Control System Engineering

SECTION B : MATHEMATICAL MODELING

Topic Covered : Concept of transfer function, relationship between transfer function and impulse response, order of a system, block diagram algebra, signal flow graphs : Mason s gain formula & its application, characteristic equation, derivation of transfer functions of electrical and electromechanical systems. Transfer functions of cascaded and nonloading cascaded elements. Introduction to state variable analysis and design.

TRANSFER FUNCTION

- The transfer function of a time-invariant, linear system is defined as the ratio of the Laplace transform of the output to the Laplace transform of the input where all the initial conditions are zero.

$$Transfer\ Function = G(s) = \left.\frac{Laplace\ Transform\ of\ output}{Laplace\ Transform\ of\ Input}\right|_{zero\ initial\ conditions}$$

1^{st} order system

Consider the impulse, step, ramp responses computed earlier. Identify the steady state and the transient parts.

Impulse response
Step response
Ramp response
Relationship between impulse, step and ramp
Relationship between impulse, step and ramp responses

$$G(s)=\frac{C(s)}{R(s)}=\frac{1/T}{s+1/T},\quad T>0$$

$$r(t)=\delta(t), R(s)=1,\quad c_{\delta}(t)=\frac{1}{T}e^{-t/T}1(t)$$

$$r(t)=1(t), R(s)=\frac{1}{s},\quad c_{step}(t)=\left[1-e^{-t/T}\right]1(t)$$

$$r(t)=t1(t), R(s)=\frac{1}{s^2},\quad c_{ramp}(t)=\left[t-T+Te^{-t/T}\right]1(t)$$

Compare steady-state part to input function, transient part to TF.

2nd order system $G(s)=\frac{C(s)}{R(s)}=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

Over damped

•(two real distinct roots = two 1st order systems with real poles)

Critically damped

•(a single pole of multiplicity two, highly unlikely, requires exact matching)

Underdamped

•(complex conjugate pair of poles, oscillatory behavior, most common)

step response

$$c_{step}(t)=K\left[1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t+\tan^{-1}\left(\sqrt{1-\zeta^2}/\zeta\right)\right)\right]1(t)$$

$$c_\delta(t)=K\left[\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_d t\right)\right]1(t)$$

2^{nd} Order System

Prototype parameters:

undamped natural frequency,

damping ratio

Relating problem specific parameters to prototype parameters

Transient vs Steady state

Consider the step, responses computed earlier. Identify the steady state and the transient parts.

2nd order system $G(s) = \frac{C(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

Over damped

•(two real distinct roots = two 1st order systems with real poles)

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$$c_{step}(t) = K\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\left(\sqrt{1-\zeta^2}/\zeta\right)\right)\right]1(t)$$

$$c_\delta(t) = K\left[\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_d t\right)\right]1(t)$$

Use of Prototypes

Too many examples to cover them all
We cover important prototypes
We develop intuition on the prototypes
We cover how to convert specific examples to prototypes
We transfer our insight, based on the study of the prototypes to the specific situations.

Transient-Response Spedifications

1. Delay time, t_d: The time required for the response to reach half the final value the very first time.
2. Rise time, t_r: the time required for the response to rise from 10% to 90% (common for overdamped and 1st order systems);
   5% to 95%;
   or 0% to 100% (common for underdamped systems);
   of its final value
1. Peak time, t_p:
2. Maximum (percent) overshoot, M_p:
3. Settling time, t_s

Derived relations for 2nd Order Systems

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

$$\sigma = \zeta \omega_n$$

$$t_r = \frac{\pi - \beta}{\omega_d} \qquad t_p = \frac{\pi}{\omega_d} \qquad \beta = \tan^{-1}\left(\frac{\omega_d}{\sigma}\right)$$

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

See book for details. (Pg. 232)

$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n} \quad 2\% \qquad t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n} \quad 5\%$$

Allowable M_p determines damping ratio.

Settling time then determines undamped natural frequency.

Theory is used to derive relationships between design specifications and prototype parameters.

Which are related to problem parameters.

Higher order system

PFEs have linear denominators.

•each term with a real pole has a time constant

•each complex conjugate pair of poles has a damping ratio and an undamped natural frequency.

Routh's Stability Criterion

How do we determine stability without finding all poles?

Actual poles provide more info than is needed.

All we need to know if any poles are in LHP.

Routh's stability criterion (Section 5-7).

$$q(s) = s^4 + 2s^3 + 3s^2 4s + 5$$

$$q(s) = s^3 + 2s^2 + s + 2$$

$$q(s) = s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50$$

What values of K produce a stable system?

$$G(s) = \frac{K}{s(s^2 + s + 1)(s + 2)}, \quad T(s) = \frac{G(s)}{1 + G(s)}$$

BLOCK DIAGRAM ALGEBRA

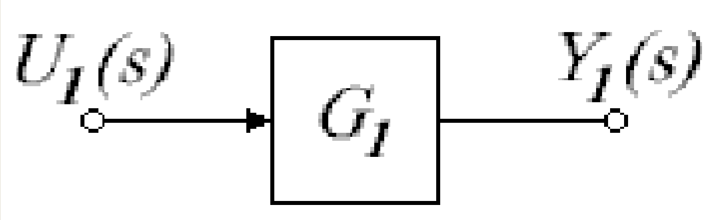
- *A graphical tool* can help us to ***visualize the model*** of a system and ***evaluate the mathematical relationships between their elements***, using their transfer functions.

- In many control systems, the system of equations can be written so that their components do not interact ***except by having the input of one part be the output of another part***.

- In these cases, it is very easy to draw a block diagram that represents the mathematical relationships in similar manner to that used for the component block diagram.

BLOCK DIAGRAM

- It represents the ***mathematical relationships*** between the elements of the system.



$$U_1(s)\,G_1(s) = Y_1(s)$$

- The ***transfer function*** of each component is placed ***in box***, and the ***input-output relationships*** between components are indicated by ***lines and arrows***.

