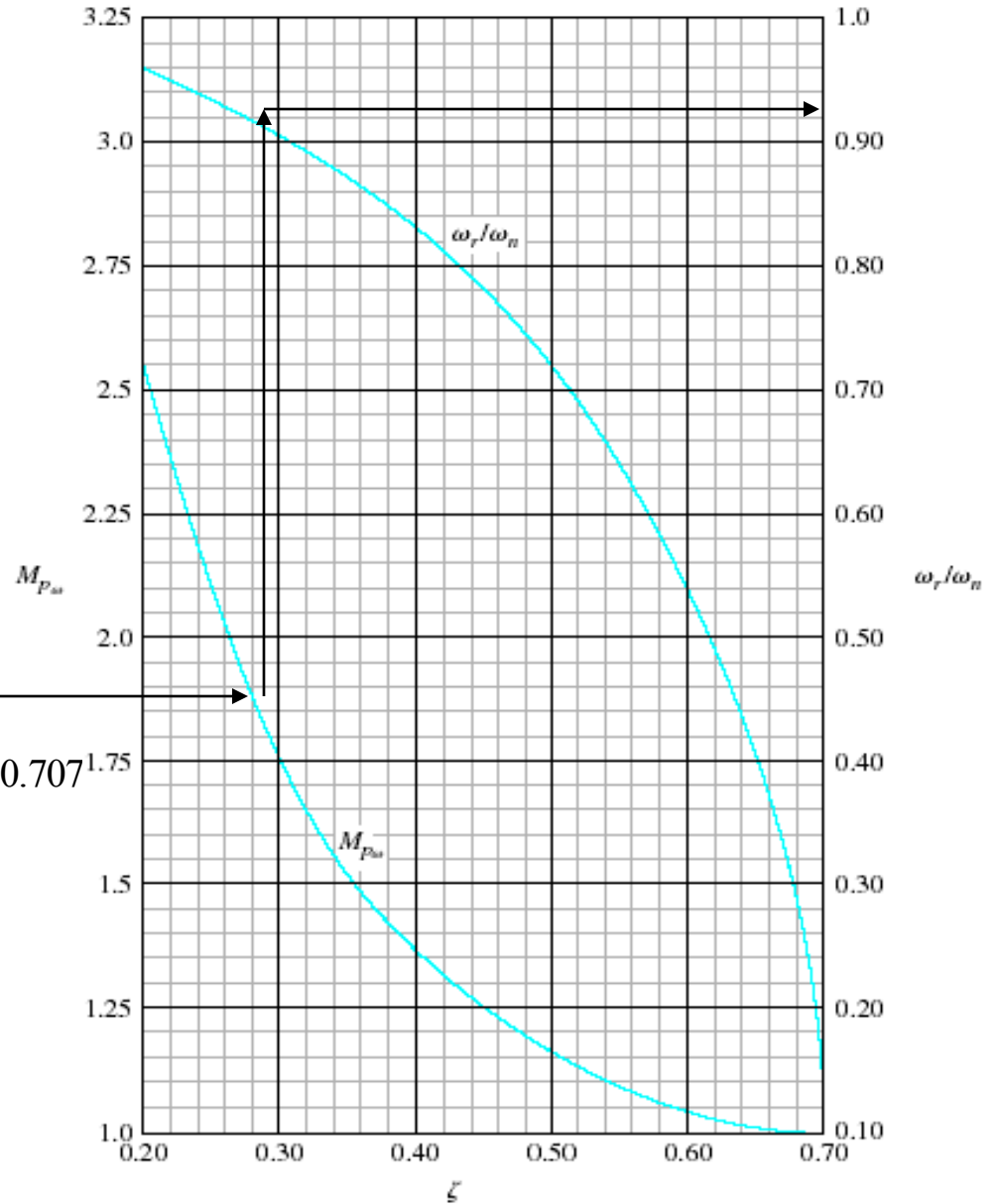
Frequency Response Plots

Bode Plots – Complex Poles

$$\omega_r = \omega_n \cdot \sqrt{1 - 2 \cdot \zeta^2} \quad \zeta < 0.707$$

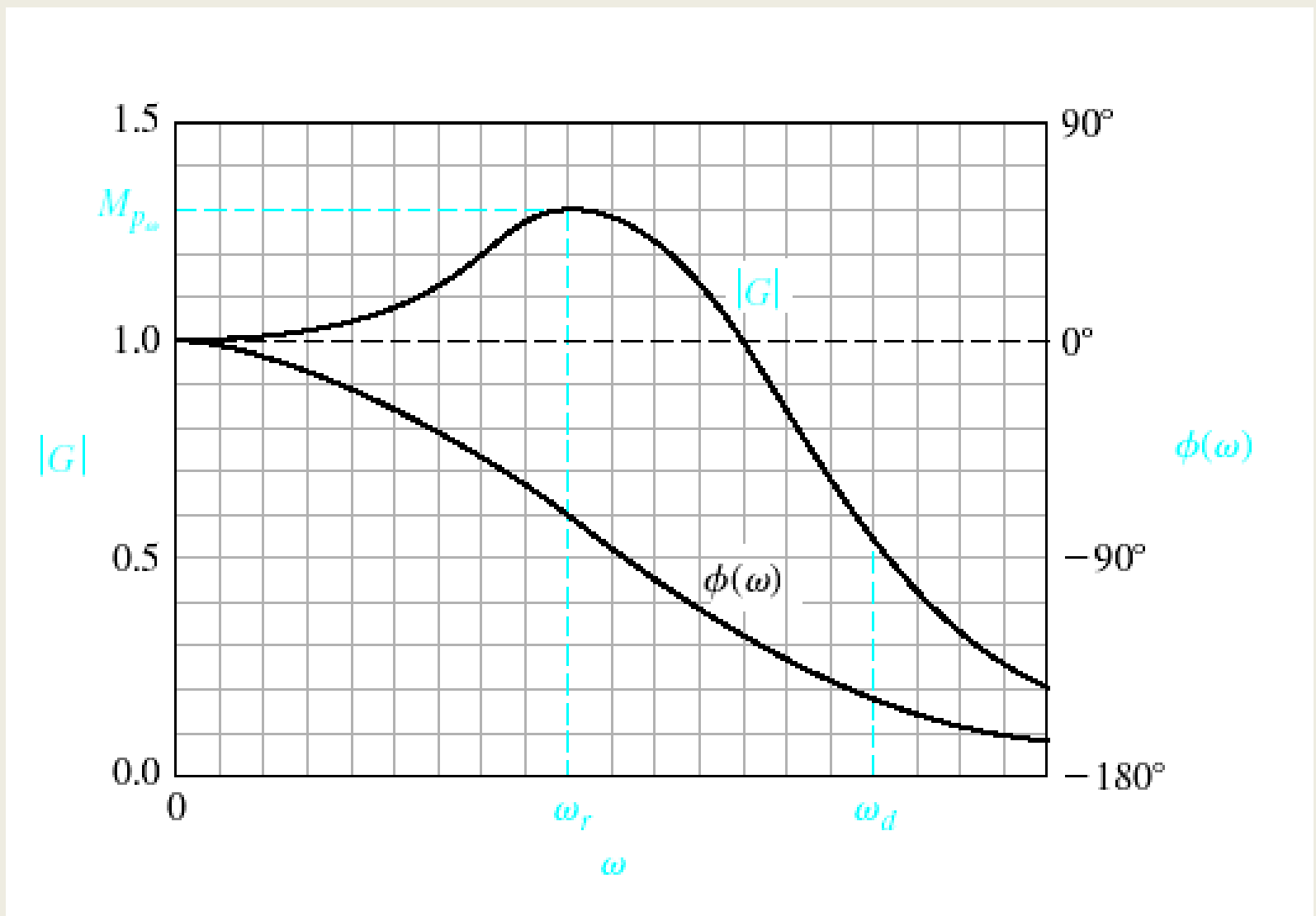
$$M_{p\omega} = |G(\omega_r)| = \frac{1}{(2 \cdot \zeta \cdot \sqrt{1 - \zeta^2})}$$

