Control System Engineering

SECTION C :
TIME DOMAIN ANALYSIS

Topic Covered : Typical test signals, time response of first order systems to various standard inputs, time response of 2nd order system to step input, relationship between location of roots of characteristics equation, w and wn, time domain specifications of a general and an under-damped 2nd order system, steady state error and error constants, dominant closed loop poles, concept of stability, pole zero configuration and stability, necessary and sufficient conditions for stability, Hurwitz stability criterion, Routh stability criterion and relative stability.

INTRODUCTION

- The time response of the ‘system’ is the output of the closed loop system as a function of time.
- The time response of a control system refers to control system behaviour over the time for a specified input test signal. The time esponse of a control system is made up of two parts:

1. Transient response

2. Steady-state response

TEST INPUT SIGNALS

1. Impulse Function: (Dirac Delta Signal)

The impulse signal $f_\delta(t)$ is defined as

$$f_\delta(t) = \begin{cases} A & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$$

Here A is the area of the impulse signal. Impulse signal $f_\delta(t)$ is shown in Fig. 4.1.

If the area of impulse signal $A = 1$ then impulse signal $f_\delta(t)$ is called as unit impulse signal $\delta(t)$ which is shown in Fig. 4.2

$$\delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$$

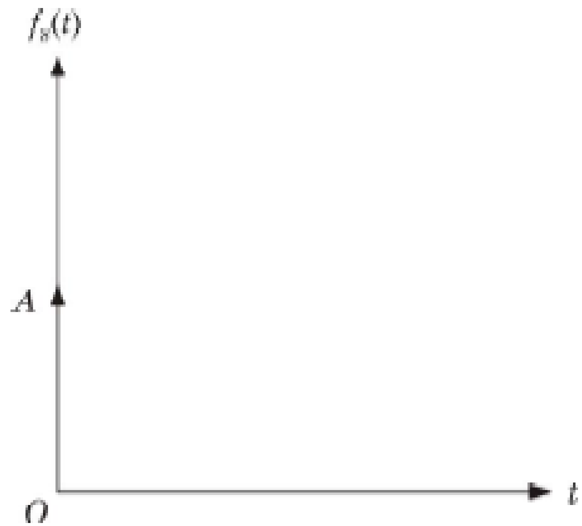


Fig. 4.1

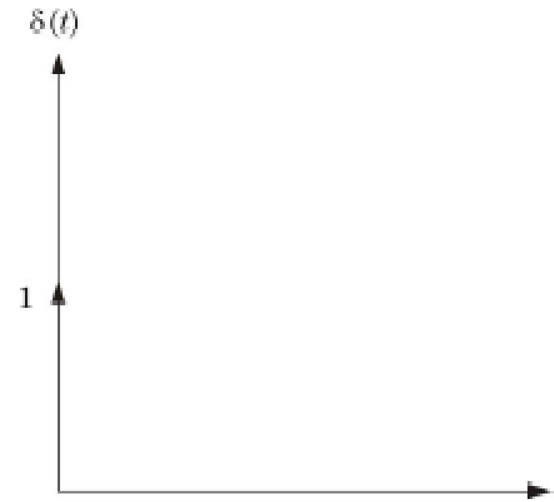


Fig. 4.2

The area of the unit impulse signal is defined as

$$\text{Area} = \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

The impulse function described mathematically as

$$\delta(t) = \lim_{T \to 0} \frac{1}{T}[U(t) - U(t-T)]$$

The Laplace transform of $\delta(t)$ is

$$\pounds[\delta(t)] = 1$$

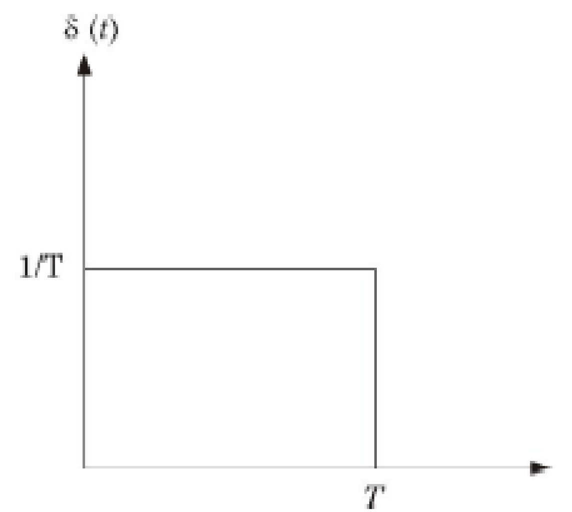


Fig. 4.3 Impulse function

2. Step Signal

The step signal $f_s(t)$ is defined by

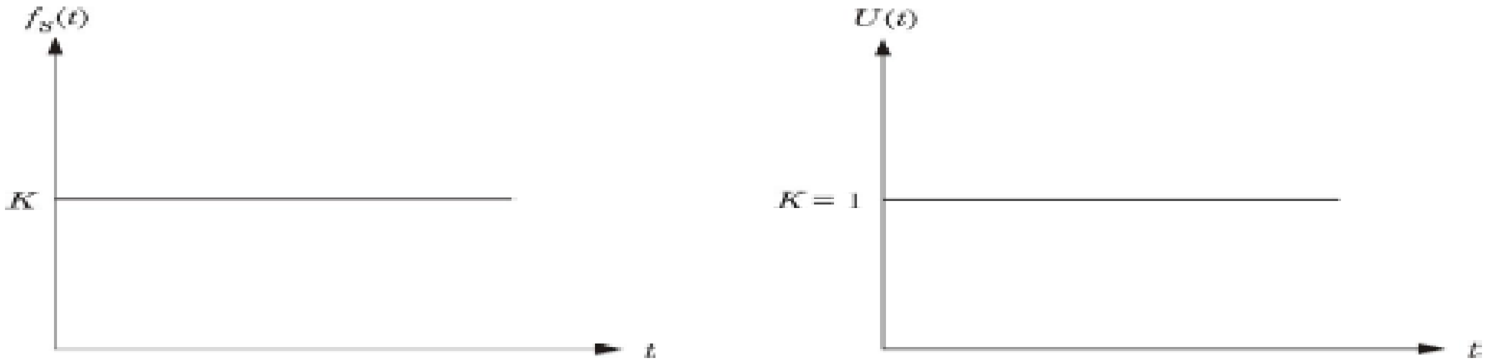
$$f_s(t) = \begin{cases} 0 & ; \quad t \leq 0 \\ K & ; \quad t > 0 \end{cases}$$

where K is the amplitude of the step signal

The signal $f_s(t)$ is shown in Fig. 4.4.

A unit step function $U(t)$ is defined as

$$U(t) = \begin{cases} 0 & ; \quad t \leq 0 \\ 1 & ; \quad t > 0 \end{cases}$$



(*a*) Step signal (*b*) Unit step signal

Fig. 4.4

Laplace Transform of $U(t)$:

$$\pounds[U(t)] = \int_0^\infty f(t)\, e^{-st} = \int_0^\infty U(t)\, e^{-st}\, dt$$

$$= \int_0^\infty 1 \cdot e^{-st} dt = \left[\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s}$$

Step function is also called *displacement function*.

3. Ramp Function

A ramp function is shown in Fig. 4.5.

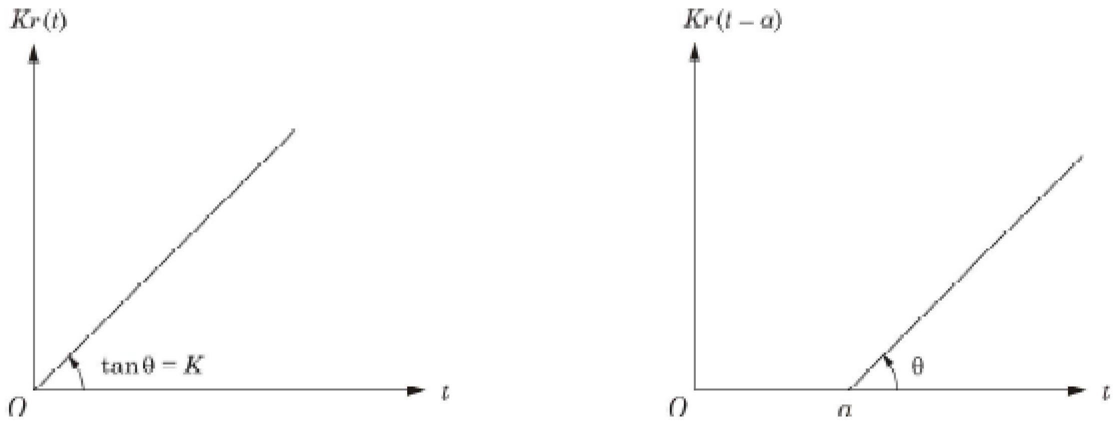
$$r(t) = \begin{cases} 0 \; ; & t \le 0 \\ Kt \; ; & t > 0 \end{cases}$$

where K is the slope of ramp signal

A unit function is denoted by $r(t)$.

Laplace Transform:

$$\pounds[r(t)] = \int_0^\infty r(t)\, e^{-st}\, dt = \int_0^\infty Kte^{-st}\, dt = \frac{K}{s^2}$$



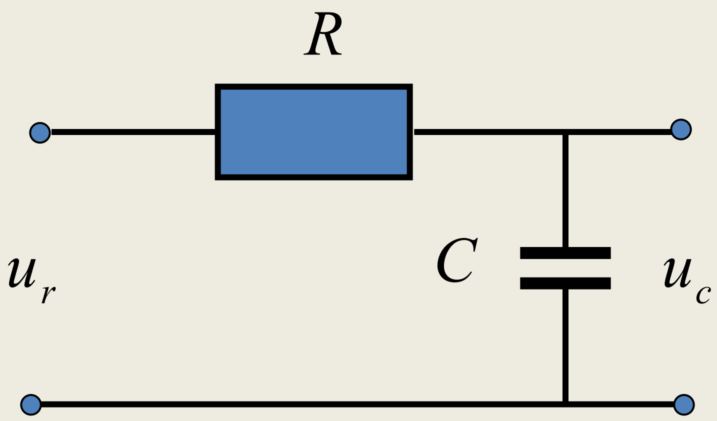
(a) Ramp function

(b) Delayed Ramp function

Fig. 4.5

Example. Electrical first-order system

$$RC\frac{du_c}{dt} + u_c = u_i$$



Taking the *Laplace* transform of the equation with zero initial condition yields

$$\frac{U_c(s)}{U_r(s)} = \frac{1}{RCs+1} = \frac{1}{Ts+1}$$

where the time constant is *T*=*RC*.

Example. A simplified mathematical model of a missile in linear motion can be described by a first-order system:

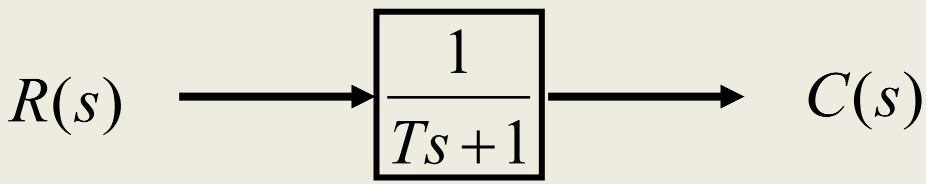
$$m\frac{dv}{dt} = f(t) - kv$$

where $f(t)$ is the force, v is the velocity, and kv denotes the force of air resistance which is proportional to the velocity.

Taking the *Laplace* transform of the equation with zero initial condition yields

$$V(s) = \frac{1}{ms + k}F(s) = \frac{1}{k}\frac{1}{(m/k)s + 1}F(s)$$

2. Unit-Step Response of First-order Systems



The input signal: $$R(s) = \frac{1}{s}$$

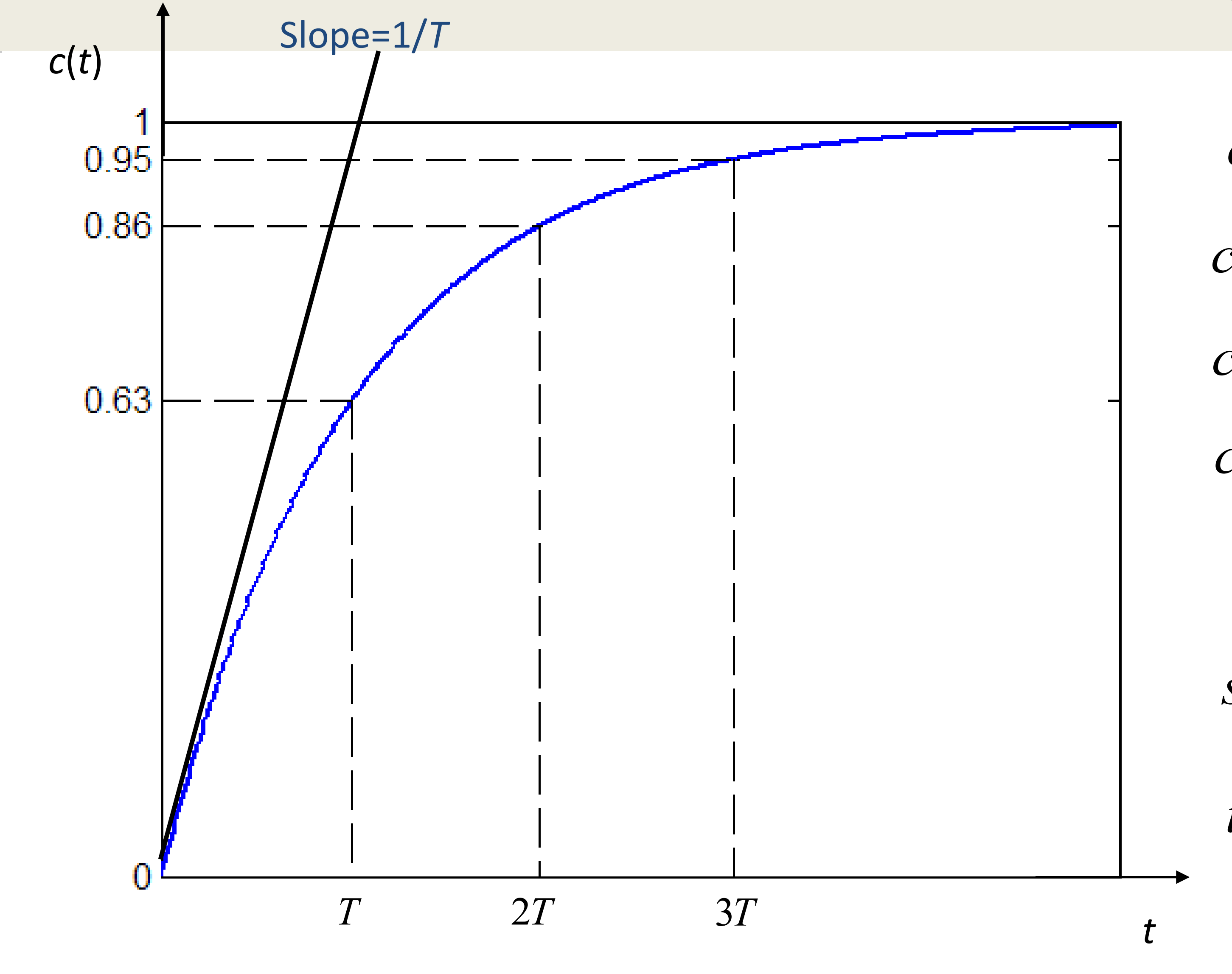
Hence, the output

$$C(s) = \frac{1}{s(Ts+1)} = \frac{1}{s} - \frac{T}{Ts+1} = \frac{1}{s} - \frac{1}{s+1/T}$$

from which, we obtain the time response

$$c(t) = 1 - e^{-t/T}, t \geq 0$$

Step response $c(t) = 1 - e^{-t/T}, \quad \left.\frac{dc}{dt}\right|_{t=0} = \frac{1}{T}$



$c(T) = 0.63$

$c(2T) = 0.86$

$c(3T) = 0.95$

$c(4T) = 0.98$

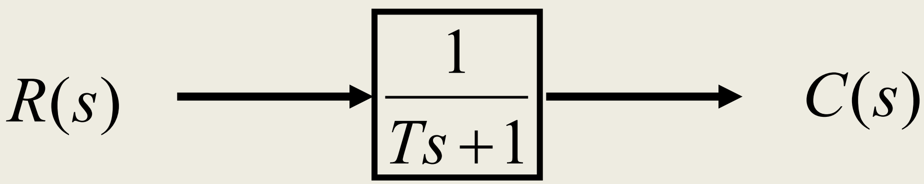
$s\% = 0$

$t_s = 3T$

Example. The response of a temperature sensor known as a thermocouple (TC) can be modeled as a first-order system. When the TC is subjected to a rapid temperature change, it will take some time to respond. If the response time is slow in comparison with the rate of change of the temperature that you are measuring, then the TC will not be able to faithfully represent the dynamic response to the temperature fluctuations.

$$t \frac{dT}{dt} + T = T_{¥}$$

3. Unit-Ramp response of First-order Systems



If the input is the unit ramp $r(t) = t \cdot 1(t)$

then $R(s) = \dfrac{1}{s^2}$

Expanding *C*(*s*) into partial fractions gives

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T}{s + 1/T}$$

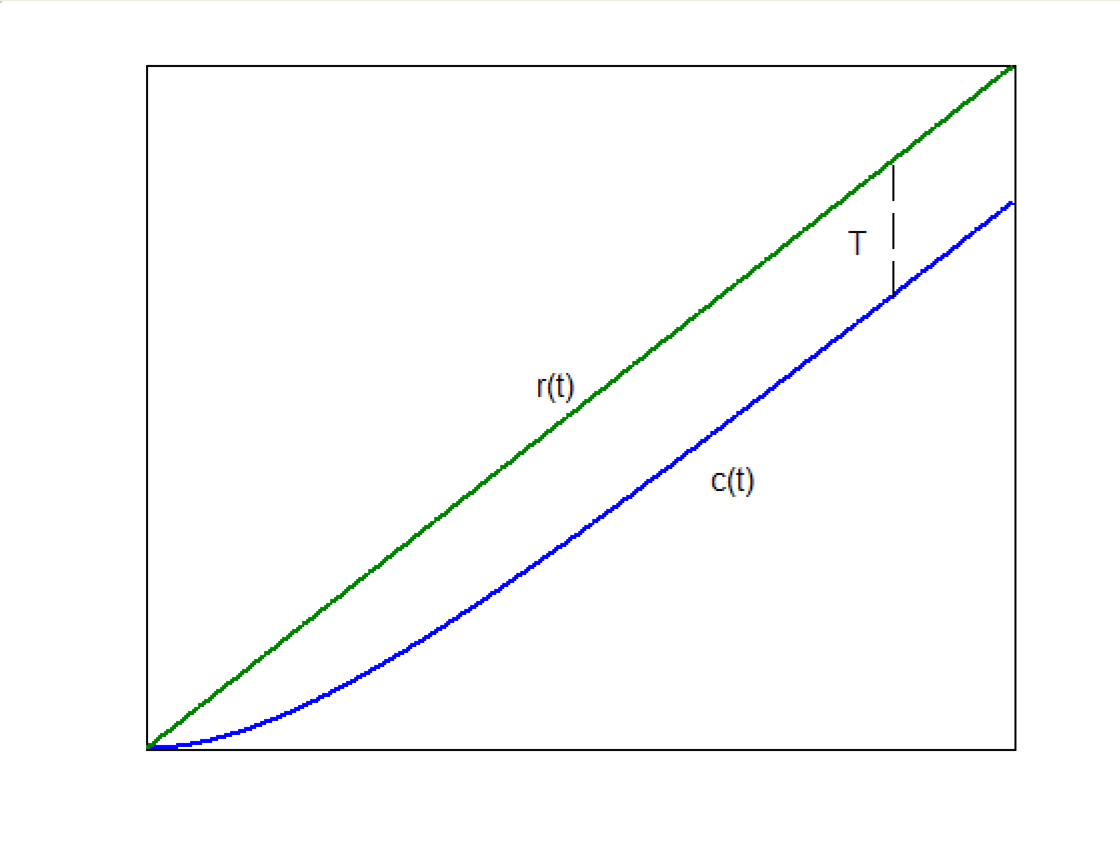
Thus, the time-domain response is

$$c(t) = t - T + Te^{-\frac{t}{T}}, t \geq 0$$

The error signal *e*(*t*) is then

$$e(t) = r(t) - c(t) = T(1 - e^{-\frac{t}{T}}), t \geq 0$$
$$\Rightarrow e(\infty) = T$$

The error in following the unit-ramp input is equal to *T* for sufficiently large *t*. The smaller the time constant *T*, the smaller the steady-state error.