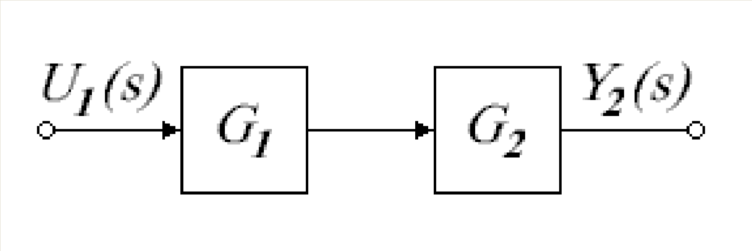
- Using block diagram, we can ***solve the equations by graphical simplification***, which is often easier and more informative than algebraic manipulation, even though the methods are in every way equivalent.

- It is convenient to think of ***each block as representing an electronic amplifier*** with the transfer function printed inside.

- The interconnections of blocks include summing points, where any number of signals may be added together.

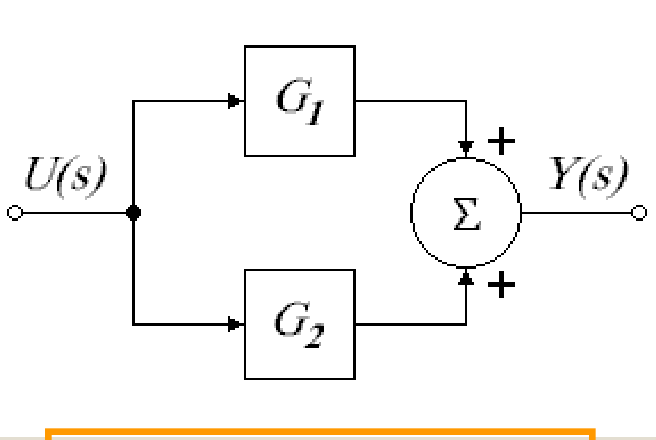
1st & 2nd Elementary Block Diagrams

- Block in series:



$$\frac{Y_2(s)}{U_1(s)} = G_1 G_2$$

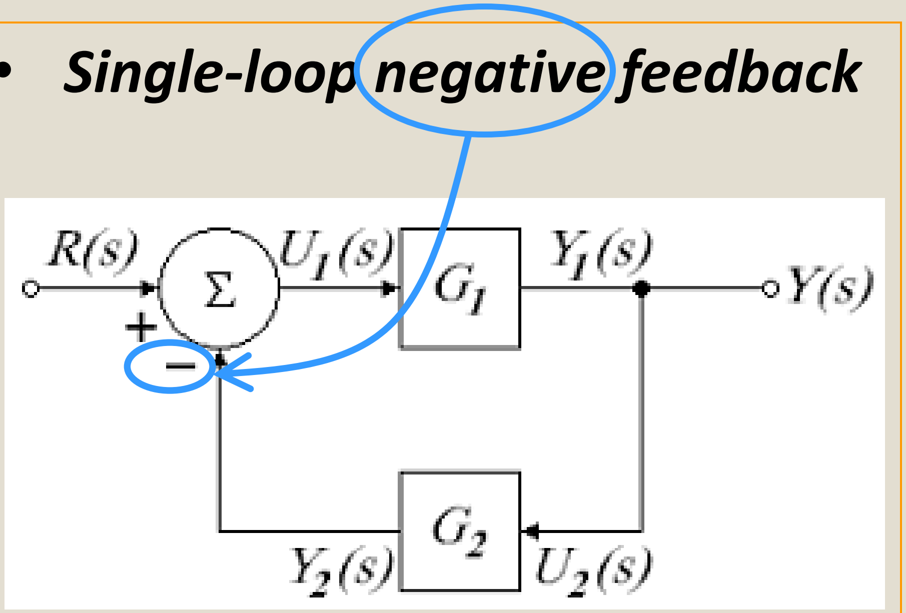
- Blocks in parallel with their outputs added:



$$\frac{Y_2(s)}{U_1(s)} = G_1 + G_2$$

3rd Elementary Block Diagram

- ***Single-loop negative feedback***

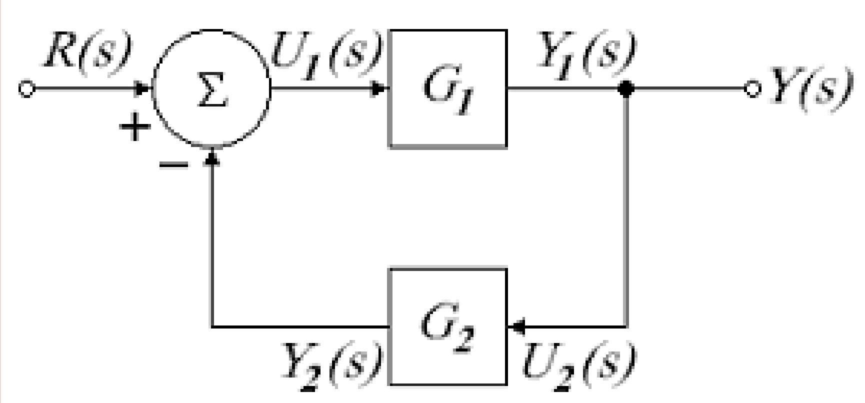


Two blocks are connected in a feedback arrangement so that each feeds into the other:

- The *overall transfer function* is given by:

$$\frac{Y(s)}{R(s)} = \frac{G_1}{1+G_1G_2}$$

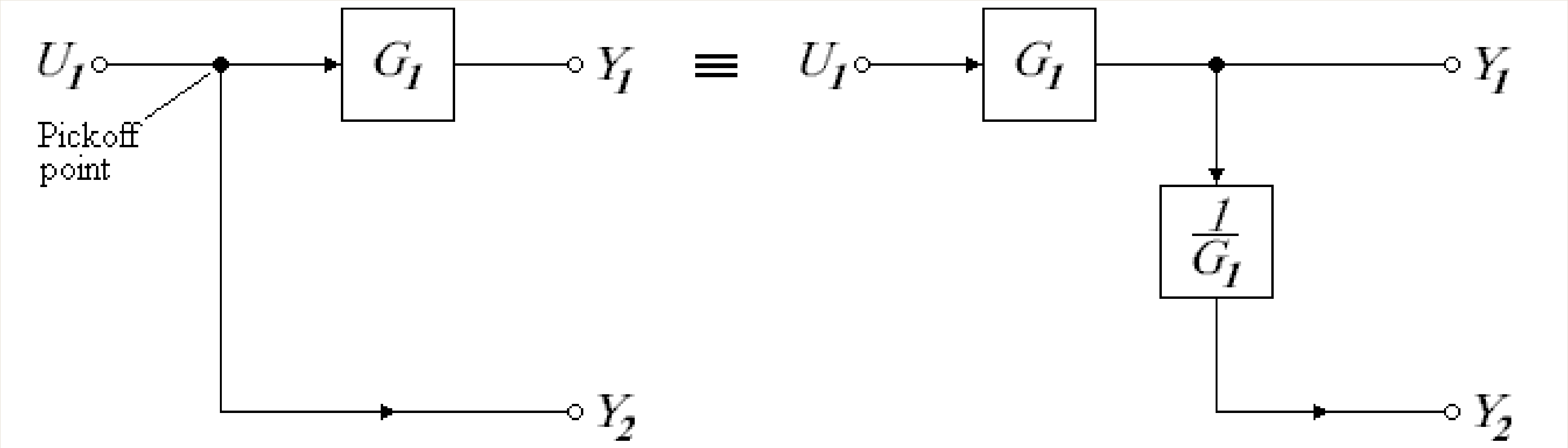
Feedback Rule



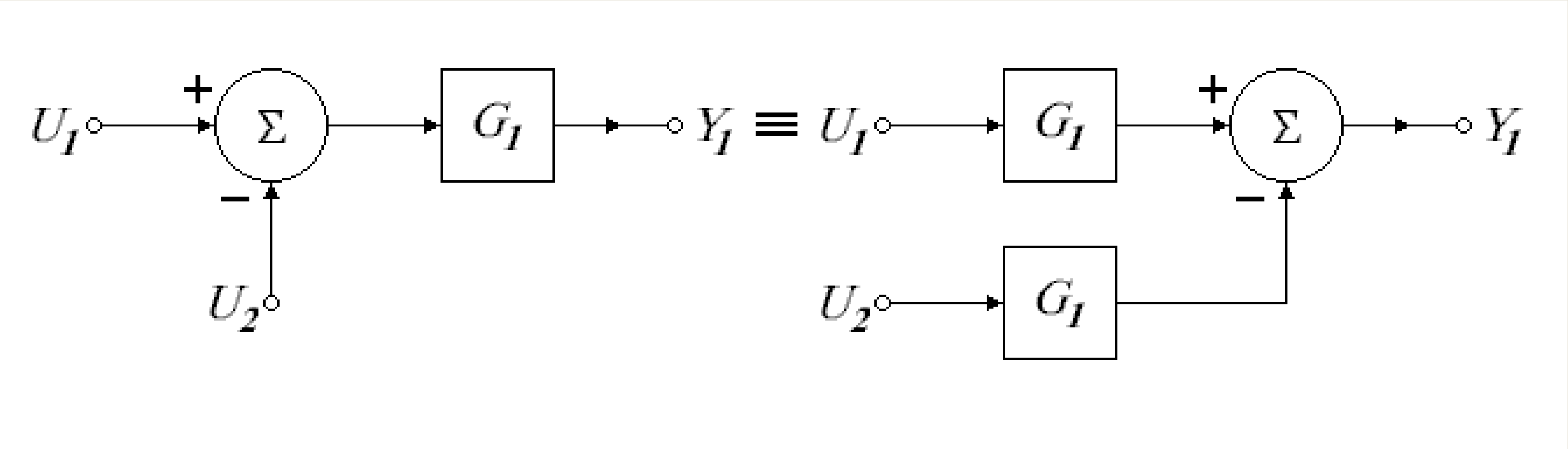
$$\frac{Y(s)}{R(s)} = \frac{G_1}{1 + G_1 G_2}$$

The gain of a single-loop negative feedback system is given by the forward gain divided by the sum of 1 plus the loop gain

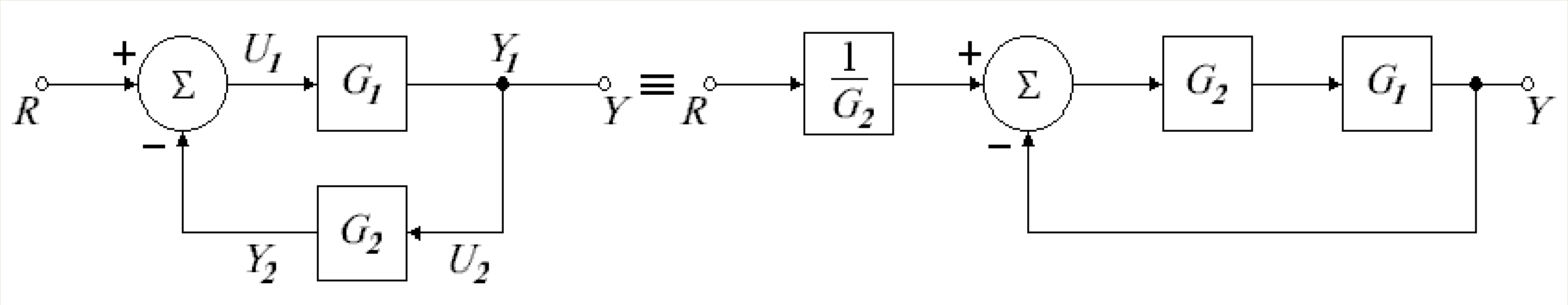
1st Elementary Principle of Block Diagram Algebra