$\zeta < 0.707$



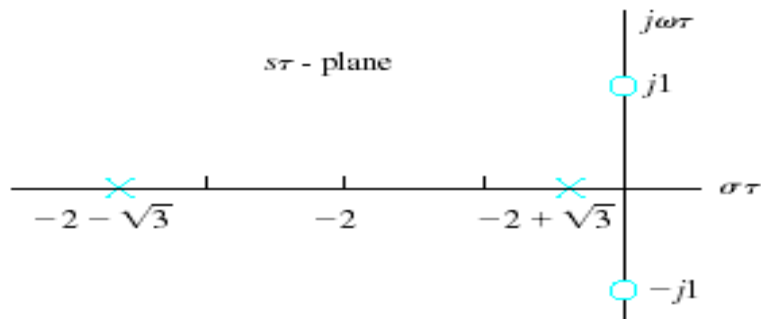
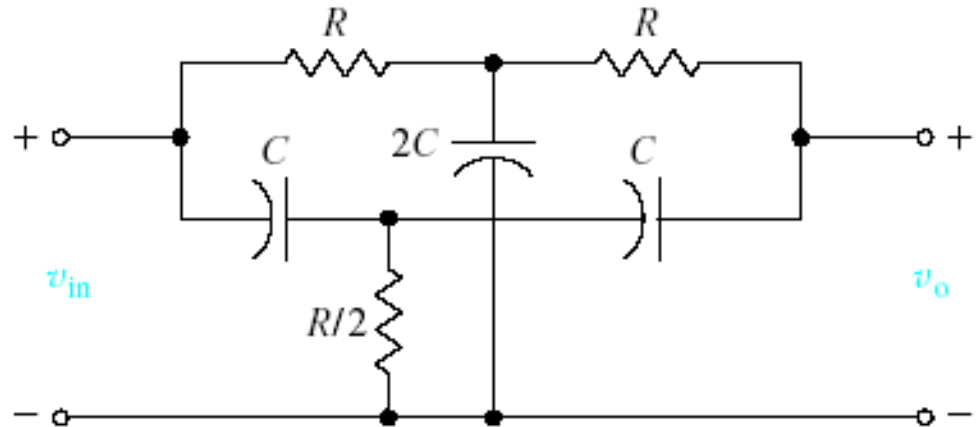
Frequency Response Plots

Bode Plots – Complex Poles

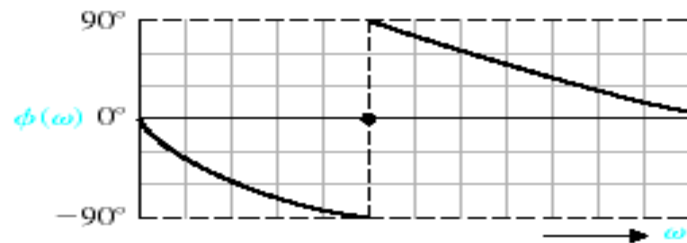
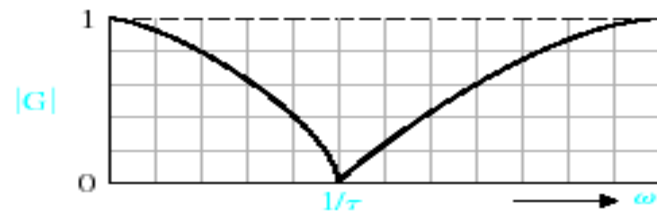


Frequency Response Plots

Bode Plots – Complex Poles



(a)



(b)

Ultimate Gain and Ultimate Period

- Ultimate Gain: K_{CU} = maximum value of $|K_C|$ that results in a stable closed-loop system when proportional-only control is used.

$$P_U \equiv \frac{2\pi}{\omega_C} \quad \bullet \text{ Ultimate Period:}$$

- K_{CU} can be determined from the *OLFR* when proportional-only control is used with $K_C = 1$. Thus

$$K_{CU} = \frac{1}{AR_{OL} \big|_{\omega=\omega_C}} \quad \text{for } K_C = 1$$

- Note: First and second-order systems (without time delays) do not have a K_{CU} value if the PID controller action is correct.

Gain and Phase Margins

- The gain margin (GM) and phase margin (PM) provide measures of how close a system is to a stability limit.