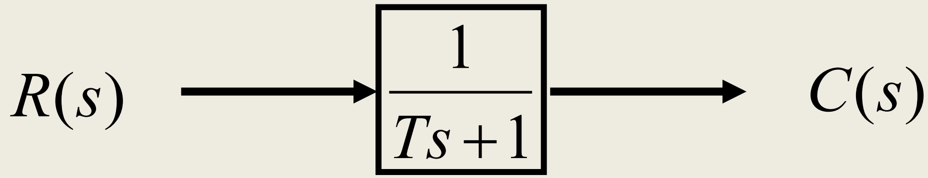
Notice that for first-order systems, the steady-state errors are different when input signals are unit-step function and unit-ramp function, respectively. For unit-step function,

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} [r(t) - c(t)] = \lim_{t\to\infty} e^{-t/T} = 0$$

while for unit-ramp function,

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} [r(t) - c(t)] = \lim_{t\to\infty} T(1 - e^{-\frac{t}{T}}) = T$$

4. Unit-Impulse response of First-order Systems



If the input is a unit-impulse, then

$$R(s) = 1$$

Therefore,

$$C(s) = \frac{1}{Ts+1}$$

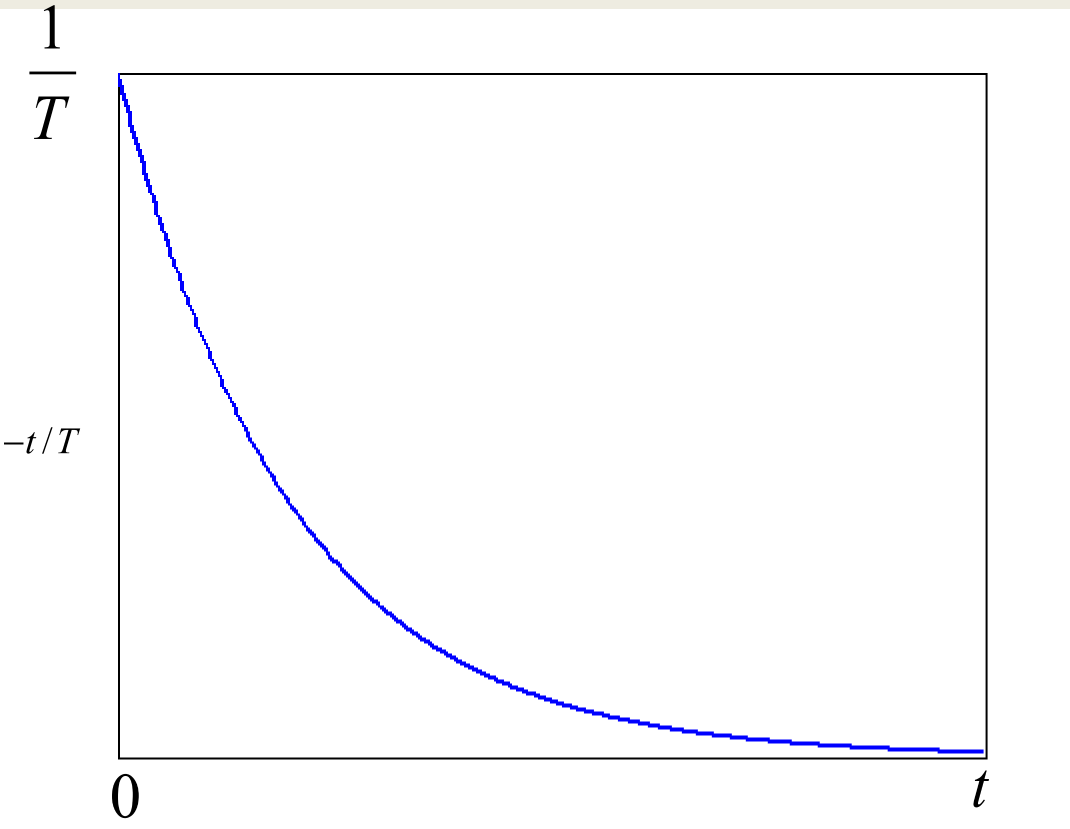
Taking the inverse Laplace transform gives

$$c(t) = \frac{1}{T}e^{-t/T}, t \geq 0$$

Unit-Impulse response

Since we have assumed zero initial conditions, the output must change instantaneously from 0 at t=0 to 1/T at t=0$^+$ (why?).

$$c(t) = \frac{1}{T} e^{-t/T}$$



5. An important property of LTI systems

It is seen that the responses of first-order systems to the three inputs are given below:

$$c_t = t - T + Te^{-\frac{t}{T}}, t \geq 0$$

$$c_{1(t)} = 1 - e^{-t/T}, t \geq 0$$

$$c_d = \frac{1}{T}e^{-t/T}, t \geq 0$$

which have the following property:

$$\frac{dc_t}{dt} = c_{1(t)}$$

$$\frac{dc_{1(t)}}{dt} = c_d$$

On the other hand, notice that for the input signals

$$\frac{1}{s^2} \xrightarrow{\;s\;} \frac{1}{s} \xrightarrow{\;s\;} 1$$

$$t \xrightarrow{\;\frac{d}{dt}\;} 1(t) \xrightarrow{\;\frac{d}{dt}\;} \delta(t)$$

Conclusion: for unit ramp, step and impulse inputs, the derivative of the *output* is equivalent to the derivative of the *input*.

Such a conclusion can be readily extended to higher-order LTI systems with respect to unit ramp, step and impulse inputs. In fact, let

$$\frac{1}{s^2} \longrightarrow \boxed{G(s)} \longrightarrow C(s)$$

Then

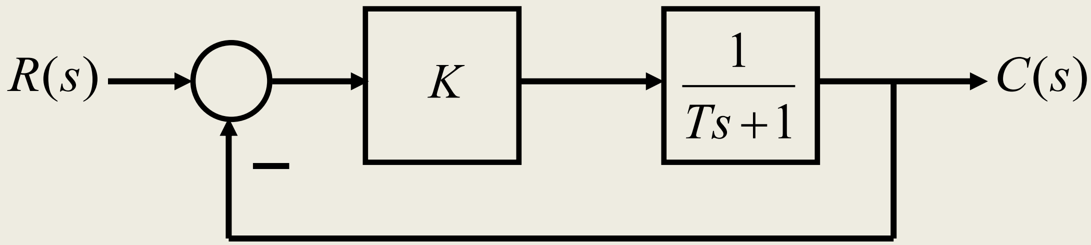
$$C(s) = G(s)\frac{1}{s^2}$$

Obviously,

$$sC(s)\frac{1}{s^2} = sG(s)\frac{1}{s^2} = G(s)\left(s\frac{1}{s^2}\right) = G(s)\frac{1}{s}$$

The left hand side is the derivative of the *output* and the right hand side is the derivative of the *input.*

Example: Determine the unit-step response of the following closed-loop system:



Solution: The closed-loop transfer function is:

$$C(s) = \frac{K}{s(Ts+1+K)} = \frac{K/T}{s\left[s+(1+K)/T\right]} = \frac{a}{s} + \frac{b}{s+(1+K)/T}$$

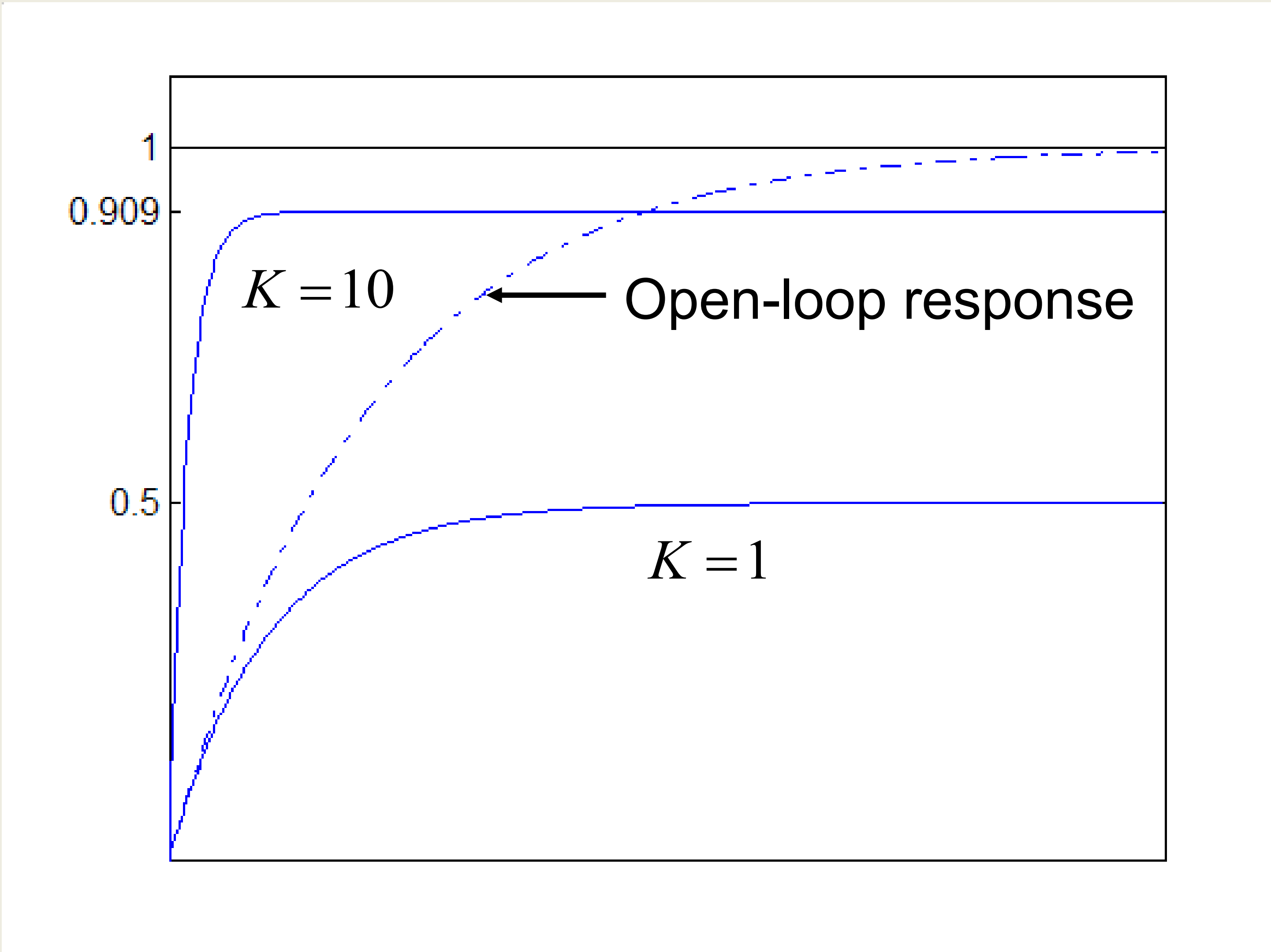
Evaluating the partial-fraction coefficients yields

$$a = \frac{K}{1+K}, \quad b = -\frac{K}{1+K}$$

Hence, the time response

$$c(t) = \frac{K}{1+K}\left(1 - e^{-[(1+K)/T]t}\right), t \geq 0$$

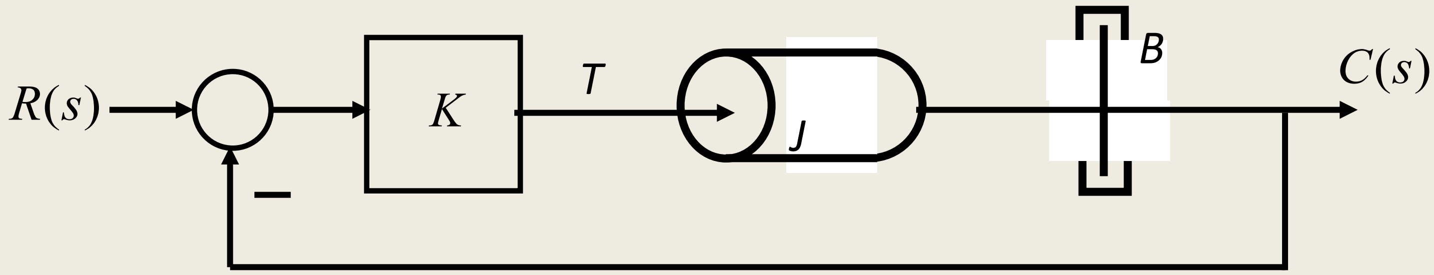
$$c(t) = \frac{K}{1+K}\left(1 - e^{-[(1+K)/T]t}\right), \quad t \geq 0$$



5-3. Second-order systems

Many physical control systems can be described by or approximated to a second-order differen- tial equation.

1. Servo System: A second-order system example



The system consists of a proportional control and load elements (inertia and viscous friction elements). Control objective: output position *C* tracks the desired position *R*.

The equation for the load elements is

$$J\ddot{c} + B\dot{c} = T$$

where *T* is the torque produced by *K, J* and *B* are the moment of inertia and viscous friction referred to the motor shaft, respectively. Hence,

$$\frac{C(s)}{T(s)} = \frac{1}{s(Js + B)}$$

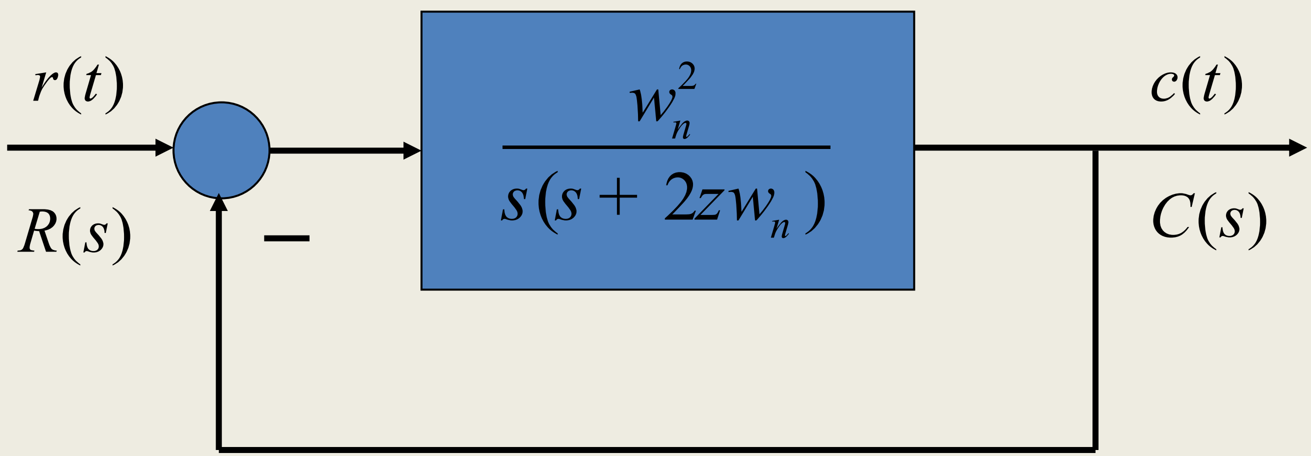
and the closed-loop transfer function

$$\frac{C(s)}{R(s)} = \frac{K/J}{s^2 + (B/J)s + K/J}$$

We can write it in *standard form*

$$\frac{C(s)}{R(s)} = \frac{w_n^2}{s^2 + 2zw_n s + w_n^2}$$

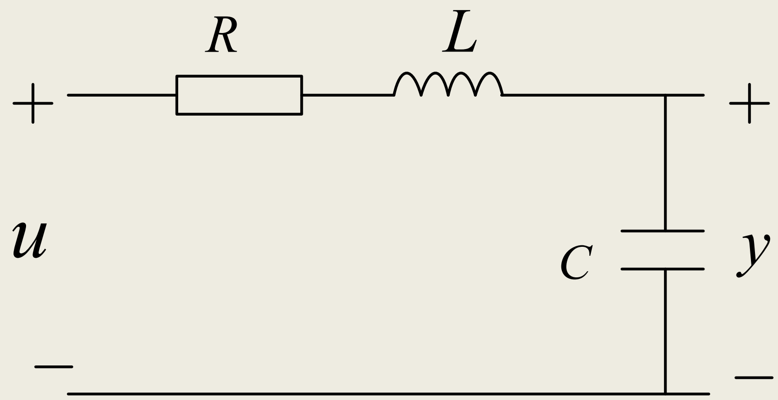
where ω_n is undamped natural frequency and ζ, the damping ratio.



The characteristic equation

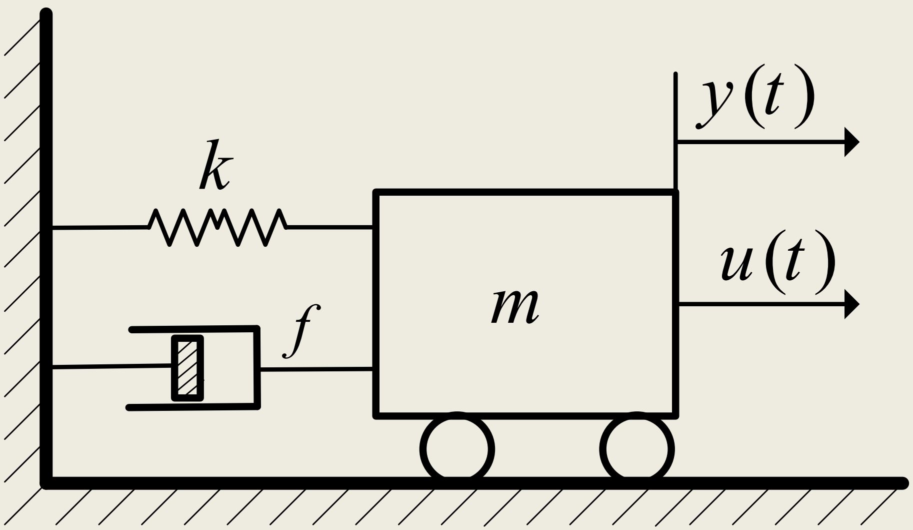
$$s^2 + 2zw_n s + w_n^2 = 0$$

RLC circuit: A second-order system example.



$$\frac{Y(s)}{U(s)} = \frac{1}{LCs^2 + RCs + 1} = \frac{1/LC}{s^2 + (R/L)s + 1/LC}$$

Mass-Spring-Damper System: A second-order system example.



k : Spring constant

f : Damping coefficient

$$m\ddot{y} + f\dot{y} + ky = u$$

$$\frac{Y(s)}{U(s)} = \frac{1}{ms^2 + fs + k} = \frac{1}{k}\left(\frac{k/m}{s^2 + (f/m)s + k/m}\right)$$

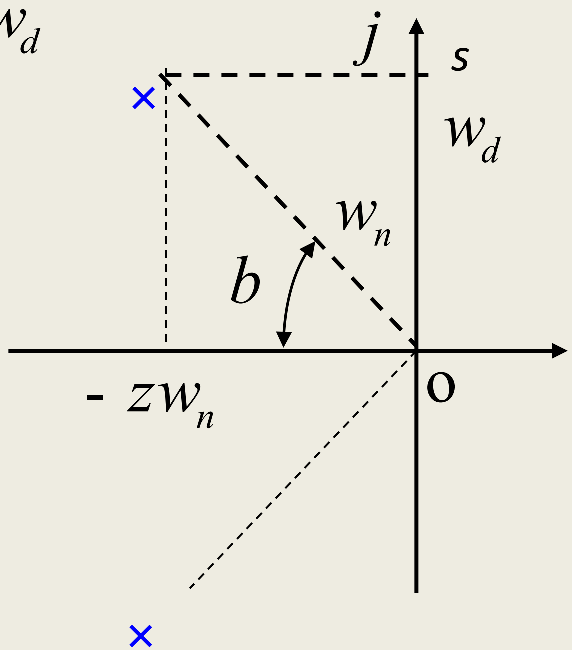
2. The step response of second-order system

(1) Underdamped case (0<ζ<1): From

$$s^2 + 2zw_n s + w_n^2 = 0$$

The two poles can be expressed as

$$s_{1,2} = - zw_n \pm jw_n\sqrt{1 - z^2} = - zw_n \pm jw_d$$

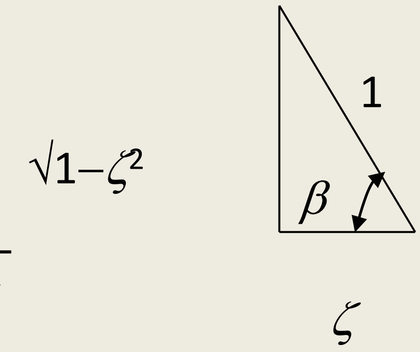


where

$$w_d = w_n\sqrt{1 - z^2}$$

ω_d is called *damped frequency*.