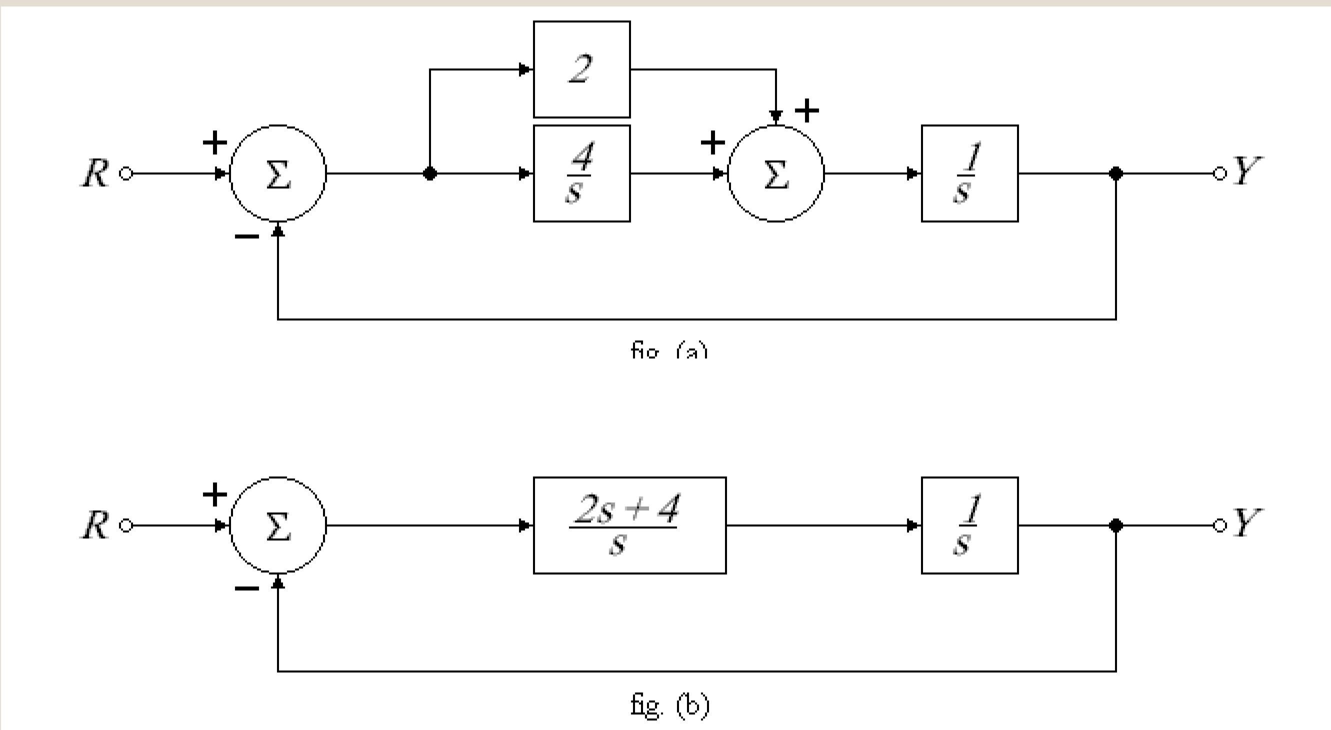
2nd Elementary Principle of Block Diagram Algebra