- Gain Margin:

Let $A_C = AR_{OL}$ at $\omega = \omega_C$. Then the gain margin is defined as: $GM = 1/A_C$

According to the Bode Stability Criterion, $GM > 1 \Leftrightarrow$ stability

- Phase Margin:

Let ω_g = frequency at which $AR_{OL} = 1.0$ and the corresponding phase angle is ϕ_g . The phase margin is defined as: $PM = 180^\circ + \phi_g$

According to the Bode Stability Criterion, $PM > 0 \Leftrightarrow$ stability

See Figure 14.12.

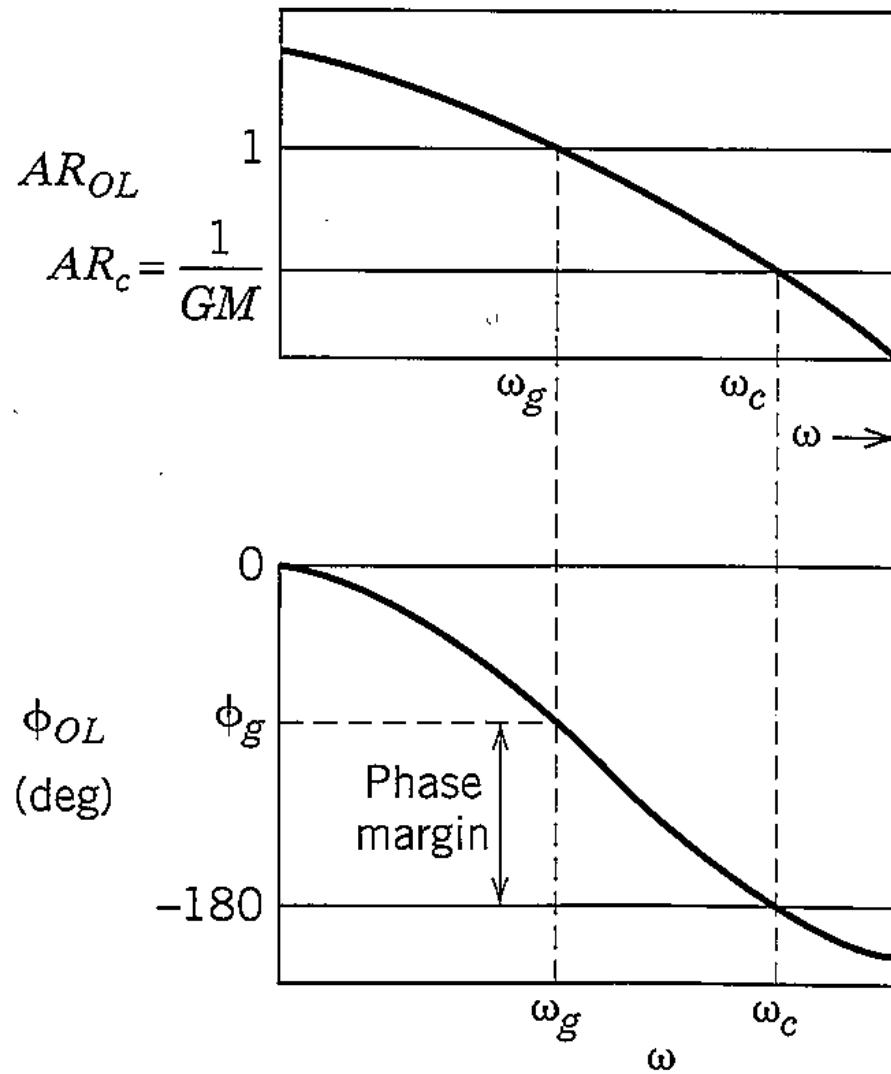


Figure 14.12 Gain and phase margins on a Bode plot

Rules of Thumb:

A well-designed FB control system will have:

$$1.7 \leq GM \leq 2.0$$

$$30^\circ \leq PM \leq 45^\circ$$

Closed-Loop FR Characteristics:

An analysis of CLFR provides useful information about control system performance and robustness. Typical desired CLFR for disturbance and setpoint changes and the corresponding step response are shown in Appendix J.