The step response



$$C(s) = \frac{w_n^2}{s(s^2 + 2zw_n s + w_n^2)} = \frac{1}{s} - \frac{s + 2zw_n}{s^2 + 2zw_n s + w_n^2}$$

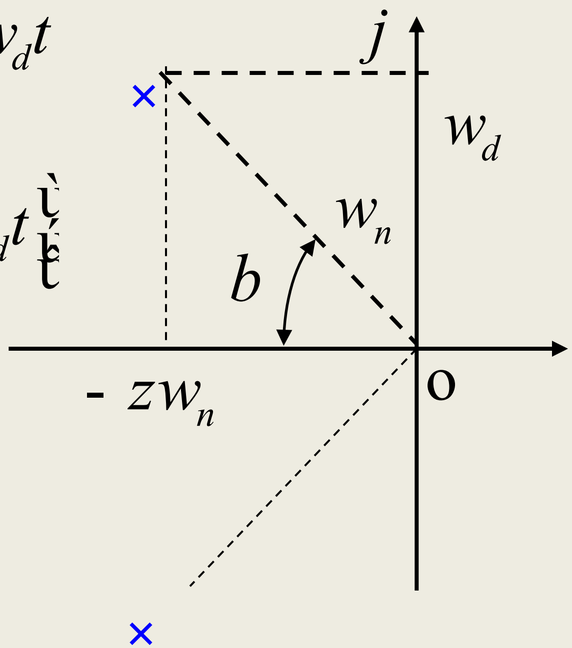
$$= \frac{1}{s} - \frac{s + zw_n}{(s + zw_n)^2 + w_d^2} - \frac{z}{\sqrt{1 - z^2}} \frac{w_d}{(s + zw_n)^2 + w_d^2}$$

Hence,

$$c(t) = 1 - e^{-zw_n t} \cos w_d t - e^{-zw_n t} \frac{z}{\sqrt{1 - z^2}} \sin w_d t$$

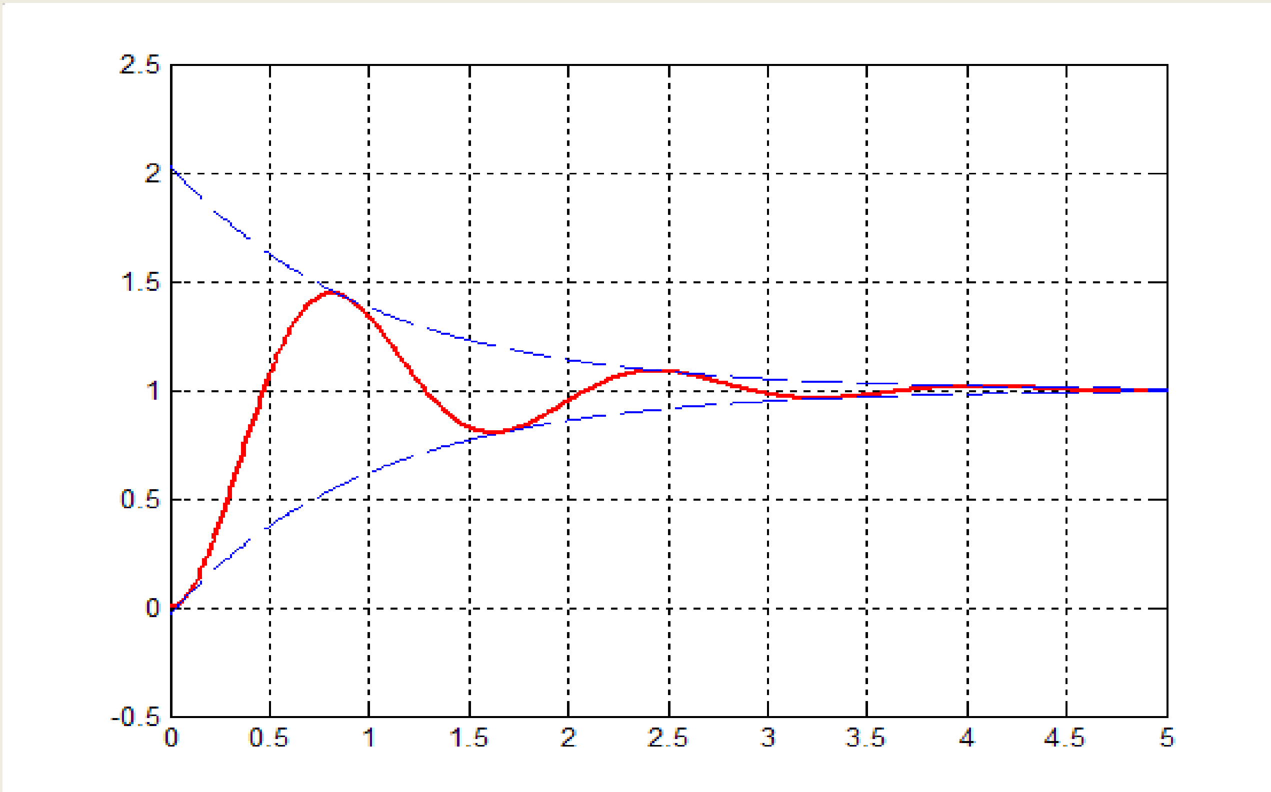
$$= 1 - \frac{e^{-zw_n t}}{\sqrt{1 - z^2}} \left[\sqrt{1 - z^2} \cos w_d t + z \sin w_d t \right]$$

$$= 1 - \frac{e^{-zw_n t}}{\sqrt{1 - z^2}} \sin(w_d t + b), t ³ 0$$



where cosβ=ζ or tgβ=$\sqrt{1-\zeta^2}/\zeta$.

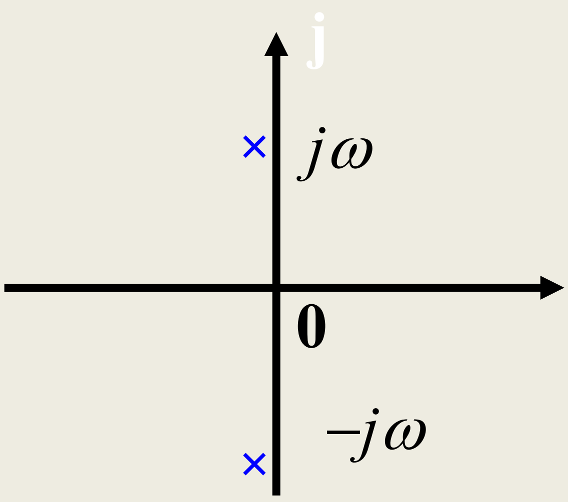
$$c(t) = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \beta), t \geq 0$$

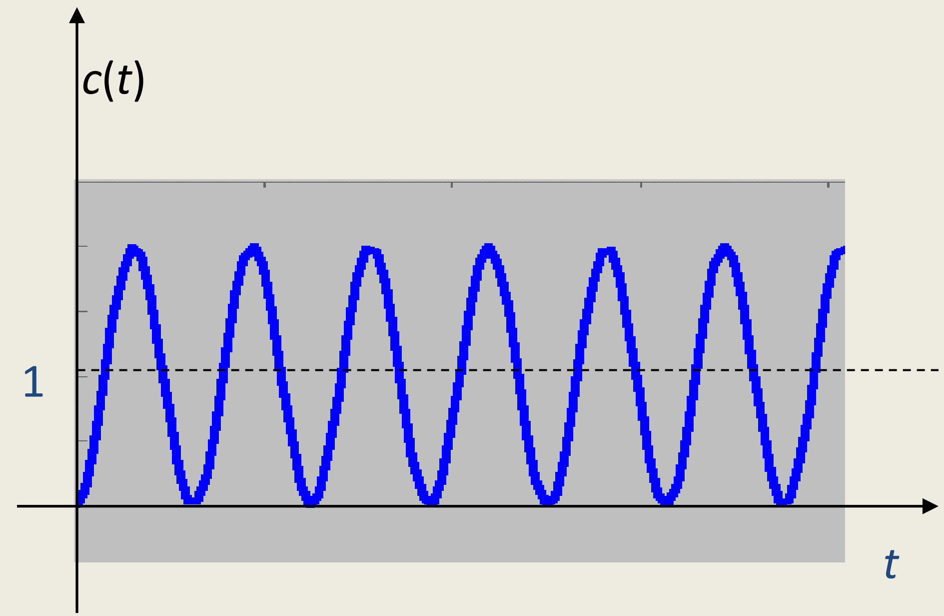


The response oscillatorily decays to one (0<ζ<1)

In particular, if the damping ratio ζ=0, the response becomes undamped and oscillations continue indefinitely:

$$c(t) = 1 - \cos w_n t, \quad t \geq 0$$





This is why we call ω_n as undamped natural frequency, which is in fact cannot be observed experimentally; what we are able to observe is damped frequency

$$w_d = w_n \sqrt{1- z^2}$$

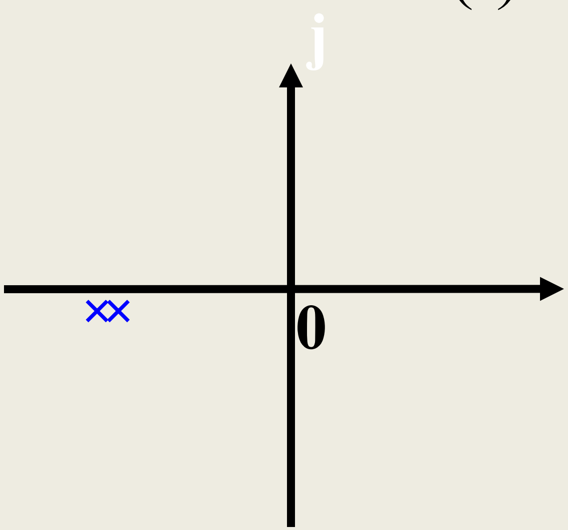
which is always lower than ω_n when $0<\zeta<1$. If ζ is increased beyond unity, that is, $\zeta>1$, the response becomes *overdamped* and will not oscillate.

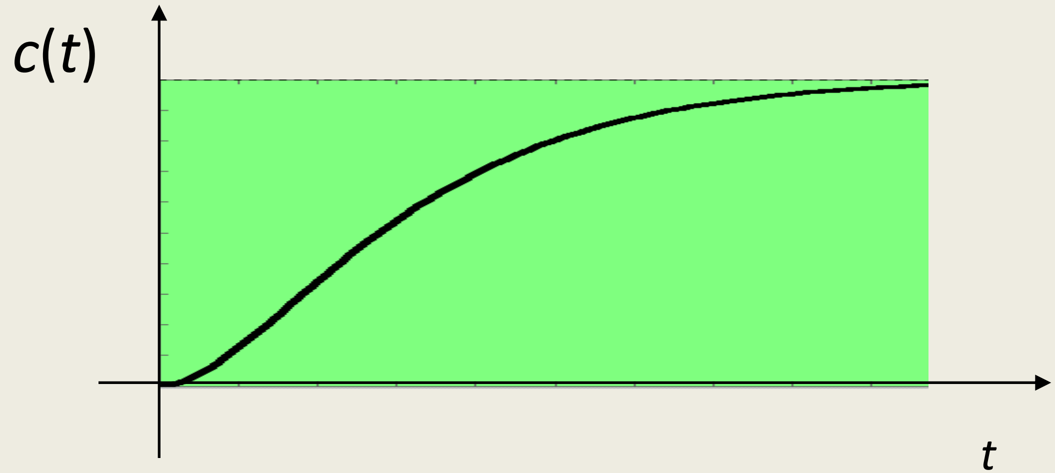
(2) Critical damped case (ζ=1): The unit-step response

$$C(s) = \frac{w_n^2}{s(s+w_n)^2} = \frac{1}{s} - \frac{1}{s+w_n} - \frac{w_n}{(s+w_n)^2}$$

Taking the inverse Laplace transform of both sides of the above equation yields

$$c(t) = 1 - e^{-w_n t}(1 + w_n t), \quad t \geq 0$$





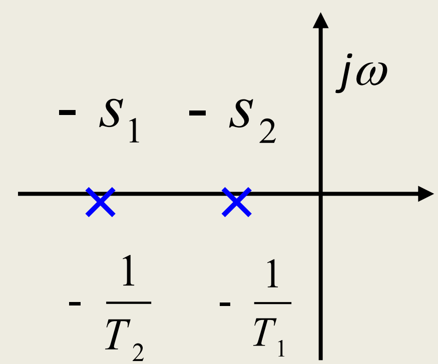
(3) Overdamped Case (ζ>1):

The characteristic equation has two negative real poles:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = (s + s_1)(s + s_2) = \left(s + \frac{1}{T_1}\right)\left(s + \frac{1}{T_2}\right)$$

where

$$\begin{cases} T_1 = \dfrac{1}{\underbrace{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}_{s_2}} := \dfrac{1}{s_2} \\[2ex] T_2 = \dfrac{1}{\underbrace{\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}}_{s_1}} := \dfrac{1}{s_1} \end{cases}$$



Therefore,

$$T_1 > T_2 \quad (s_2 < s_1), \;\; \omega_n^2 = \frac{1}{T_1 T_2}$$

Hence,

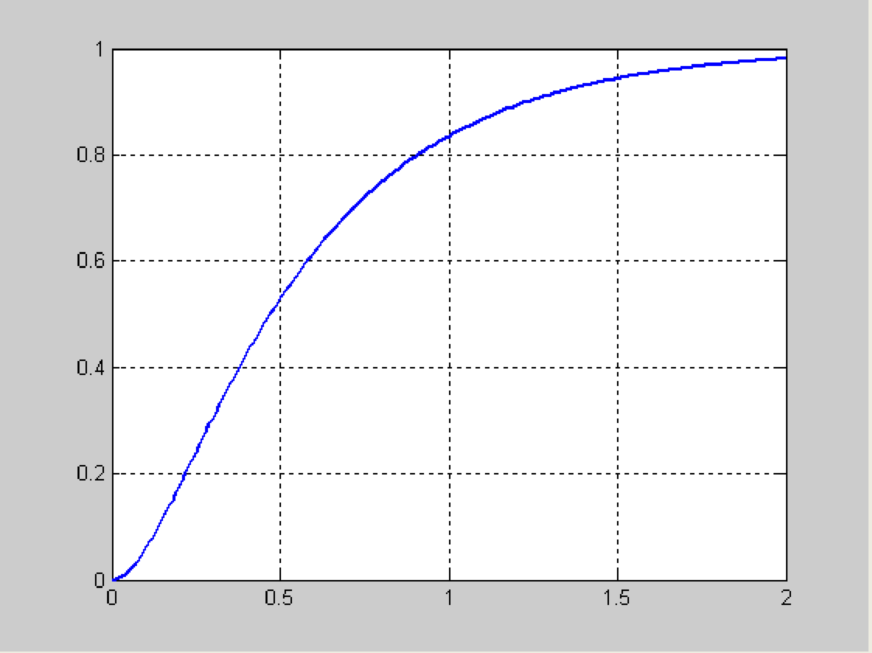
$$C(s) = \frac{\frac{1}{T_1 T_2}}{\left(s + \frac{1}{T_1}\right)\left(s + \frac{1}{T_2}\right)}\frac{1}{s} = \frac{1}{s(T_1 s + 1)(T_2 s + 1)}$$

$$= \frac{1}{s} + \frac{1}{\frac{T_2}{T_1} - 1} \times \frac{1}{\left(s + \frac{1}{T_1}\right)} + \frac{1}{\frac{T_1}{T_2} - 1} \times \frac{1}{\left(s + \frac{1}{T_2}\right)}$$

The unit-step response is

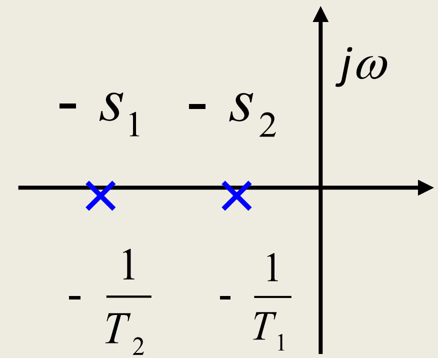
$$c(t) = 1 + \frac{1}{\frac{T_2}{T_1} - 1} e^{-\frac{1}{T_1}t} + \frac{1}{\frac{T_1}{T_2} - 1} e^{-\frac{1}{T_2}t}$$

$$= 1 + \frac{\omega_n^2}{2\sqrt{\zeta^2 - 1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right), t \geq 0$$

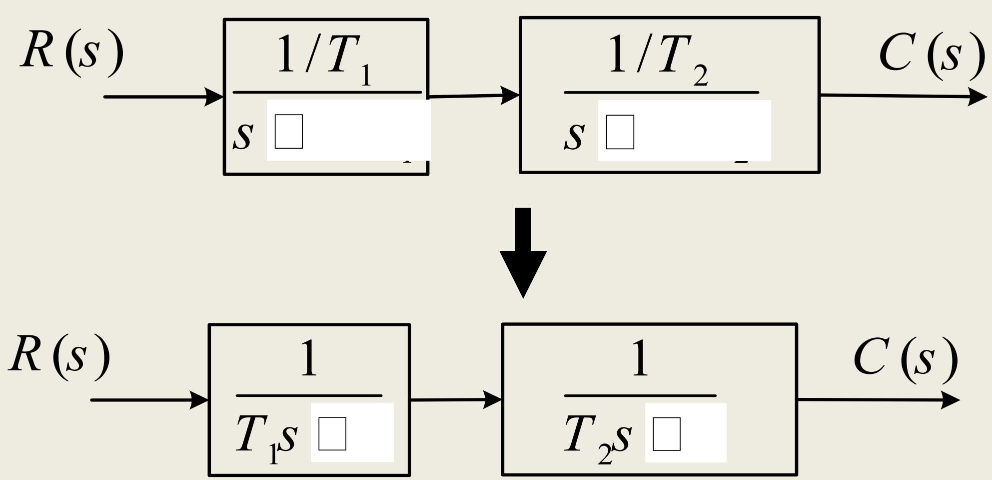


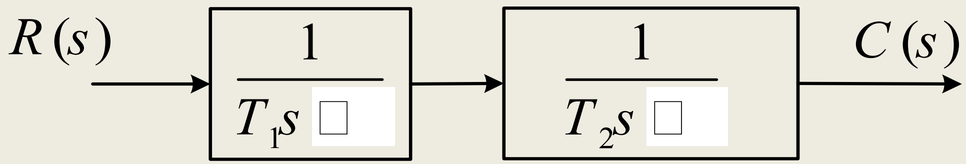
Furthermore, from

$$C(s) = \frac{\dfrac{1}{T_1 T_2}}{\left(s + \dfrac{1}{T_1}\right)\left(s + \dfrac{1}{T_2}\right)} R(s)$$