3rd Elementary Principle of Block Diagram Algebra



Example 1: Transfer function from a Simple Block Diagram

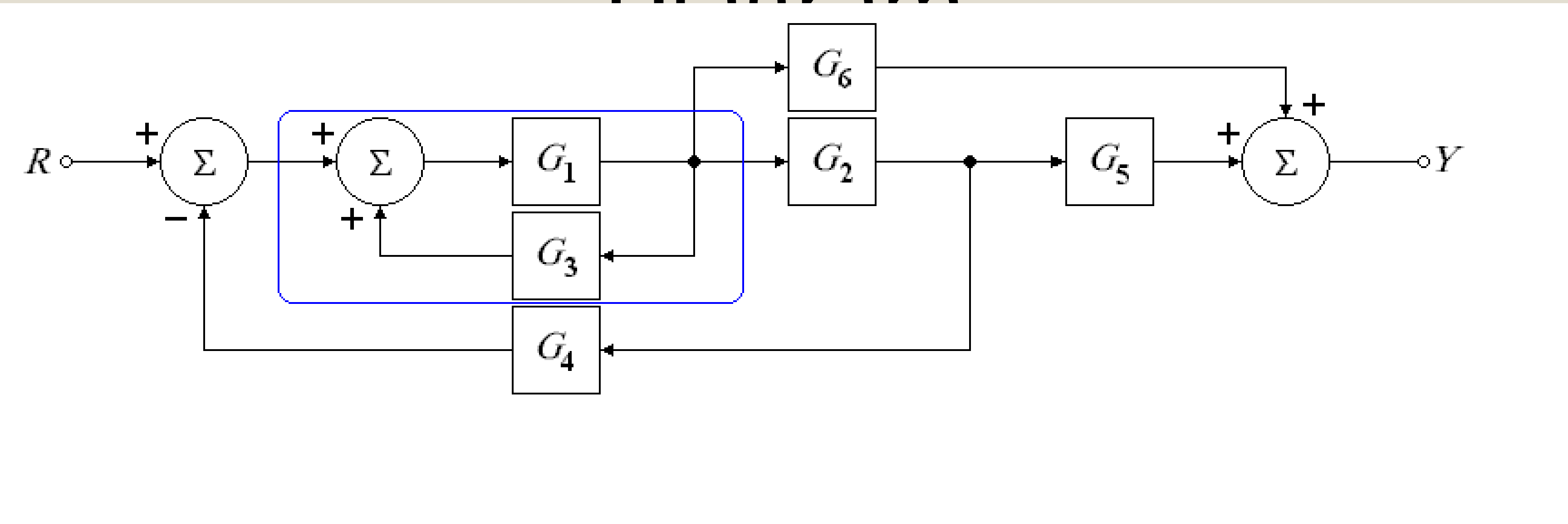


$$T(s) = \frac{Y(s)}{R(s)}$$

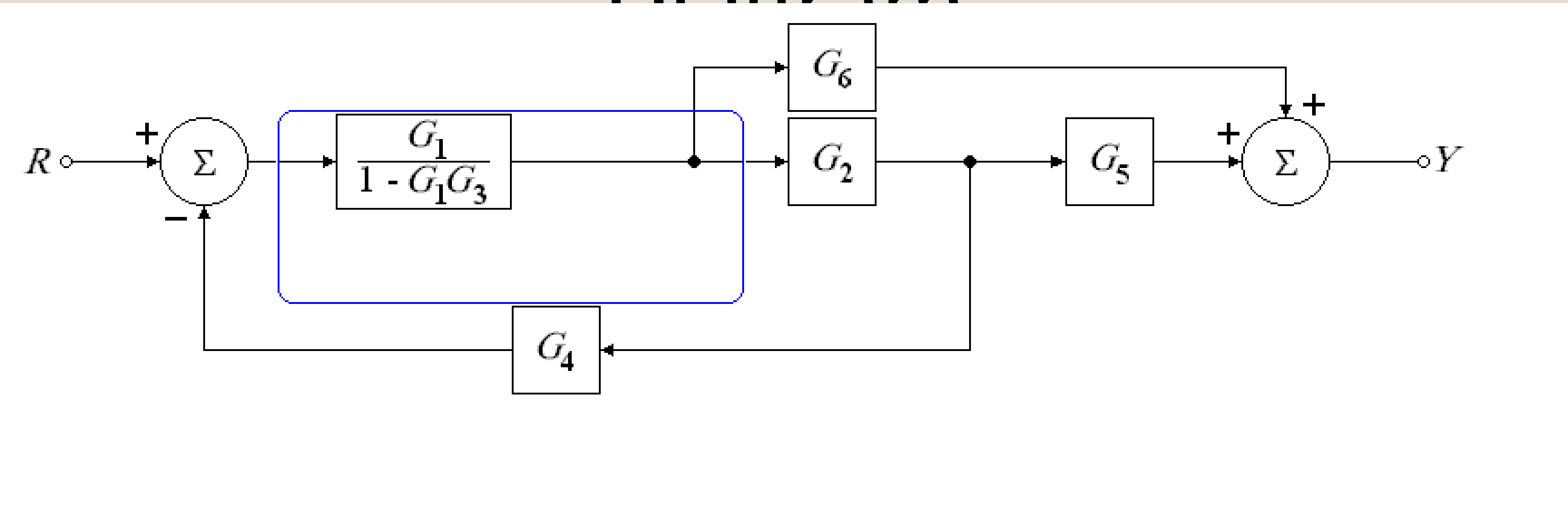
$$T(s) = \frac{\dfrac{2s+4}{s^2}}{1 + \dfrac{2s+4}{s^2}}$$

$$T(s) = \frac{2s+4}{s^2 + 2s + 4}$$

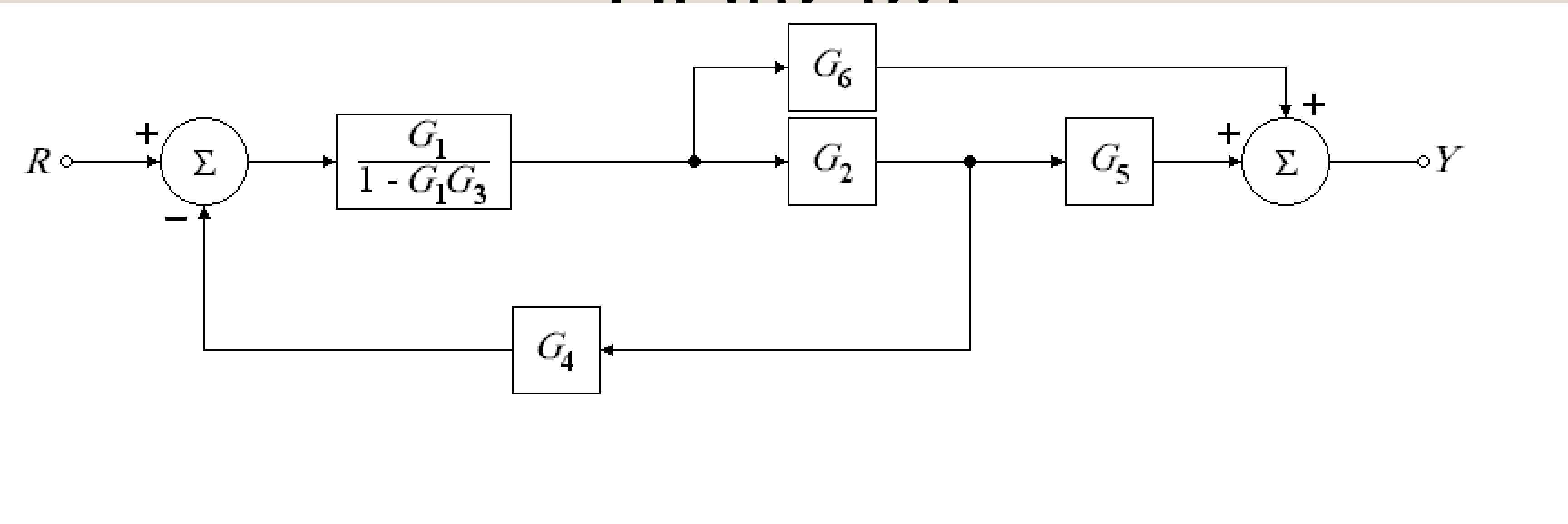
Example 2: TF from the Block Diagram



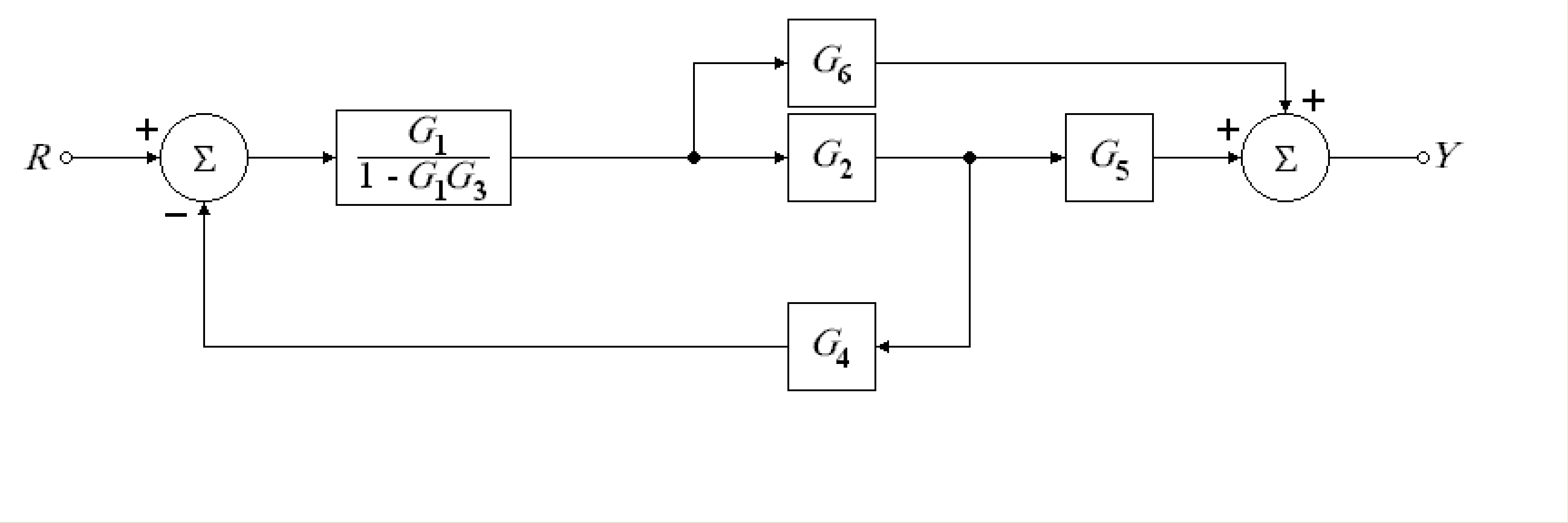
Example 2: TF from the Block Diagram



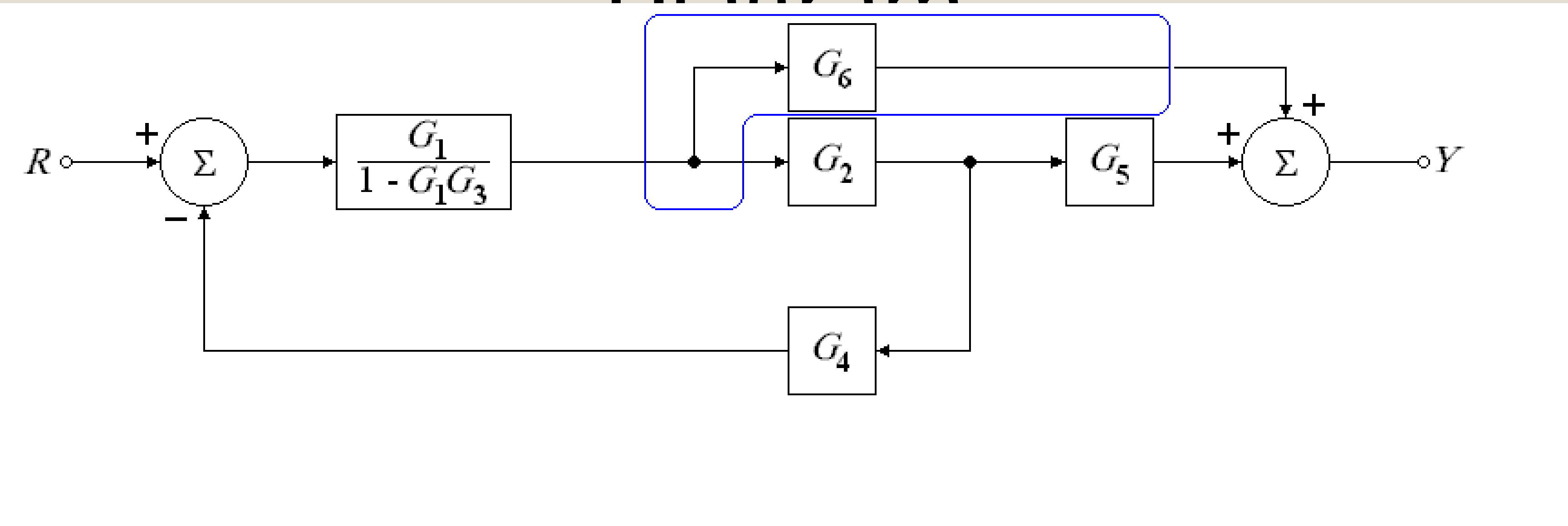
Example 2: TF from the Block Diagram