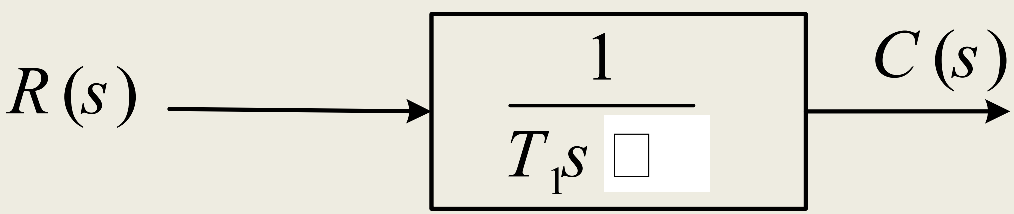


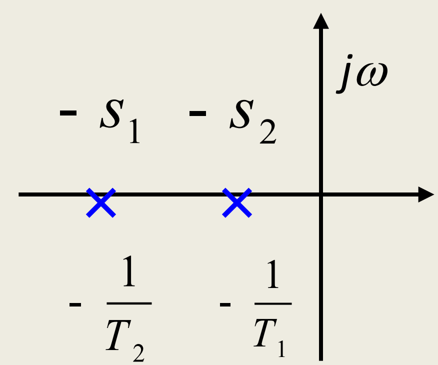
it can be seen that for overdamped case, the second-order system can be expressed as two first-order systems connected in cascade:





In particular, if $-s_2$ is located very much closer to the $j\omega$ axis than $-s_1$, (which means $s_2 << s_1$), then for an approximate solution we may neglect $-s_1$ and the second-order system can be reduced to a first-order system:



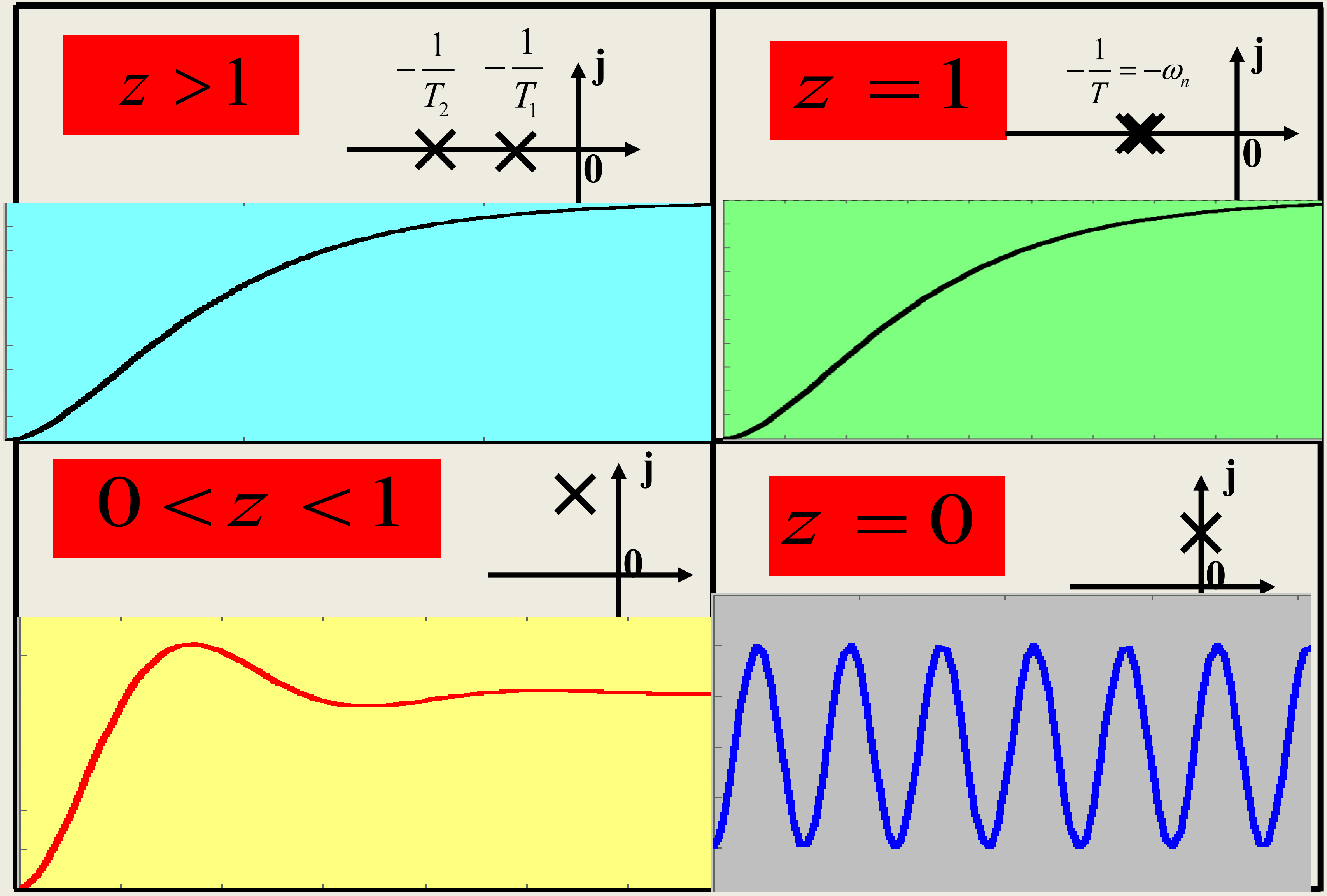


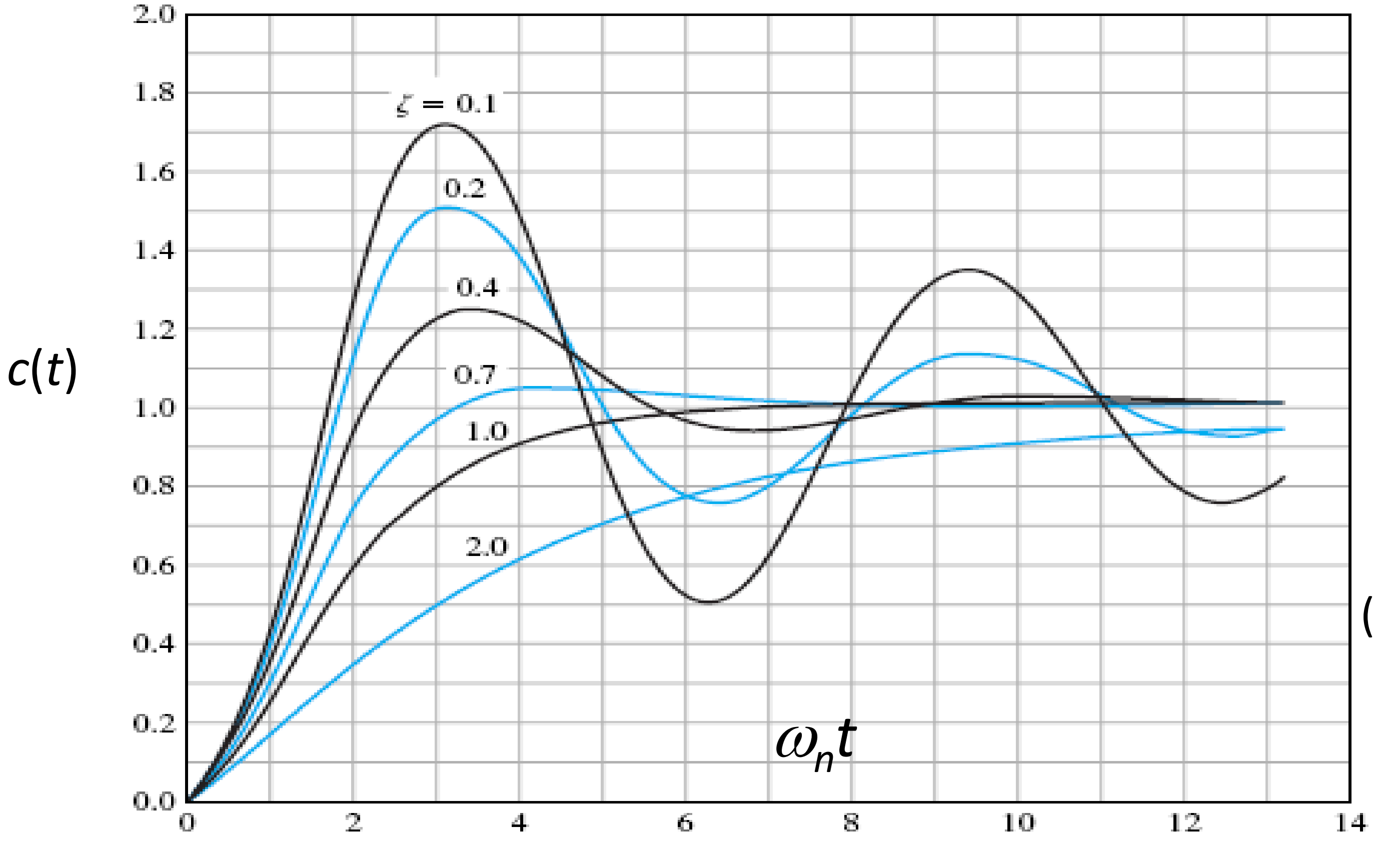
or

$$\frac{C(s)}{R(s)} = \frac{s_2}{s+s_2} = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}}{s+\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}}$$

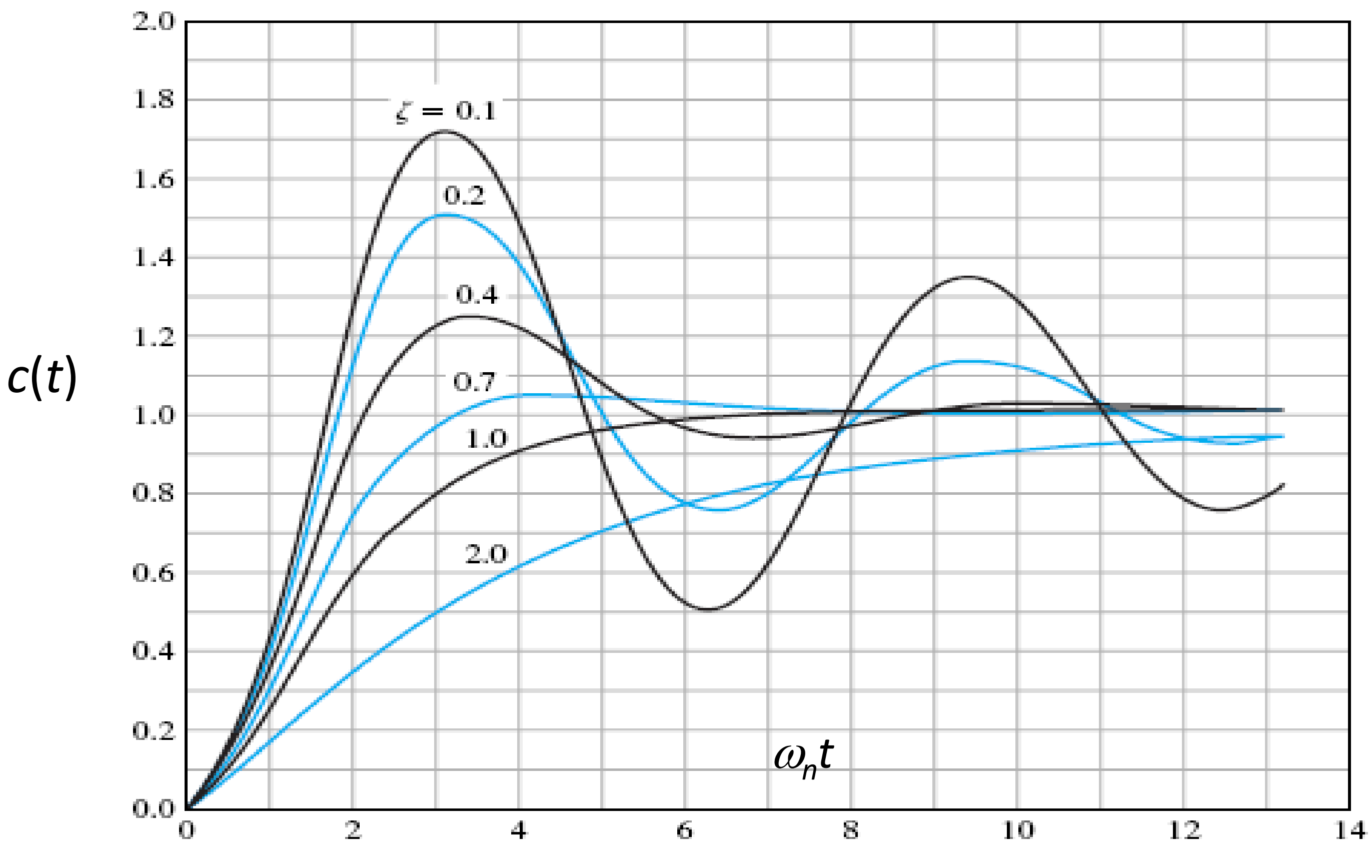
$$c(t) = 1 - e^{-(\zeta-\sqrt{\zeta^2-1})\omega_n t}, \quad t \geq 0$$

Second-order system $\Phi(s)=\frac{w_n^2}{s^2+2zw_ns+w_n^2}$



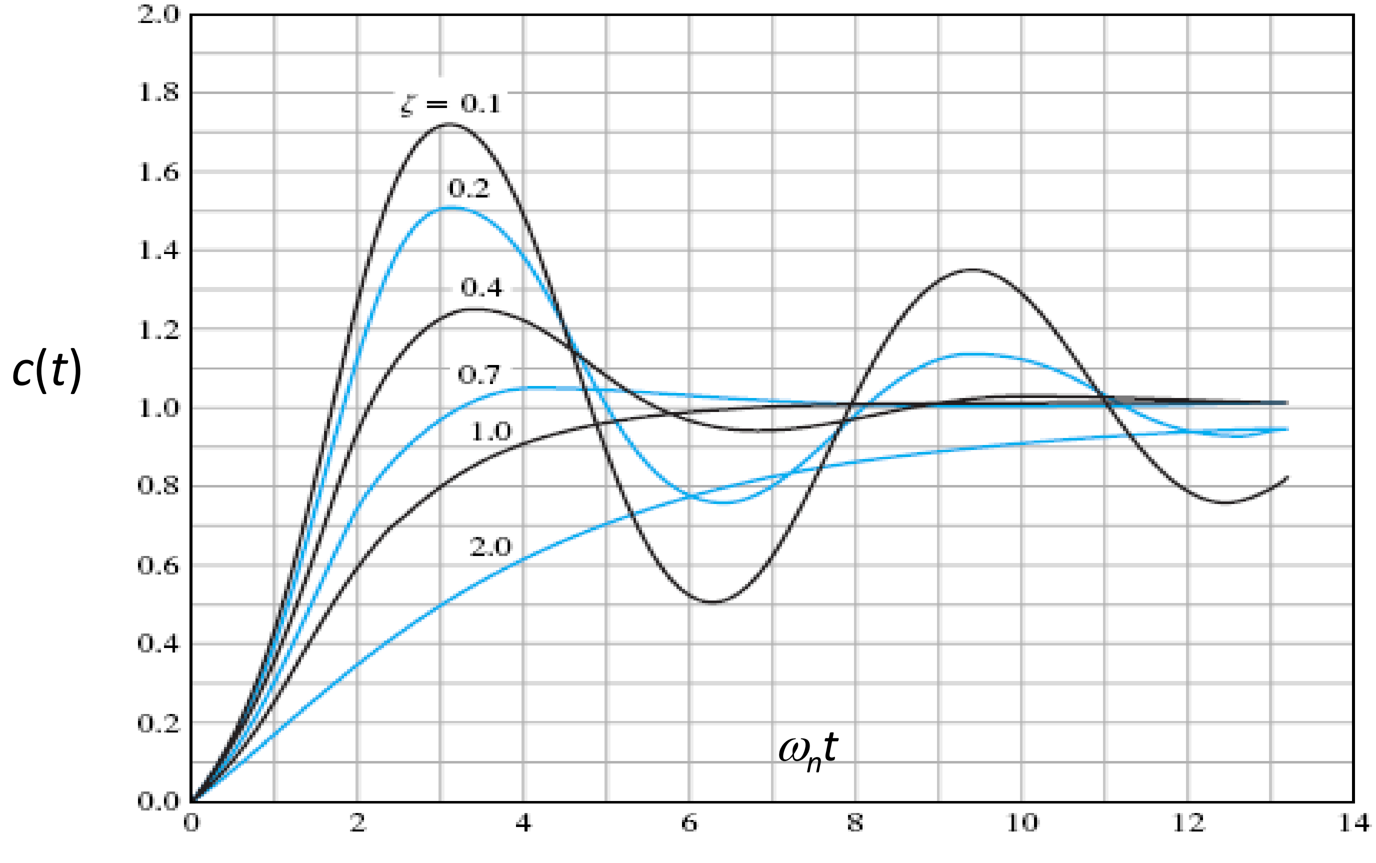


(1)When $0.4 \leq \zeta \leq 0.8$, the system gets close to the final value more rapidly than a critically damped or overdamped systems. Small values of ζ ($\zeta <$ 0.4) yield excessive overshoot in the transient response, and a system with a large value of ζ ($\zeta > 0.8$) responds sluggishly.



(a)

In particular, when $\zeta \approx 0.707$, the system exhibits fastest response with a nice overshoot (=4%).



(a)

(2) Among the systems without oscillation ($\zeta \geq 1$), the critical damped system exhibits the fastest response.

(3) An overdamped system is always sluggish.

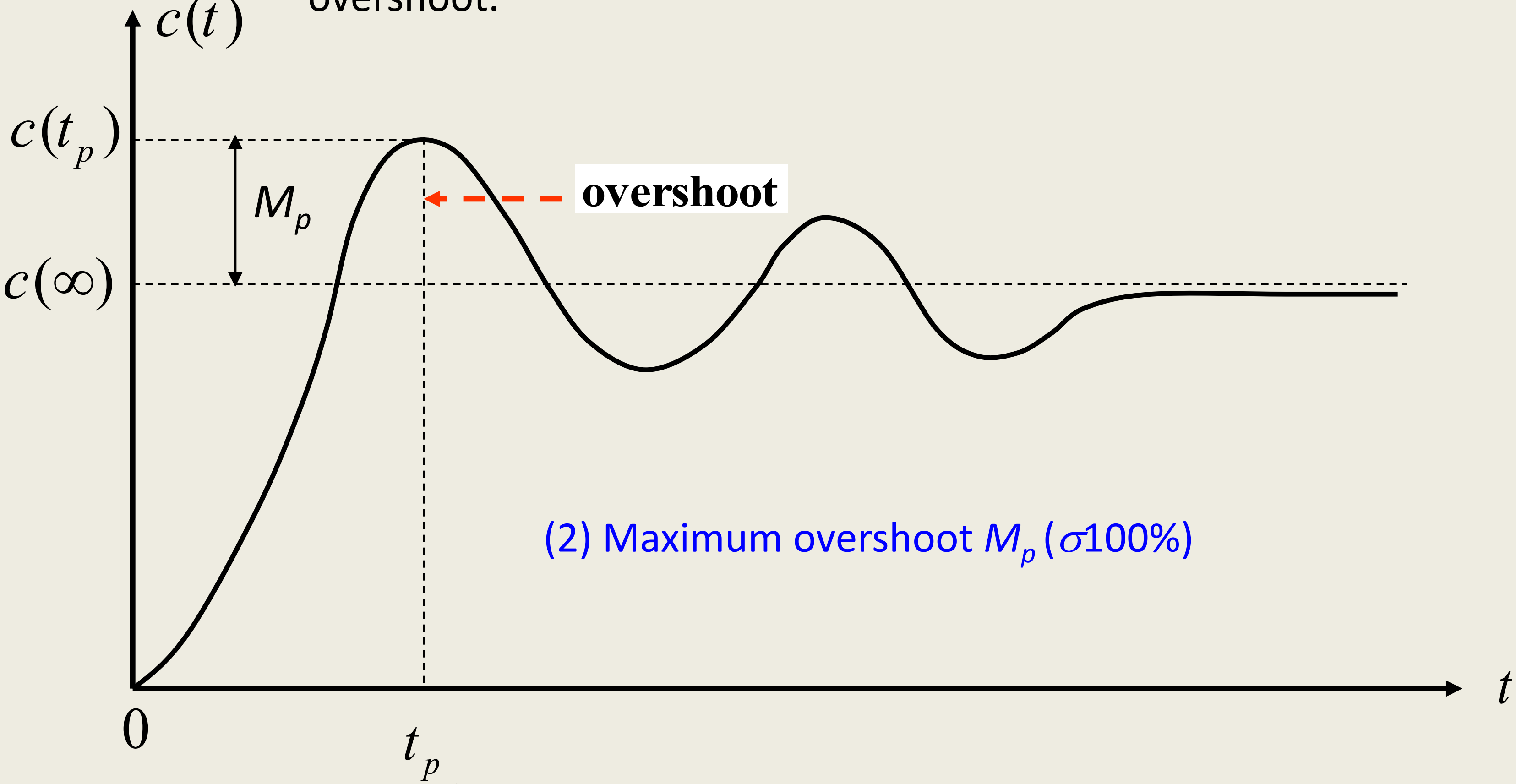
3. Definitions of Transient Response Specifications

- In many practical cases, the desired performance characteristics are specified in terms of *time-domain quantities*.

- Frequently, the performance characteristics of a control system are specified in terms of the transient response to a *unit-step input* since it is easy to generate and is sufficiently drastic.

(1) Peak time t_p and maximum percent overshoot M_p

t_p is the time required for the response to reach the first peak of the overshoot.



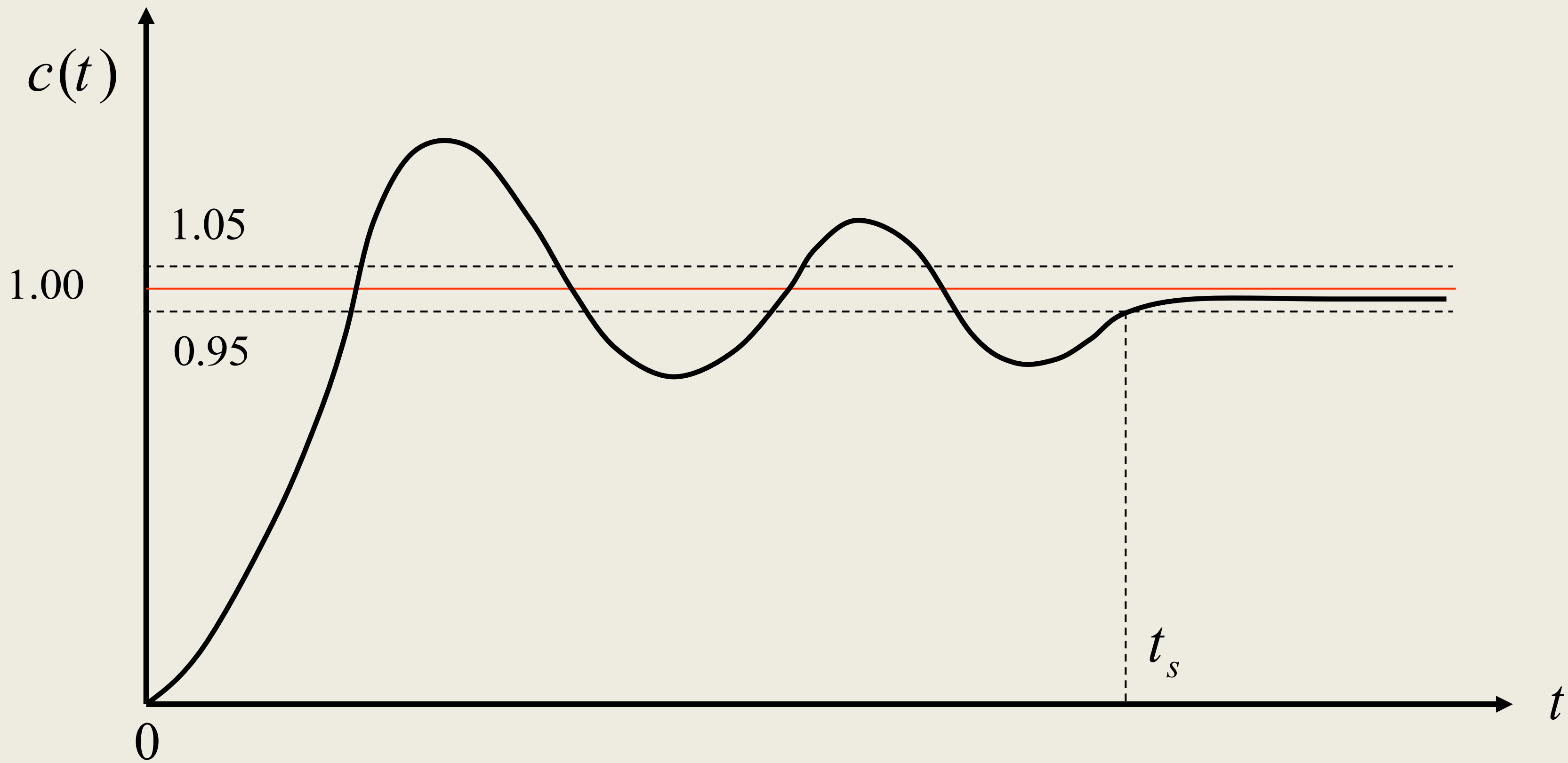
(2) Maximum overshoot M_p (σ100%)

Maximum percent overshoot:

$$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

(2) Settling time t_s

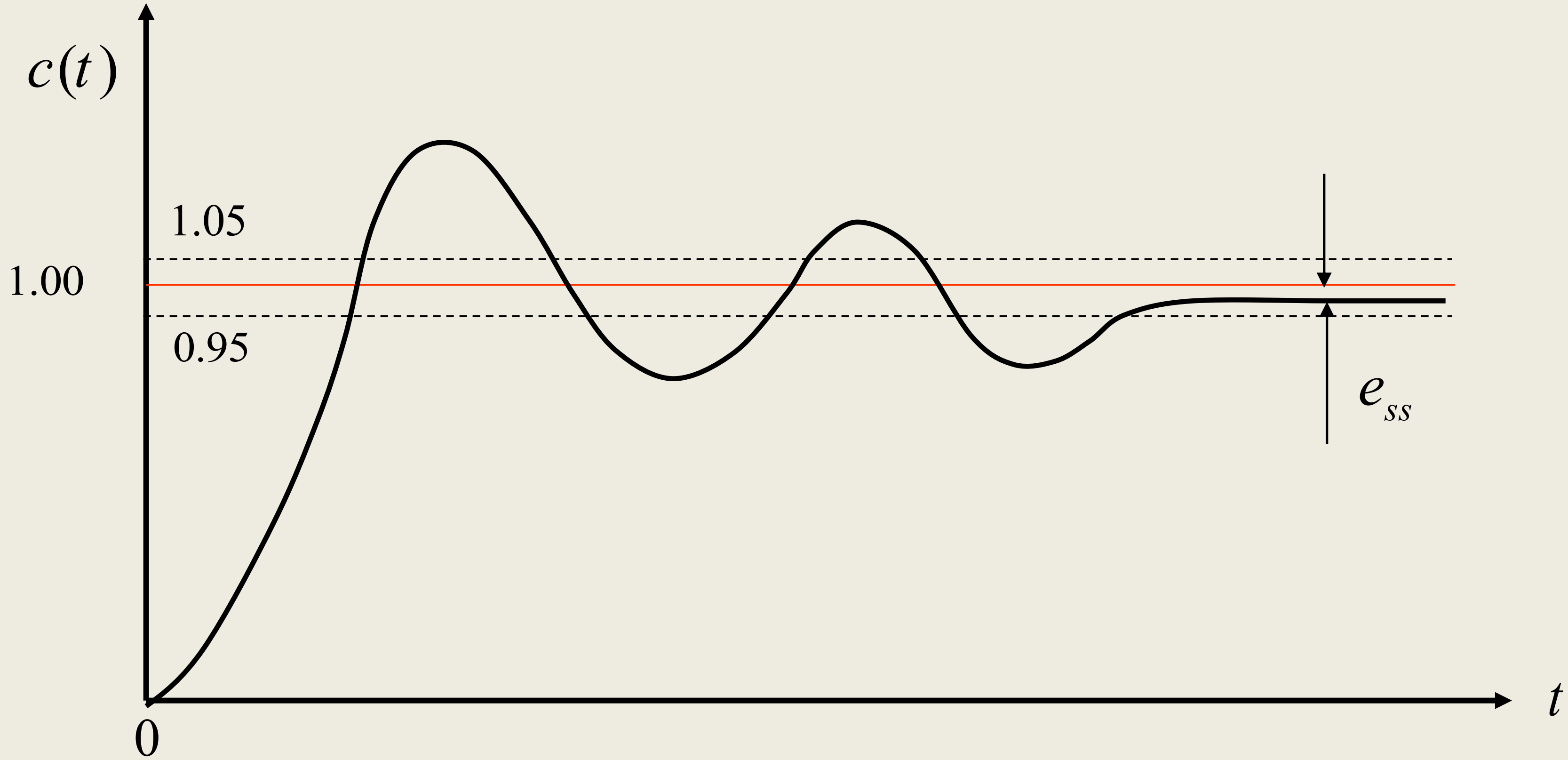
The settling time is defined as the time required for the step response to reach and stay within a specified percentage of its final value. A frequently used figure is 5% or 2% tolerance band.



t_s corresponding to a ±5% tolerance band

(3) Steady-state error e_{ss}

$$e_{ss} := \lim_{t\to\infty}(r(t) - c(t)) = \lim_{t\to\infty}(1 - c(t))$$



Steady-state error

4. Transient Response Specifications for second-order systems

(1) Peak time t_p (0<ζ<1):

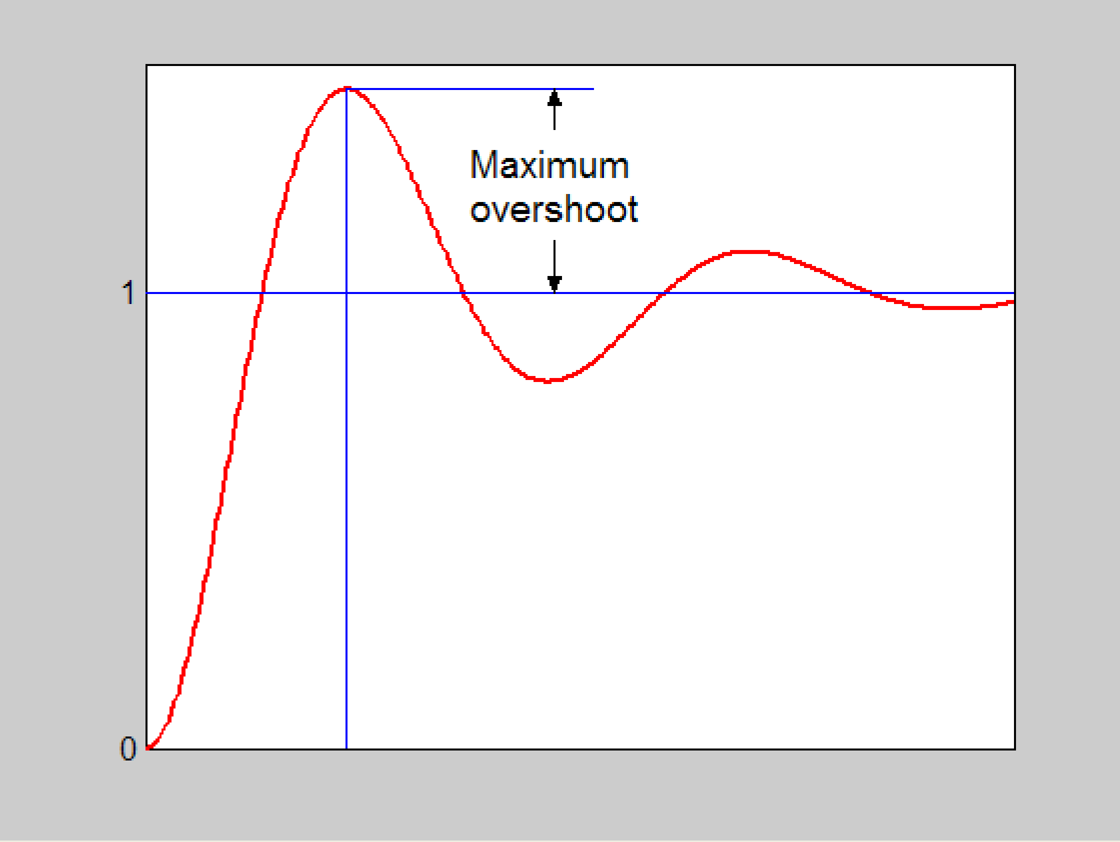
$$c(t) = 1 - \frac{e^{-zw_n t}}{\sqrt{1-z^2}} \sin\left(w_d t + b\right)$$

$$\frac{dc(t)}{dt} = \frac{w_n}{\sqrt{1-z^2}} e^{-Ww_n t} \sin w_d t = 0 \Rightarrow w_d t = 0, p, 2p, \ldots$$

$$t_p = \frac{p}{w_d} = \frac{p}{w_n \sqrt{1-z^2}}$$

(2) Maximum Overshoot (0<ζ<1):

$$t_p = \frac{p}{w_d} = \frac{p}{w_n\sqrt{1-V^2}} \Rightarrow \qquad c(t_p) = 1 + e^{-\frac{pz}{\sqrt{1-z^2}}}$$

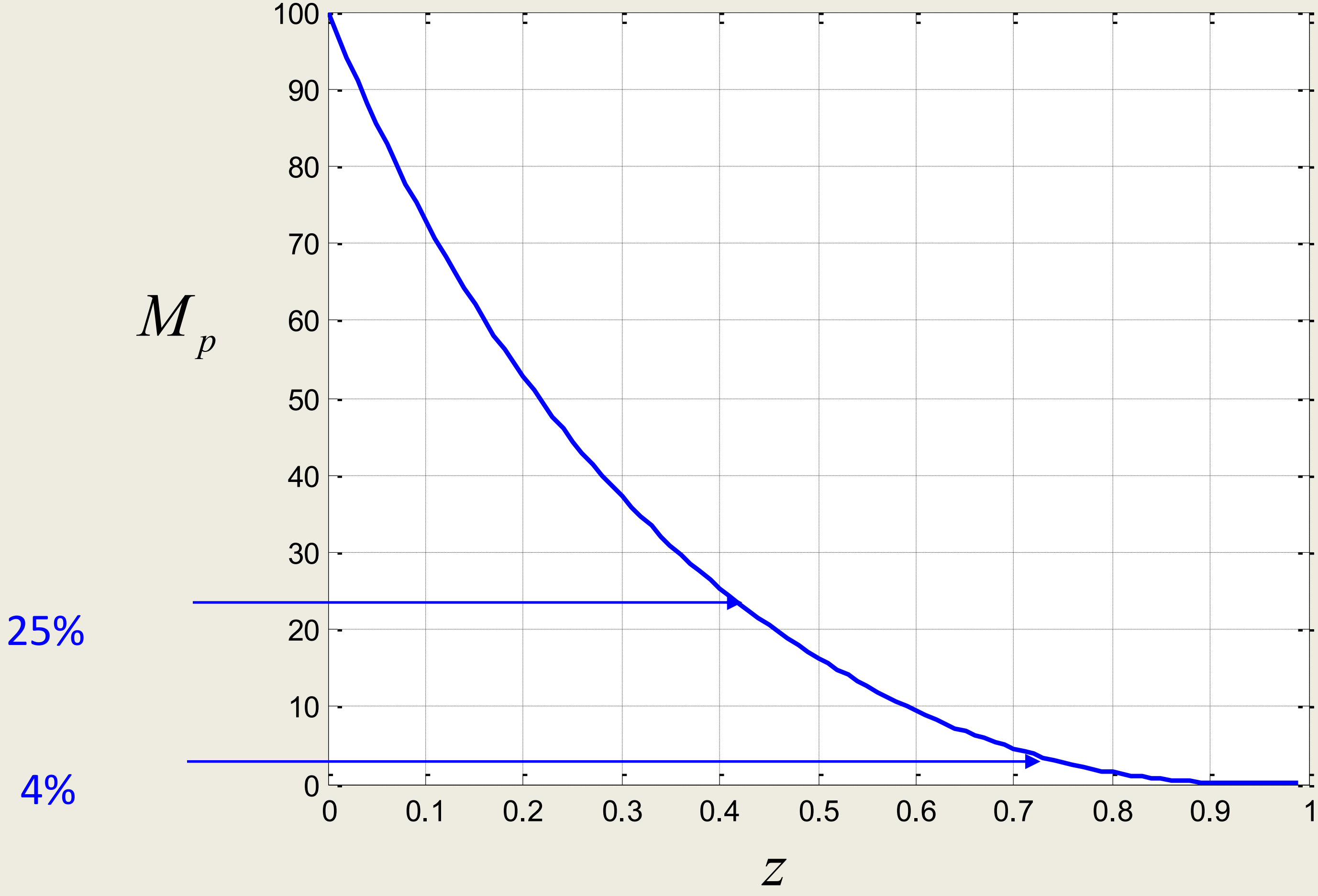


Hence,

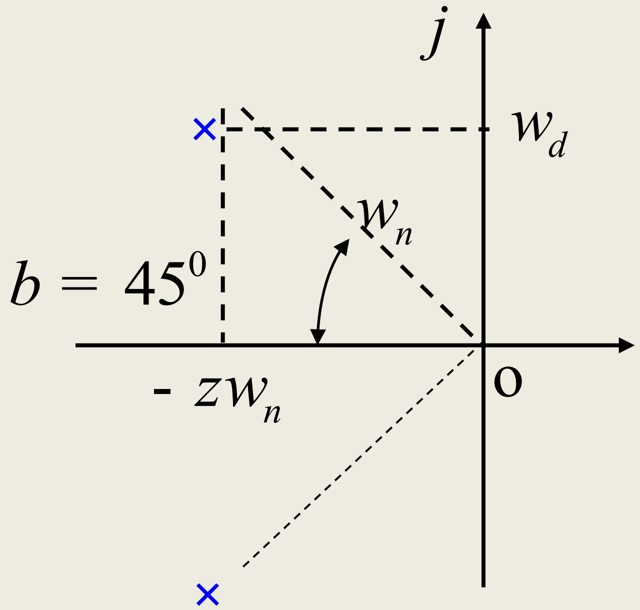
$$M_p = e^{\frac{-pz}{\sqrt{1-z^2}}} \times 100\%$$

The relationship between ζ and M_p is given below. Note that if ζ is between 0.4 and 0.7, then M_p is between 25% and 4% .

$$M_p = e^{-pz/\sqrt{1-z^2}} \times 100\%$$



In particular, when $\zeta=1/\sqrt{2}=0.707$, which corresponds to $\beta=45^0$, $M_p=4\%$! Such a ζ is called *optimal damping ratio*.