Example 2: TF from the Block Diagram



Example 2: TF from the Block Diagram



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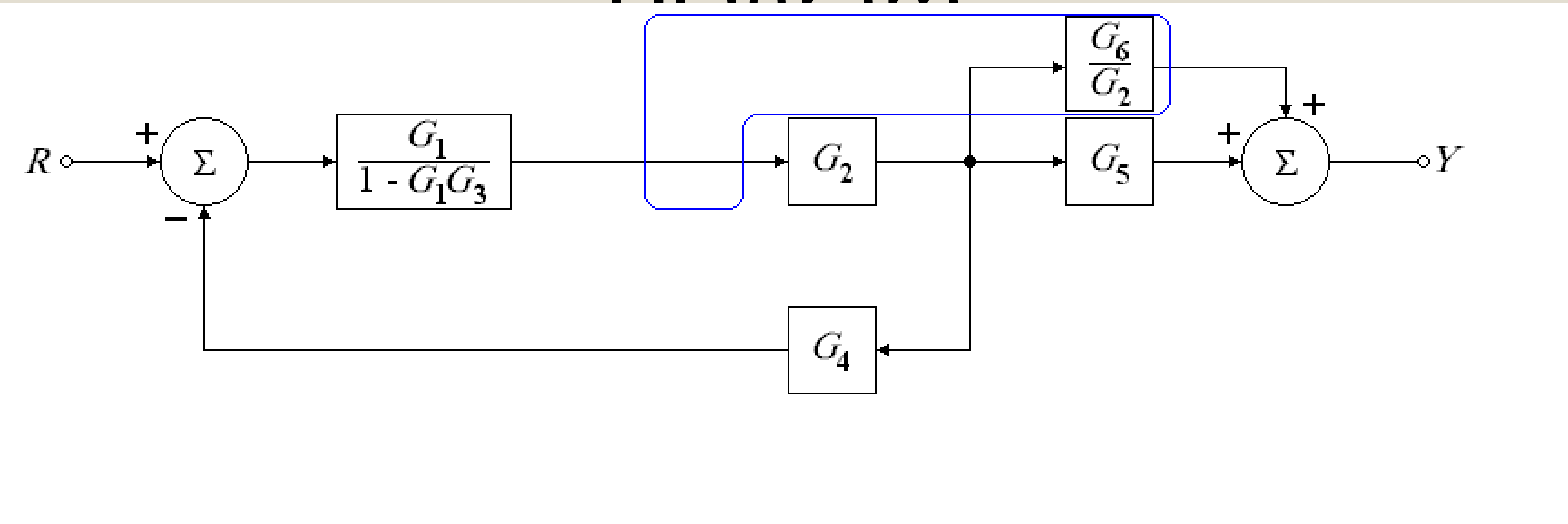
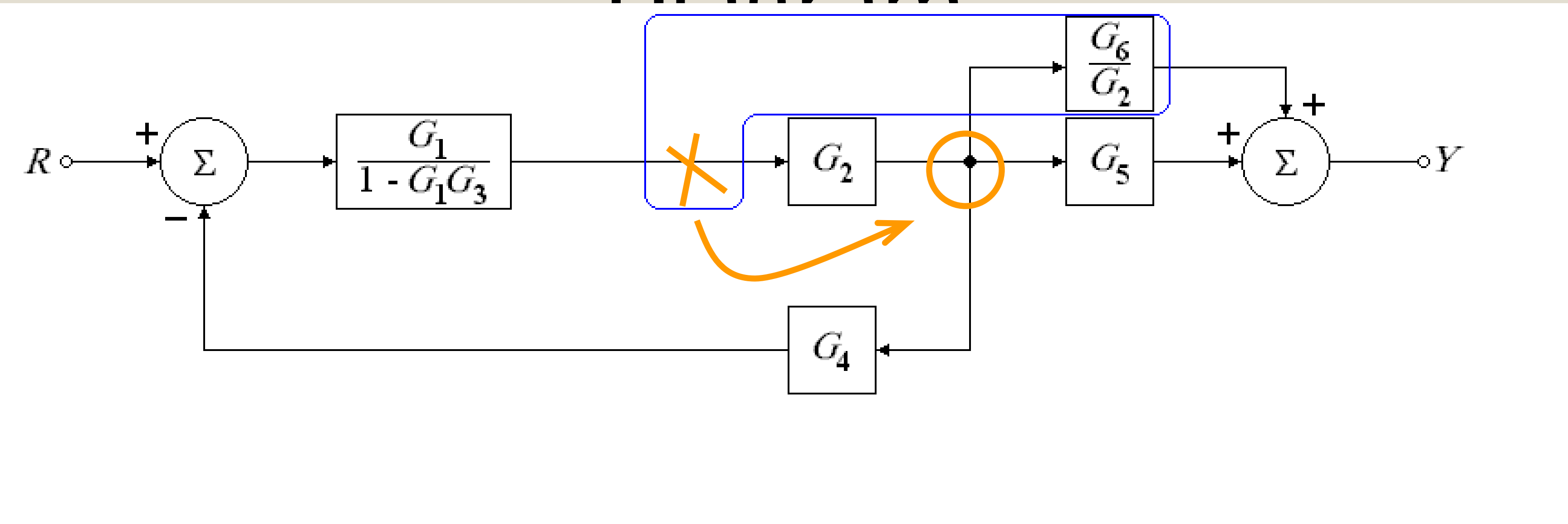
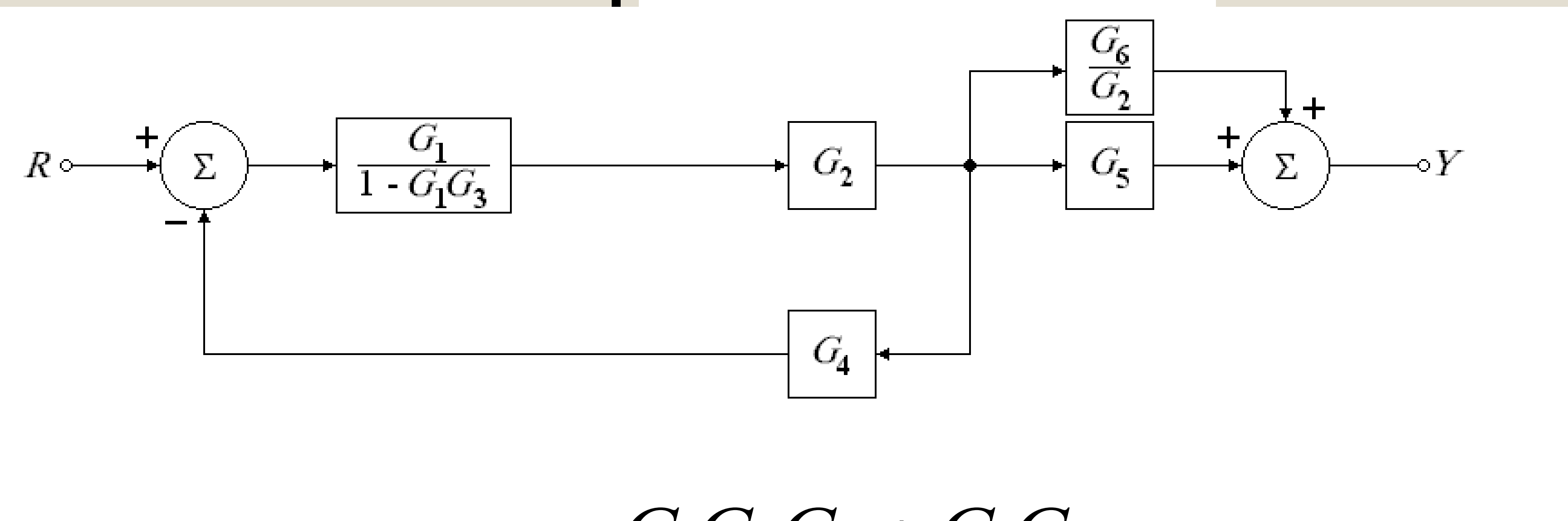