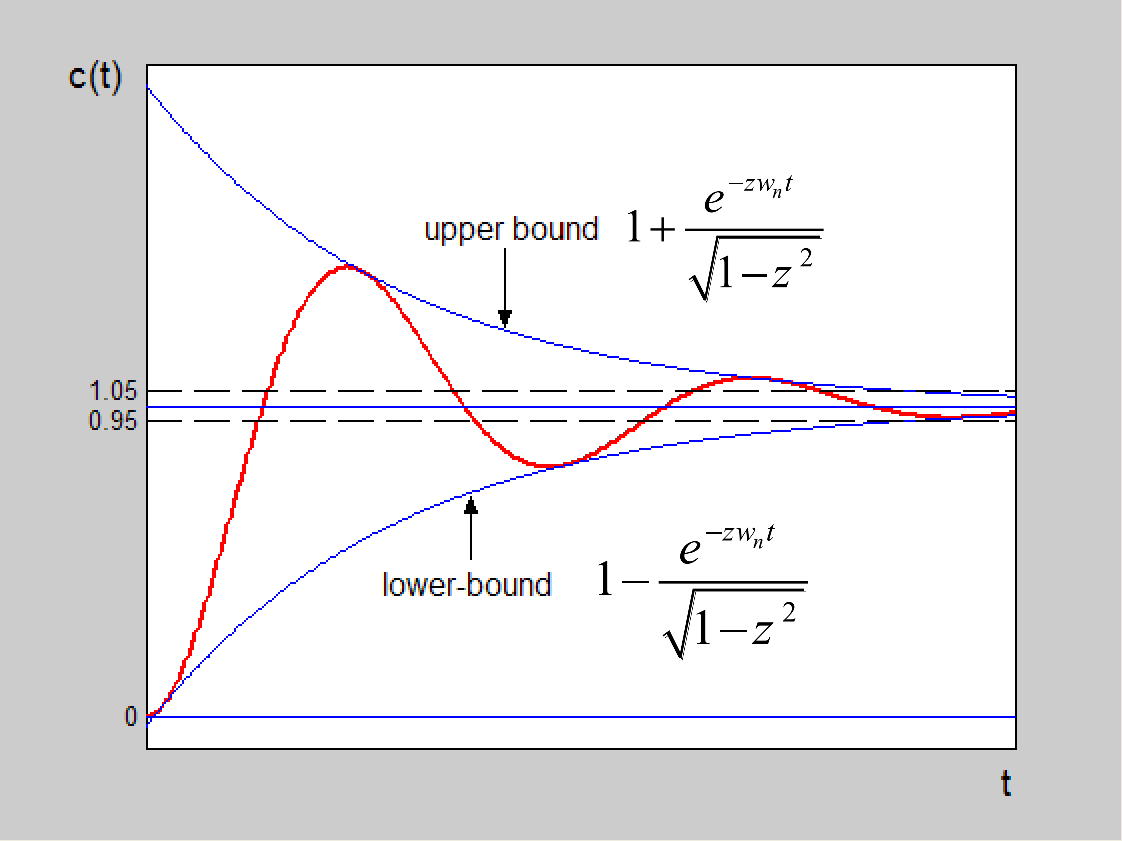


(3) Settling time ($0<\zeta<1$):

$$c(t) = 1 - \frac{e^{-zw_n t}}{\sqrt{1-z^2}} \sin\left(w_d t + b\right)$$

$$1 \pm \frac{e^{-zw_n t}}{\sqrt{1-z^2}} = \text{envelope curves}$$

whose time constant is $T=1/\zeta\omega_n$.



Therefore,

$t_s \approx 4T = 4/\zeta\omega_n$, (2% criterion)

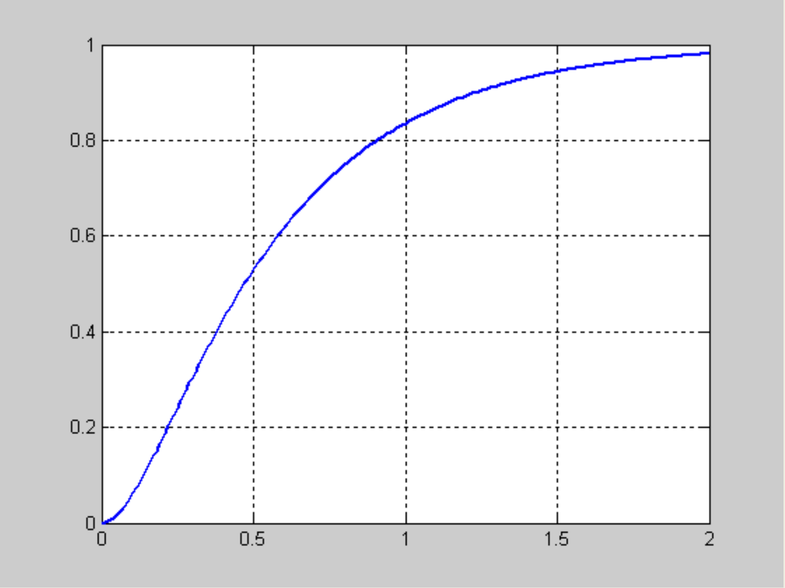
$t_s \approx 3T = 3/\zeta\omega_n$, (5% criterion)

(4) Settling time t_s for ($\zeta \geq 1$):

From

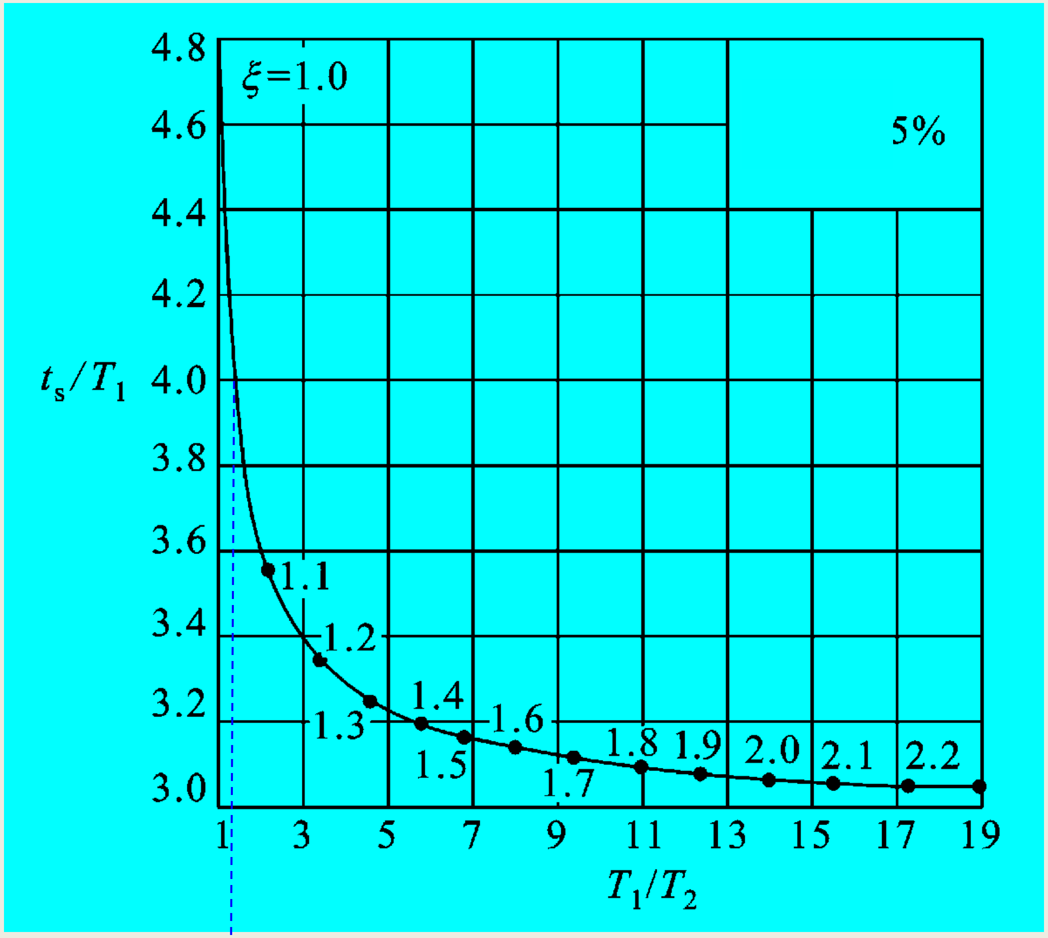
$$C(s) = \frac{\frac{1}{T_1 T_2}}{\left(s + \frac{1}{T_1}\right)\left(s + \frac{1}{T_2}\right)} \frac{1}{s}$$

$$= \frac{1}{s(T_1 s + 1)(T_2 s + 1)}$$



the system can be considered as two first-order subsystems connected in cascade. Since no oscillation occurs, only settling time t_s is concerned.

When $\zeta \geq 1$, the settling time t_s can be obtained by looking up the following table (for 5% tolerance band):



1.5

For example:

1) $T_1 = T_2$ Û $z = 1$

$t_s = 4.75T_1$

2) $T_1 / T_2 = 1.5$ Þ $z = 1.02$

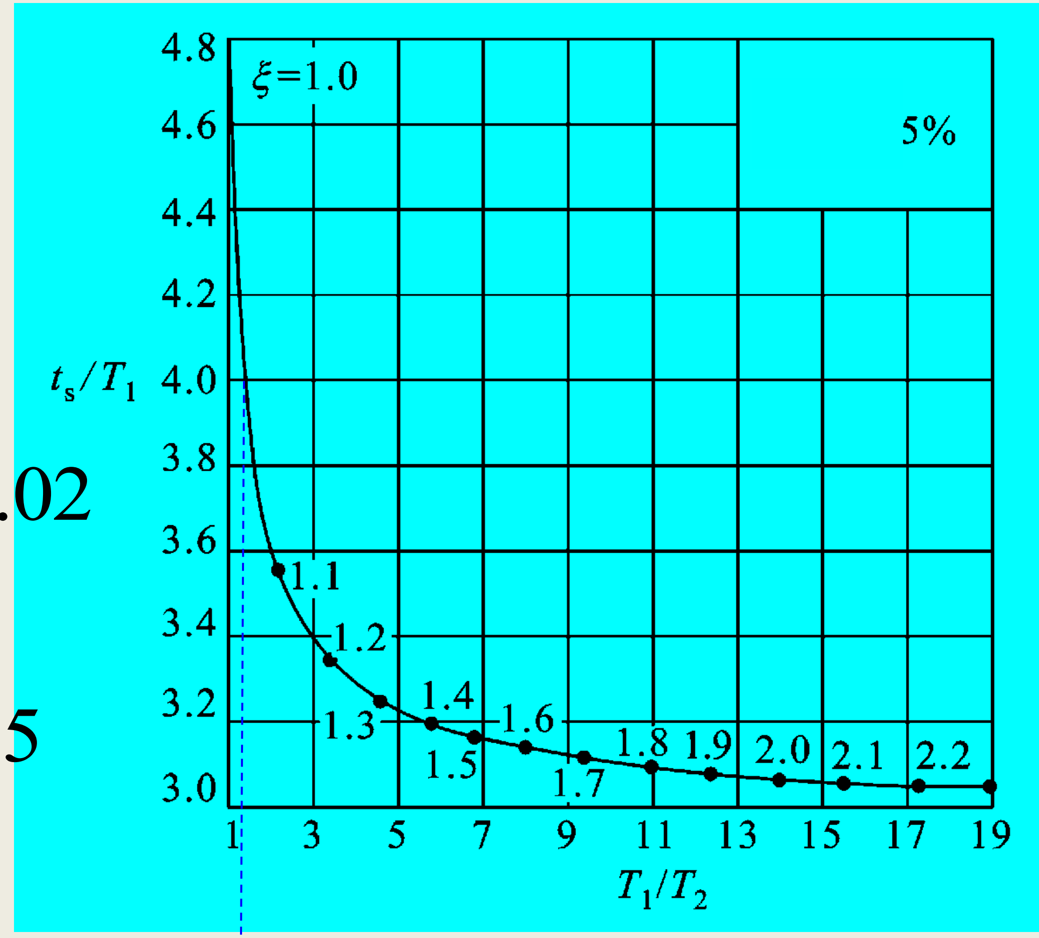
$t_s = 4T_1$

3) $T_1 / T_2 = 4$ Þ $z = 1.25$

t_s » $3.3T_1$

4) $T_1 / T_2 > 4$ Þ $z > 1.25$

t_s » $3T_1$



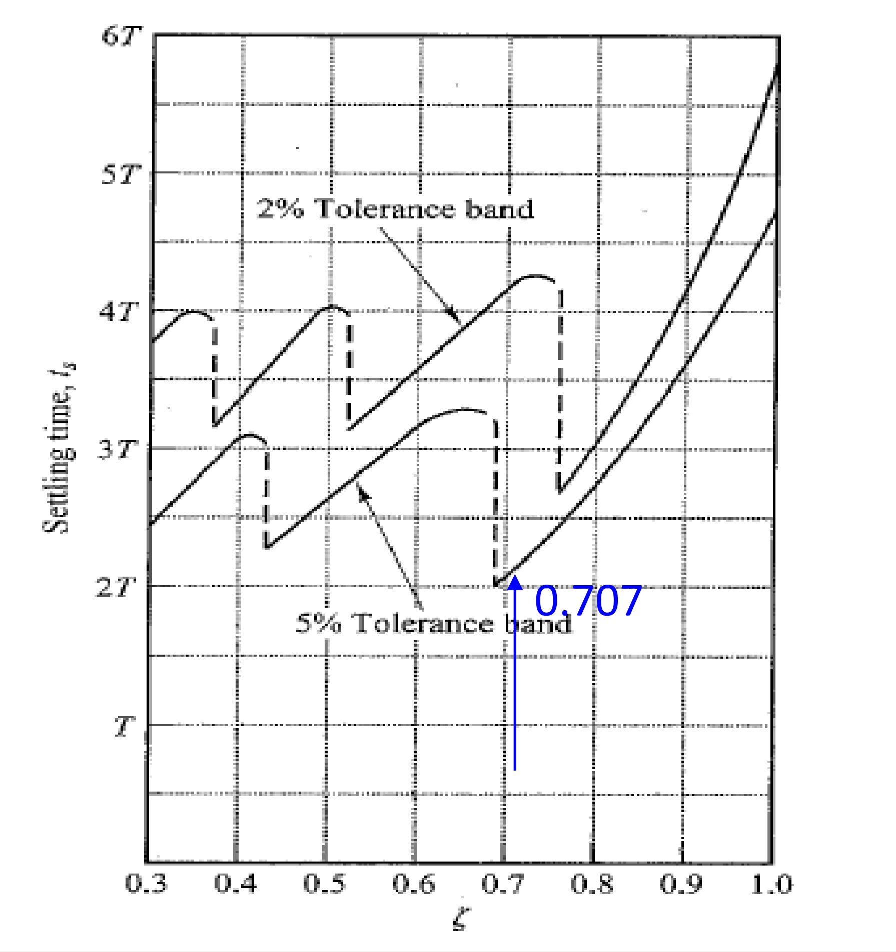
1.5

Table: t_s/T_1 versus T_1/T_2

Comments on settling time

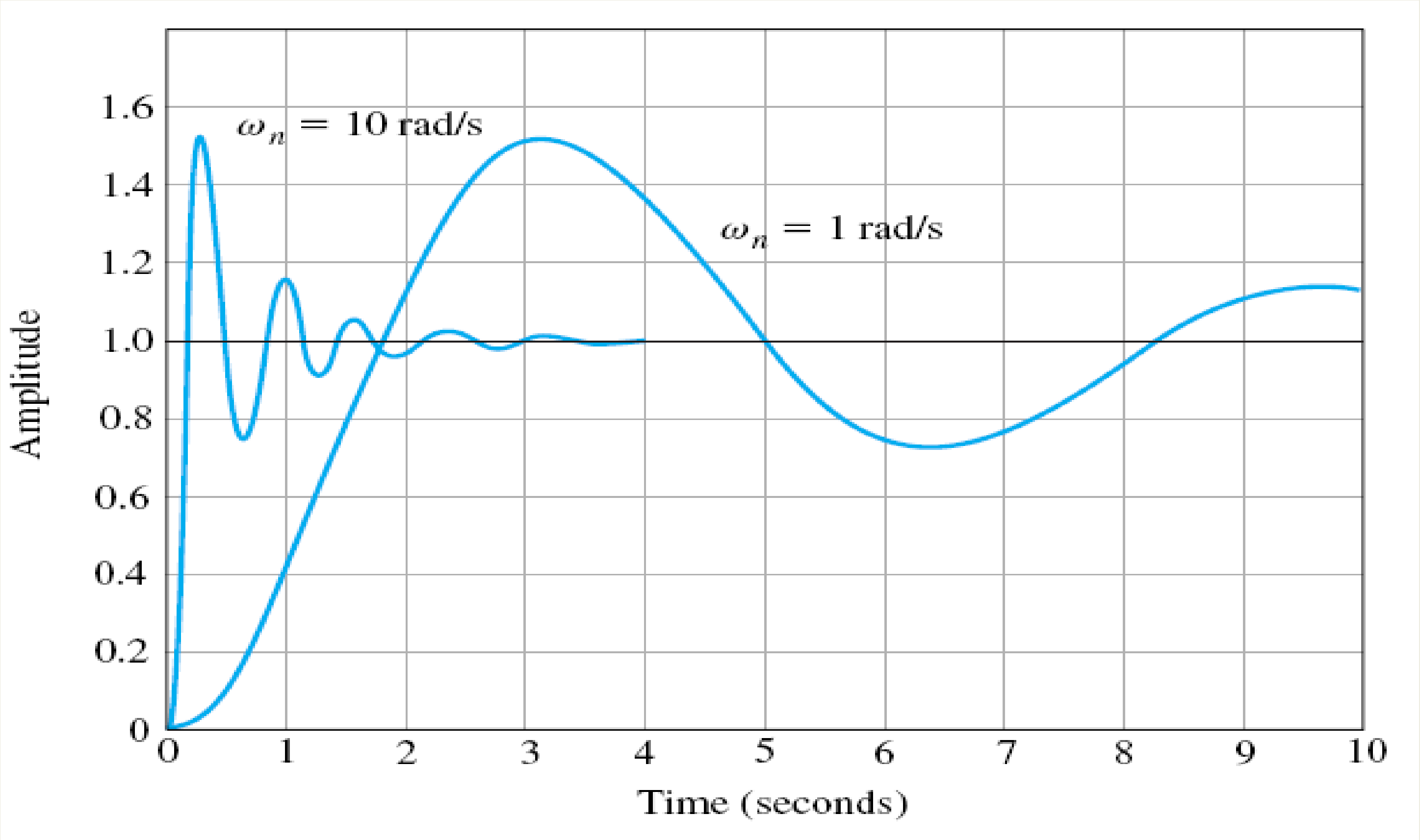
- t_s is inversely proportional to the product of ζ and ω_n. Since ζ is usually given by designer from the requirement of M_p, t_s is mainly determined by ω_n.

t_s reaches a minimum value around ζ = 0.76 (for the 2% criterion) or ζ = 0.68 (for the 5% criterion) and then increases almost linearly for large values of ζ. Note that ζ=0.707 implies that β=45^0.



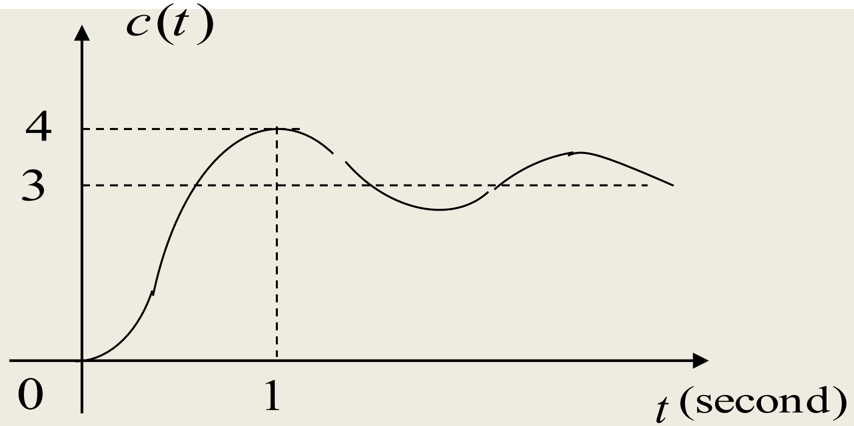
- (t_s versus ζ).

Example. The step responses for ζ=0.2 with ω_n=1 and ω_n=10, respectively, are shown below.



For a given ζ, the response is faster for larger ω_n. Note that the overshoot is independent of ω_n.

Example. The unit-step response of a second-order system is shown below, where $\lim_{t\to\infty} c(t)=3$. Determine its transfer function.



Solution: The transfer function must have the following form:

$$F(s) = \frac{K w_n^2}{s^2 + 2zw_n s + w_n^2}$$

with K=3. From the response, it is clear that t_s=1s and M_p=(1/3)100%=33.3%. Hence, by utilizing the formulas

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100\%$$

and

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

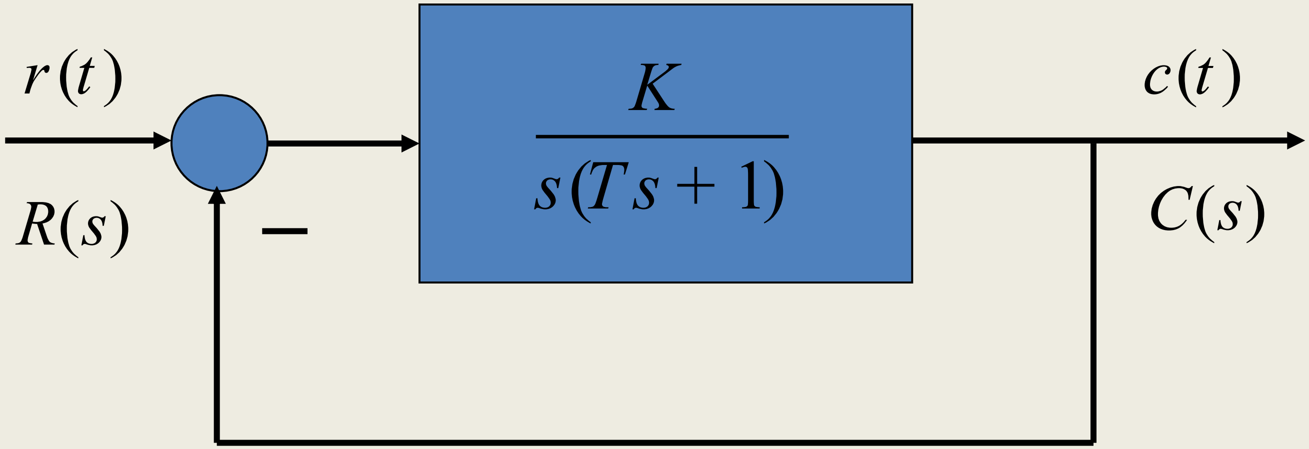
one can obtain that

$$\zeta = 0.33$$

and

$$\omega_n = 3.33\ \mathbf{rad/s}$$

Example. Consider the following system:



where T=0.1s, and K is the open-loop gain. To determine K such that no overshoot and t_s=1s.

Solution: By the requirements, ζ should satisfy $\zeta \geq 1$. Therefore, the closed-loop characteristic equation can be expanded as

$$G(s) = s^2 + \frac{1}{T}s + \frac{K}{T} = \left(s + \frac{1}{T_1}\right)\left(s + \frac{1}{T_2}\right) = s^2 + \left(\frac{1}{T_1} + \frac{1}{T_2}\right)s + \frac{1}{T_1 T_2} = 0$$

Equating the coefficients for the same power of s yields:

$$\begin{cases} \frac{K}{T} = \frac{1}{T_1 T_2} \\ \frac{1}{T} = \frac{1}{T_1} + \frac{1}{T_2} \end{cases}$$

To make the response as quickly as possible, it is required that ζ be close to 1. Looking up the table, we obtain that when

$$T_1 / T_2 = 1.5$$

we have

$$t_s / T_1 = 4$$

In that case, ζ=1.02, which is very much close to 1 and therefore, possesses a fast transient response. Since t_s=1s, we have

$$T_1 = \frac{1}{4} t_s = 0.25 \text{ s}$$

Therefore,

$$\frac{T_1}{T_2} = 1.5 \Rightarrow T_2 = \frac{T_1}{1.5} = \frac{0.25}{1.5} = 0.167 \text{ s}$$

To determine *K*, notice that

$$\begin{cases} \dfrac{K}{T} = \dfrac{1}{T_1 T_2} \\ \dfrac{1}{T} = \dfrac{1}{T_1} + \dfrac{1}{T_2} \end{cases}$$

Therefore,

$$K = \left.\frac{T}{T_1 T_2}\right|_{T=0.1} = \frac{0.1}{0.25' \ 0.167_2} = 0.24 s^{-1}$$

Finally, we should check that

$$\frac{1}{T_1} + \frac{1}{T_2} = \frac{1}{T} = \frac{1}{0.1} = 10$$

is satisfied. Otherwise, *K* should be redesigned. Fortunately, in this example, from

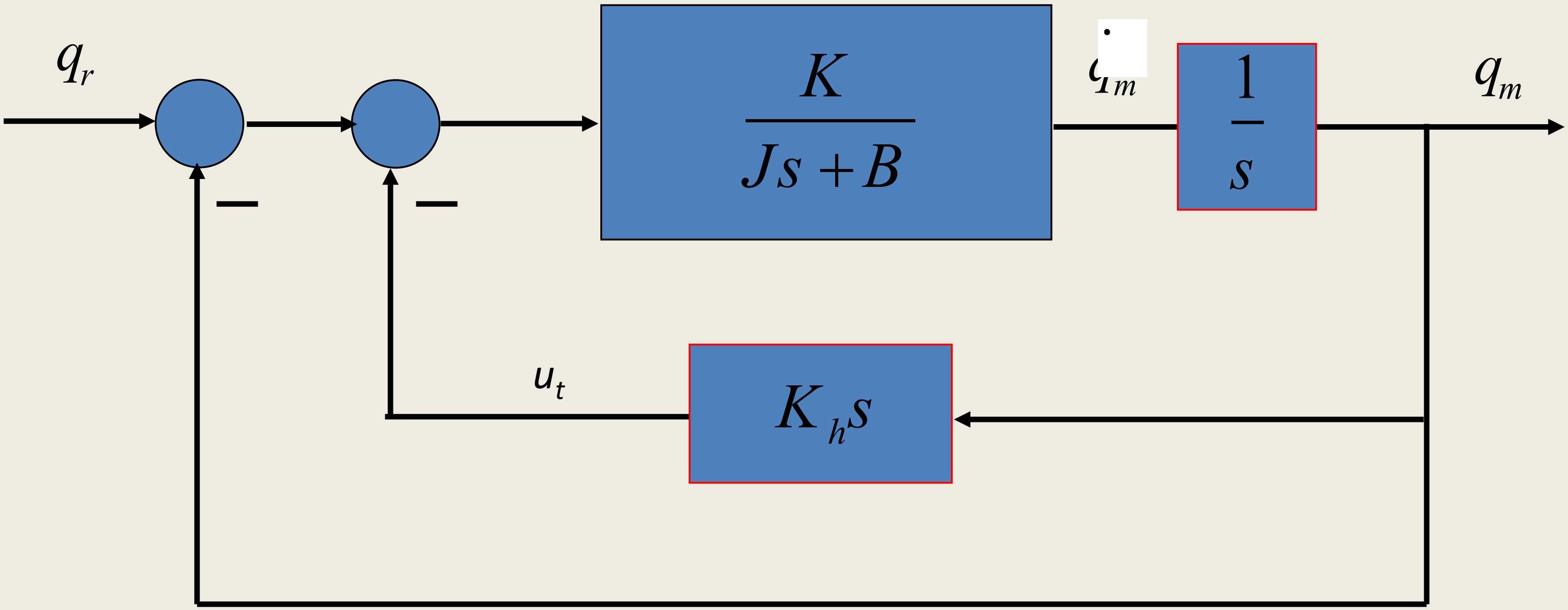
,

$$\frac{1}{T_1} = 4, \quad \frac{1}{T_2} = 6$$

such a condition is satisfied.

5. Servo system with velocity feedback

The derivative of the output signal can be used to improve system performance :



Note that without the derivative feedback (K_h=0), system may exhibit excessive overshoot. Indeed, the closed-loop transfer function in that case is

$$\frac{\Theta_m(s)}{\Theta_r(s)} = \frac{K/J}{s^2 + (B/J)s + K/J} = \frac{w_n^2}{s^2 + 2zw_n s + w_n^2}$$

where

$$w_n = \sqrt{K/J}$$

$$z = \frac{B}{2\sqrt{JK}}$$

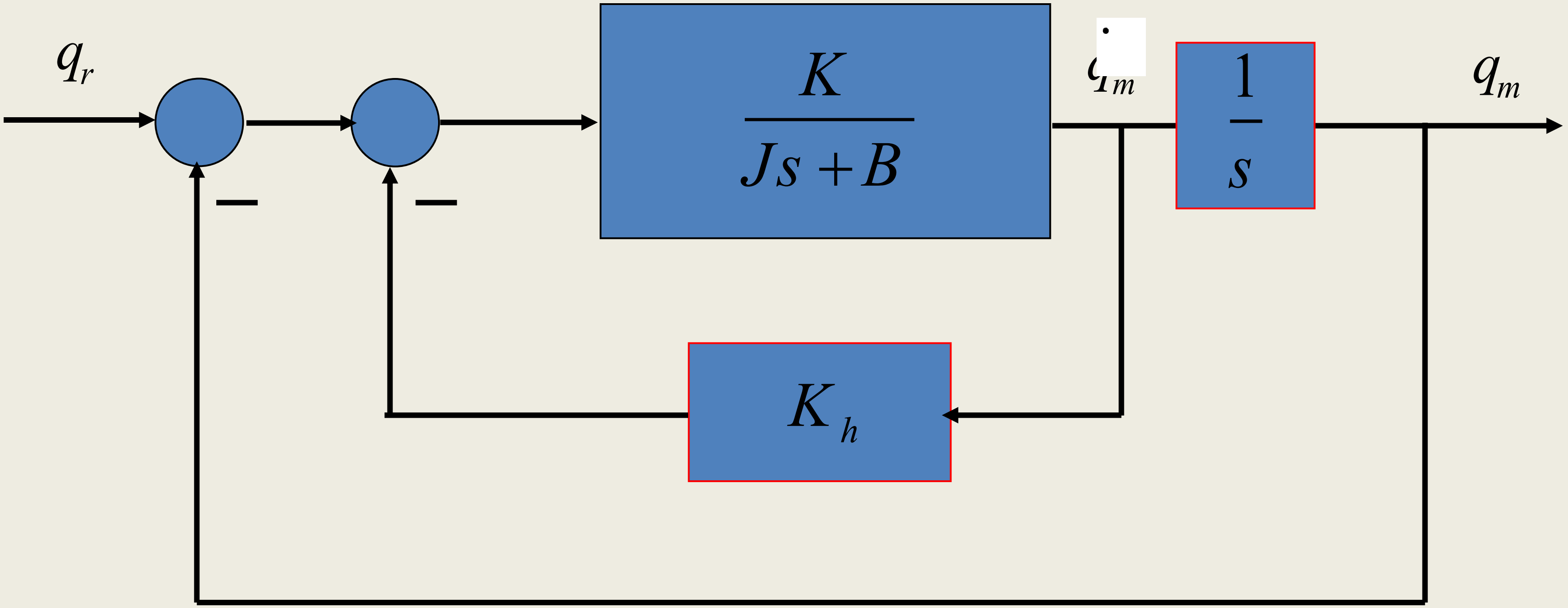
However, taking the derivative feedback into account, the closed-loop transfer function becomes

$$\frac{\Theta_m(s)}{\Theta_r(s)} = \frac{w_n^2}{s^2 + 2(z + K_h w_n / 2)w_n s + w_n^2}$$

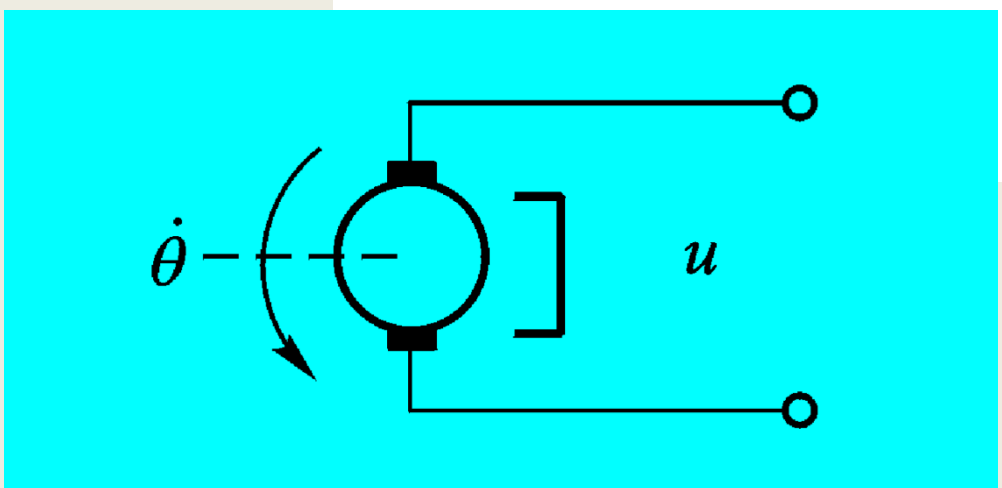
The improved damping ratio is:

$$z_t = z + \frac{1}{2}K_h w_n$$

For a servo control system, in obtaining the derivative of the output position signal, it is desirable to use a tachometer generator instead of physically differentiating the output signal (noise effect, p.175).



Servo-Tek Tachometer Generators provide a convenient means of converting rotational speed into an isolated analog voltage signal suitable for control applications.



Mathematical model:

$$u(t) = K_h \dot{q}(t)$$

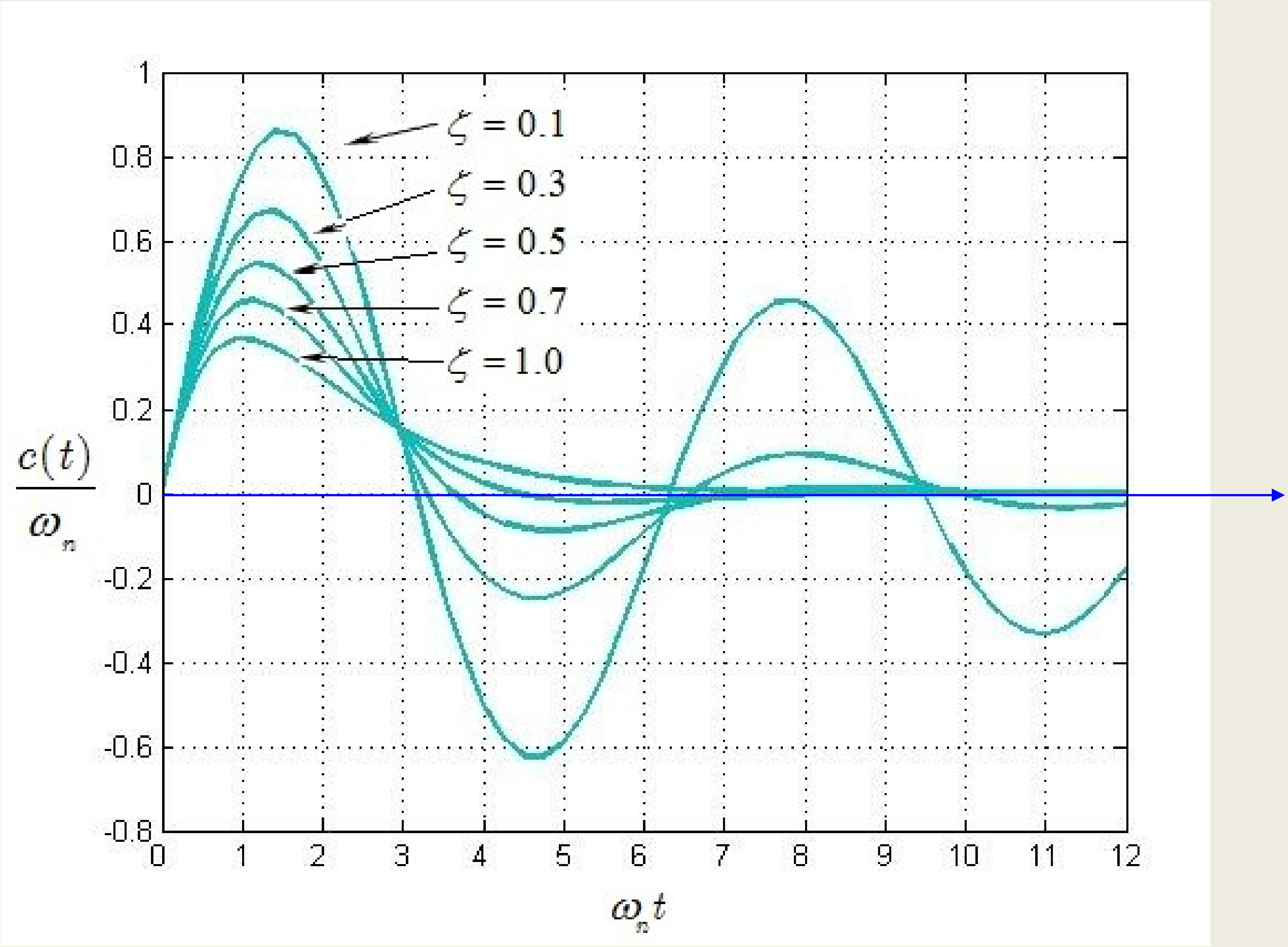
6. Impulse response of second-order systems

By the property of LTI systems, differentiating the corresponding unit-step response of the second-order system (or directly taking the inverse Laplace trans-form) yields

$$c(t)=\frac{w_n}{\sqrt{1-z^2}}e^{-zw_n t}\sin\left(w_d t\right), t\geq 0, \quad 0\leq z<1$$

$$c(t)= w_n^2 t e^{-\ w_n t}, \quad t\ ^{3}\ 0, \quad z=1$$

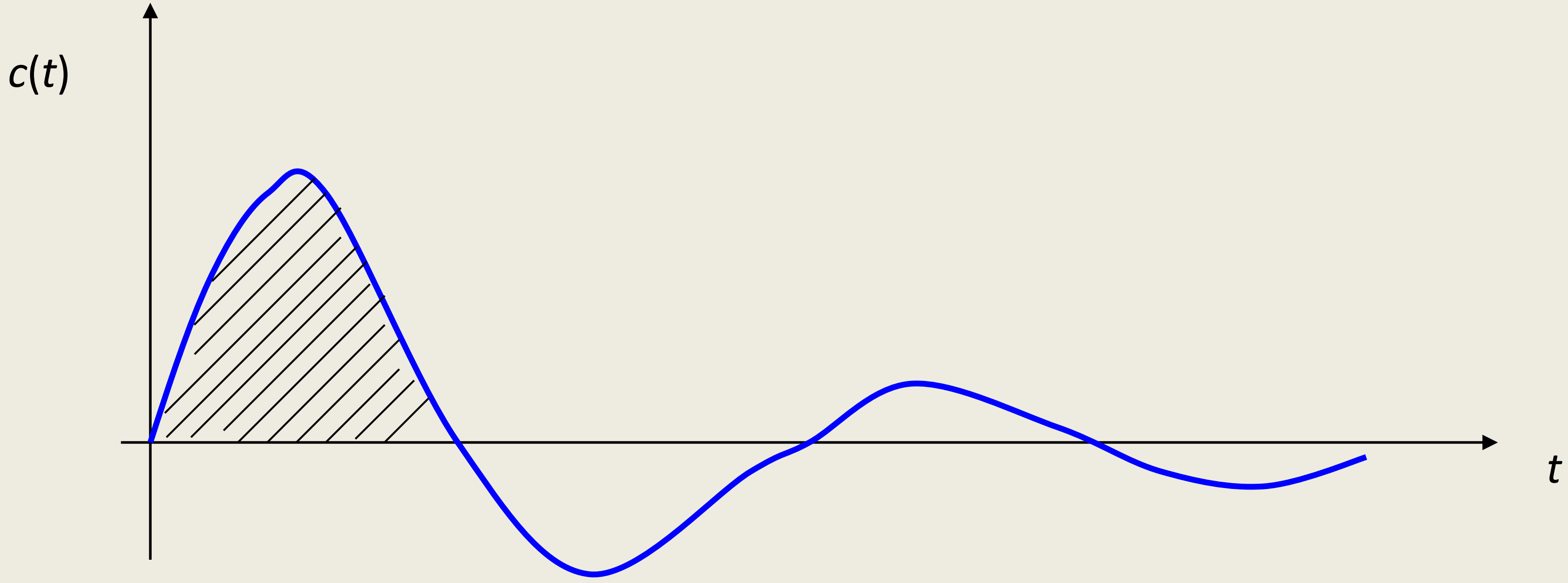
$$c(t)=\frac{w_n}{2\sqrt{z^2-1}}e^{-\left(z-\sqrt{z^2-1}\right)w_n t}-\frac{w_n}{2\sqrt{z^2-1}}e^{-\left(z+\sqrt{z^2-1}\right)w_n t}, t\geq 0, \quad z>1$$



For $\zeta<1$, $c(t)$ oscillates and takes both positive and negative values.

Questions: 1) How to obtain the unit-ramp responses for a second-order system when ζ=0, 0<ζ<1, and ζ≥1 ?

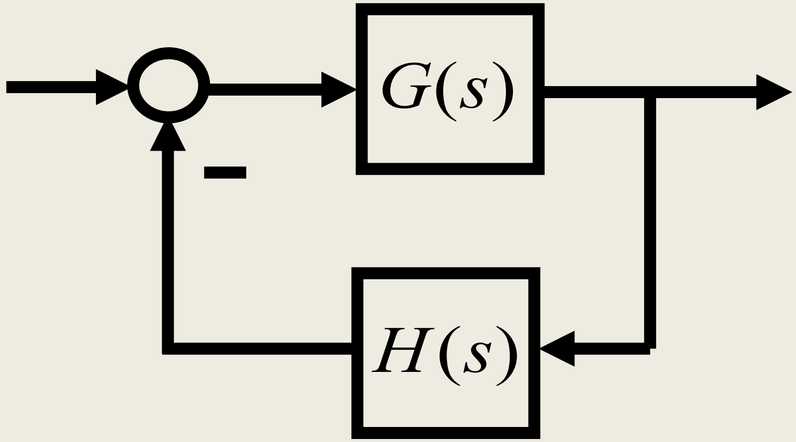
2) How to obtain the maximum overshoot from the unit-impulse response?



5-4 Higher-Order Systems

In this section, we shall present a transient response analysis of higher-order systems in general forms. We shall show that the response of higher-order systems is the sum of the responses of first-order and second-order systems

1. Transient response of higher-order systems



Consider the closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{b_0 \prod_{i=1}^{m} (s + z_i)}{\prod_{k=1}^{n} (s + p_k)}, m \leq n$$

Assumption 1: $m \leq n$.

Almost any physical control system satisfies this condition. A transfer function satisfies Assumption 1 is called a *proper transfer function*. If $m<n$, the system is called *strictly proper*. We only deal with proper or strictly proper systems.

Assumption 2: All the closed-loop poles lie in the left-half *s*-plane.

Assumption 3: All the poles are *distinct*.

Unit-step response: Let the input signal be a unit-step function. Then the output is

$$C(s) = \frac{b_0 \prod_{i=1}^{m} (s + z_i)}{\prod_{k=1}^{n} (s + p_k)} \times \frac{1}{s} = \frac{a}{s} + \sum_{k=1}^{n} \frac{a_k}{s + p_k}$$

where a and a_k are the residues of the poles at $s=0$ and $s=-p_k$, respectively:

$$a = \lim_{s \to 0} \frac{b_0 \prod_{i=1}^{m} (s + z_i)}{\prod_{k=1}^{n} (s + p_k)} = \frac{b_0 \prod_{i=1}^{m} (z_i)}{\prod_{k=1}^{n} (p_k)} = \Phi(0)$$

$$a_k = \lim_{s \to -p_k} \frac{b_0 \prod_{i=1}^{m} (s + z_i)}{\prod_{\substack{i=1 \\ i \neq k}}^{n} (s + p_i)} \frac{1}{s} = \frac{b_0 \prod_{i=1}^{m} (-p_k + z_i)}{\prod_{\substack{i=1 \\ i \neq k}}^{n} (-p_k + p_i)} \frac{1}{-p_k}$$

We consider the general case that $C(s)$ consists of real poles and pairs of complex-conjugate poles. Then

$$C(s)=\frac{a}{s}+\sum_{j=1}^{q}\frac{a_j}{s+p_j}+\sum_{k=1}^{r}\frac{b_k(s+z_k w_k)+c_k w_k\sqrt{1-z_k^2}}{s^2+2z_k w_k s+w_k^2}$$

where $n=q+2r$. For example,

$$C(s)=\frac{-s^3+3s+4}{(s+1)(s^2+2s+2)}\frac{1}{s}=\frac{2}{s}-\frac{2}{s+1}-\frac{s+1}{s^2+2s+2}$$

Therefore, for the general case, the unit-step response can be written as

$$c(t)=a+\sum_{i=1}^{q}a_i e^{-p_i t}+\sum_{k=1}^{N}b_k e^{-z_k w_k t}\cos w_k\sqrt{1-z_k^2}\,t$$

$$+\sum_{k=1}^{N}c_k e^{-z_k w_k t}\sin w_k\sqrt{1-z_k^2}\,t$$

$$=a+\sum_{i=1}^{q}a_i e^{-p_i t}+\sum_{k=1}^{N}d_k e^{-z_k w_k t}\sin(w_k t+q_k)$$

Some useful concepts: Residues and dipoles:

• Under the assumption 1, a pair of closely located closed-loop pole and zero is called a *dipole* and can be neglected (p.181).

• On the other hand, if a closed-loop pole is located very far from the imaginary axis, the transient associated with the pole lasts a short time and therefore may be neglected.

Example. A system's transfer function is

$$F(s) = \frac{(s + 1.01)}{(s + 1)(s + 2)}$$

By partial fraction expansion,

$$F(s) = \frac{(s+1.01)}{(s+1)(s+2)} = \frac{0.01}{(s+1)} + \frac{1-0.01}{(s+2)}$$

Taking the inverse Laplace transform yields

$$c(t) = 0.01e^{-t} + (1-0.01)e^{-2t}, \quad t \geq 0$$

Obviously, the contribution of the term $0.01e^{-t}$ to the response is small and therefore, can be neglected.

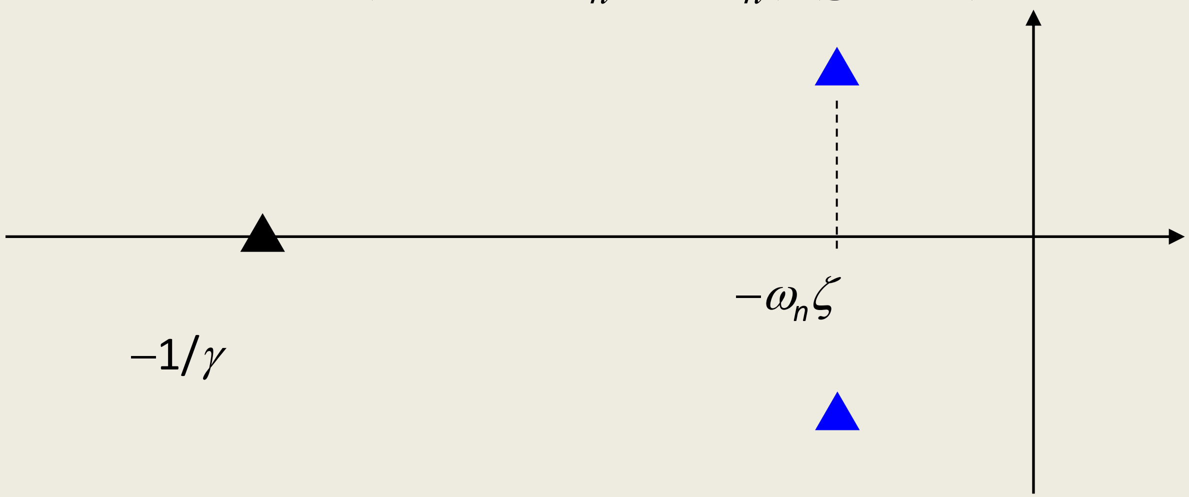
Example. A system's transfer function is

$$F(s) = \frac{1/5}{(s+1)(\frac{1}{5}s+1)} \approx \frac{1/5}{(s+1)}$$

2. Dominant poles

Example. Consider the following third-order closed-loop system:

$$\Phi(s) = \frac{w_n^2}{(s^2 + 2zw_n s + w_n^2)(gs+1)}$$



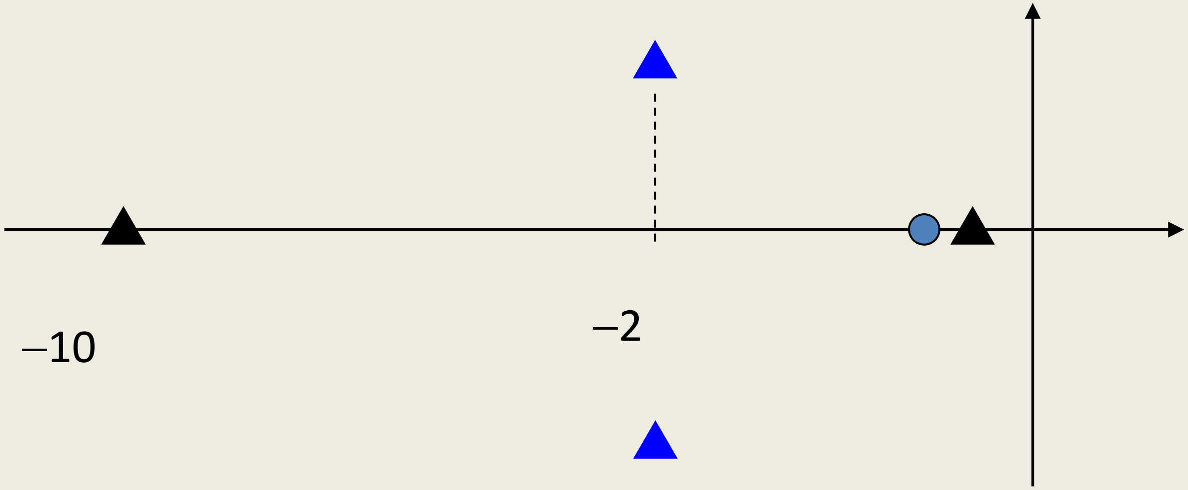
If the *real* parts satisfy $1/\gamma \geq 5\omega_n\zeta$ ($1/\gamma \geq 4\omega_n\zeta$), then the third-order system can be approximated by a second-order system

$$\Phi(s) \approx \frac{w_n^2}{(s^2 + 2zw_n s + w_n^2)}$$

whose two poles are called *dominant poles*.

In general, if the ratios of real parts exceed 5 (or 4) and there are no zeros nearby, then the closed-loop poles nearest the $j\omega$ axis will dominate in the transient response and are called *dominant poles*.

Example. The locations of closed-loop poles and zero of a higher order system are shown below. Determine its dominant poles.



The pair of zero-pole nearest the $j\omega$ axis is a dipole and therefore can be neglected. By the rule of determining the dominant poles introduced above, the system can be approximated by a second-order system with a pair of complex-conjugate dominant poles.

(1) Concept of stability

Definition: A signal $x(t)$ is said to be bounded if there is a positive real number M such that

$$|x(t)| \leq M$$

for all $t \in [0, \infty)$.

Definition: A system is said to be bounded-input-bounded-output (BIBO) stable if for each bounded input the corresponding output is bounded.

(2). Stability criterion

Theorem: An LTI system with *closed-loop* transfer function $G(s)$ is said to be stable if and only if $G(s)$ is proper and all its poles lie in the left-half s-plane (⇔*have negative real parts*).

Example. Consider the following three systems:

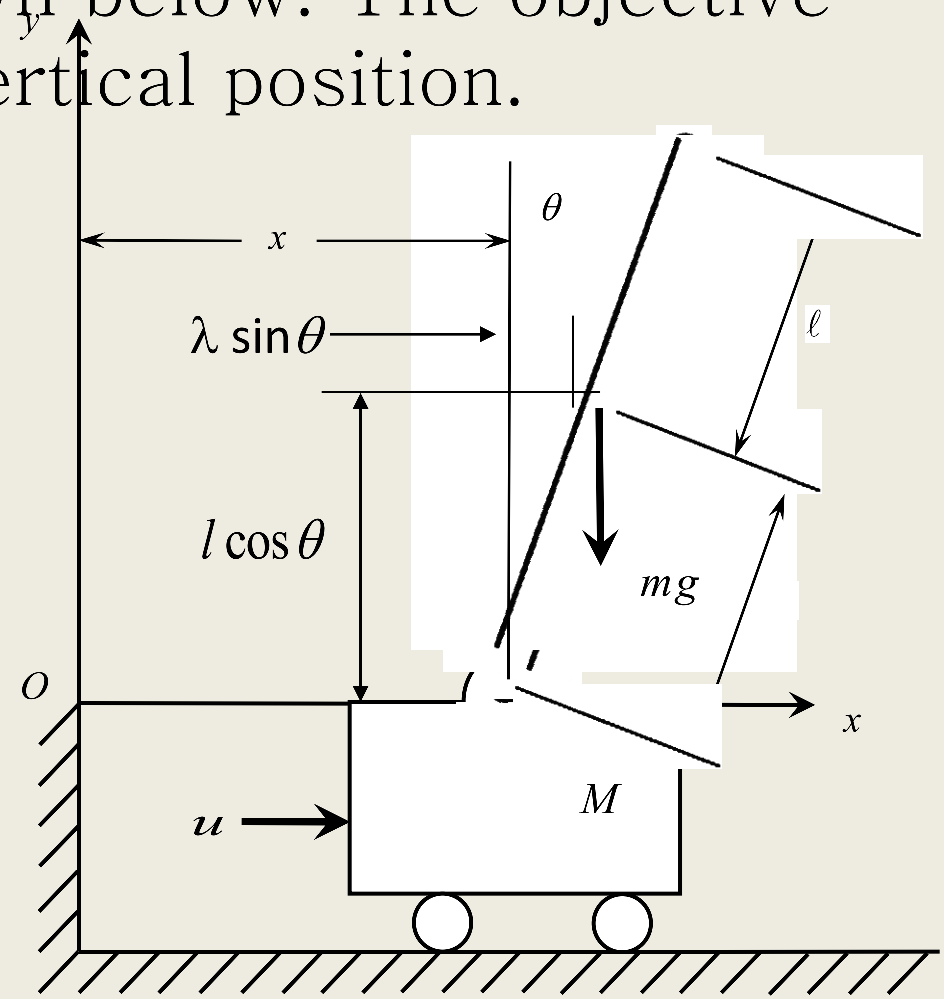
$$\Phi(s) = \frac{1}{(s+1)};$$

$$\Phi(s) = \frac{1}{(s-1)};$$

$$\Phi(s) = \frac{1}{(s^2+1)}.$$

Investigate their unit-step responses.

Example. An inverted pendulum mounted on a motor-drive cart is shown below. The objective is to keep the rod in a vertical position.

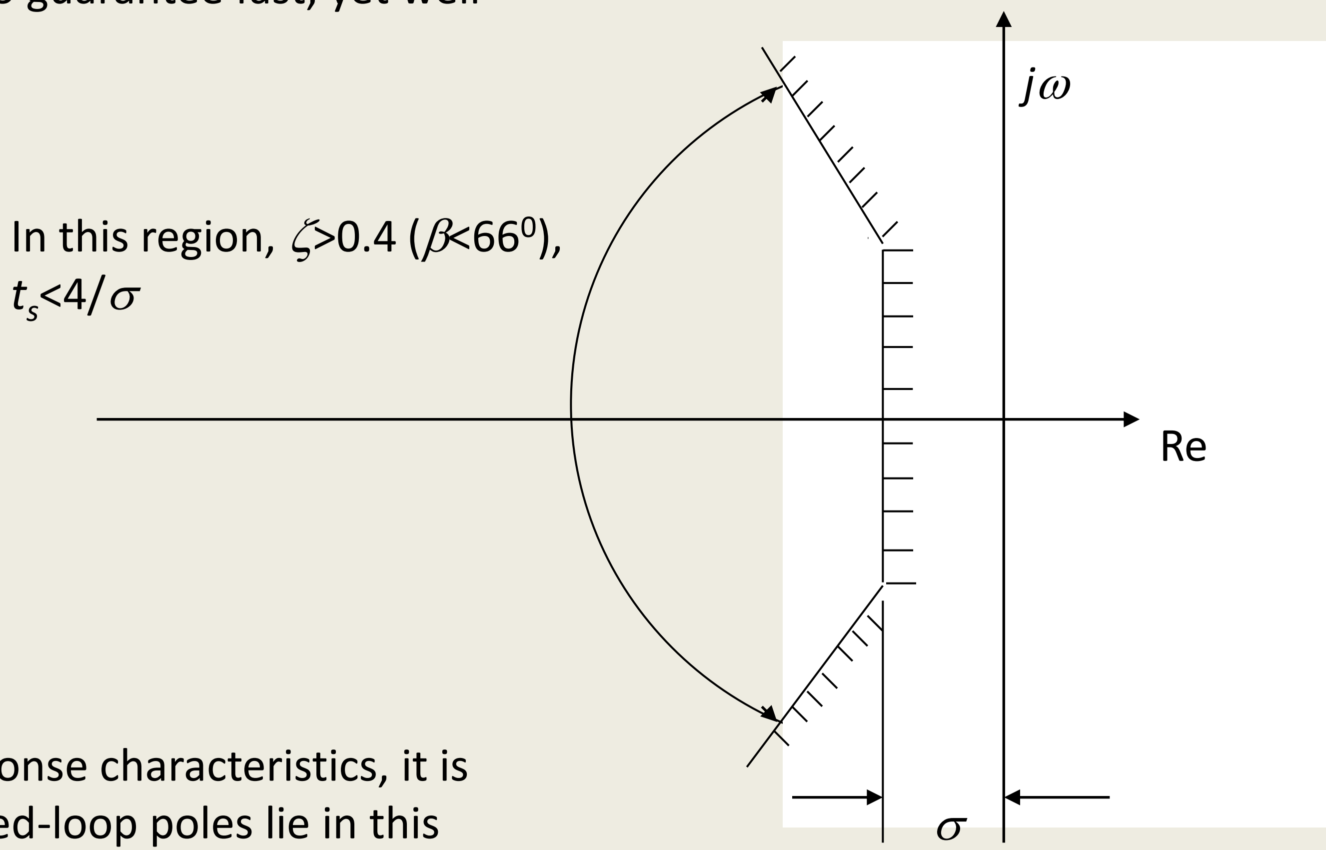


The system is unstable.

$$\frac{\Theta(s)}{U(s)} = \frac{lm}{\left[(M+m)mgl-(MI+mMl^2+mI)s^2\right]}$$

(3). Relative Stability

If dominant poles lie closed to $j\omega$ axis, the transient may exhibit excessively oscillation or very slow. Therefore, to guarantee fast, yet well



damped, transient response characteristics, it is necessary that the closed-loop poles lie in this region.

For example, if the two dominant poles are located in the region with β=45^0 as shown in the following figure. Then, ζ=0.707. The system possesses a satisfactory transient response.