Example 2: TF from the Block Diagram

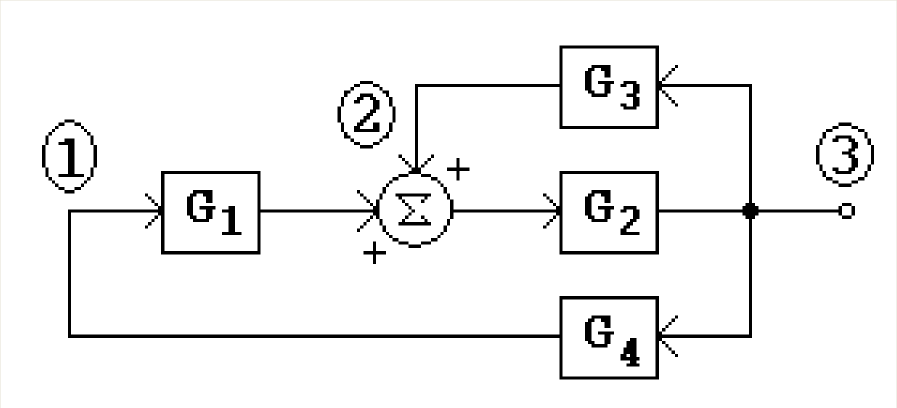


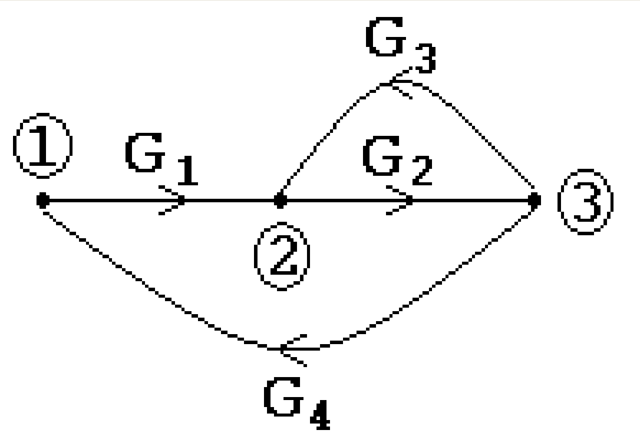
Example 2: TF from the Block Diagram



$$T(s) = \frac{G_1G_2G_5 + G_1G_6}{1 - G_1G_3 + G_1G_2G_4}$$

Block Diagram and its corresponding Signal Flow Graph





- Compact alternative ***notation to the block diagram***.
- It characterizes the system by a network of directed branches and associated transfer functions.
- The two ways of depicting signal are equivalent.

SIGNAL FLOW GRAPHS

- A signal flow graph is a pictorial representation of the simultaneous equation describing a system. Signal flow graph is applicable to the linear systems.
- The signal flow graph was introduced by S.J. Mason for the cause-and-effect representation of linear systems that are modeled by algebraic equations.

SIGNAL FLOW GRAPHS….

Let us consider the simple equation

$$y_j = A_{ij} y_i$$

- The variables y_i and y_j can be functions of time, complex frequency, or any other quantity.
- Consider a signal flow graph shown in Fig. 2.1. Here A_{ij} is a mathematical operator mapping y_j into y_i , and is called the transmission function.
- Note that the variable y_i and y_j are represented by a small dot called a node.

Definitions of SFG Terms

1. **Input Node (Source):** An input node is a node that has only outgoing branches. Node $y1$ in Fig. 2.2 is input node.
2. **Output Node (Sink):** An output node is a node that has only incoming branches and have no outgoing branches. Node $y5$ in Fig. 2.2 is output node.
3. **Branch:** The transmission function Aij is represented by a line with an arrow, called a branch.
4. **Path:** A path is a traversal of connected branches in the direction of the branch arrows. The path should not cross a node more than once. F o r E x am p le , $y1$ *to* $y2$ to $y3$ to $y4$ to $y5$
5. **Forward path:** A forward path is a path that starts at an input node and ends at an output node and no node is traversed more than once.
6. **Feedback loop or feedback path:** A loop is a path that originates and terminates on the same node and along which no other node is encountered more than once. For example, there are four loops in the SFG of Fig. 2.2. These are shown in Fig. 2.3.
7. **Path Gain:** The path gain is the product of the branch gains encountered in traversing a path. Path gain of path $y1$to $y2$ to $y3$ to $y4$ in Fig. 2.2 is $A12A23A34$.
8. **Forward Path Gain:** The forward path is the path gain of a forward path.
9. **Loop Gain:** The loop gain is the path gain of a loop. For example the loop gain of the loop $y3$ to $y4$ to $y3$ in Fig. 2.2 is $A34A43$.
10. **Nontouching loops:** If the loops does not have a common node then they are said to be nontouching loops. For example, the loop $y2$ to $y3$ to $y2$ and $y4$ to $y4$ of the SFG in Fig. 2.2 are nontouching loops.
11. **Self loop:** A self-loop is a feedback loop consisting of a single branch. For example $A44$ is a self-loop.

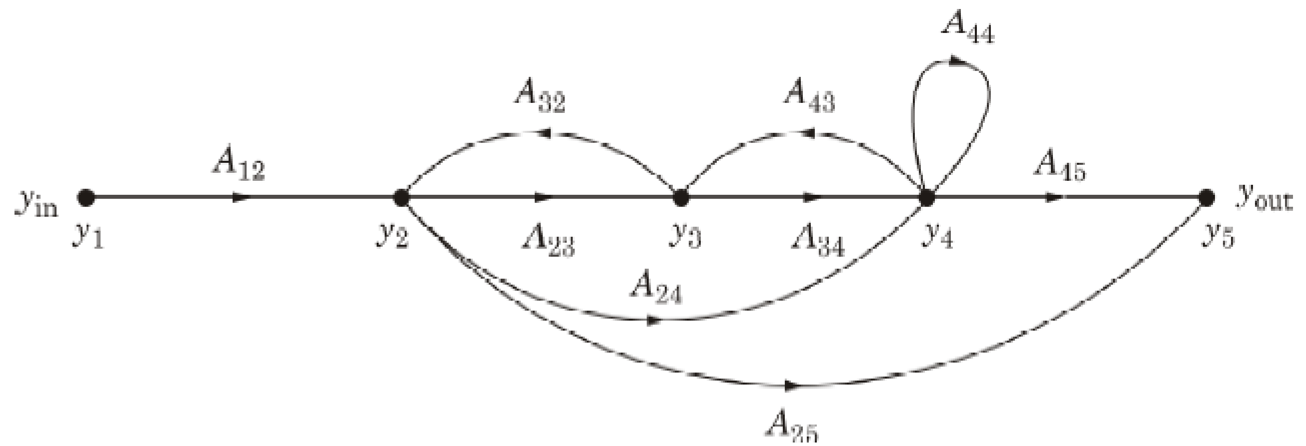


Fig. 2.2 Signal Flow Graph

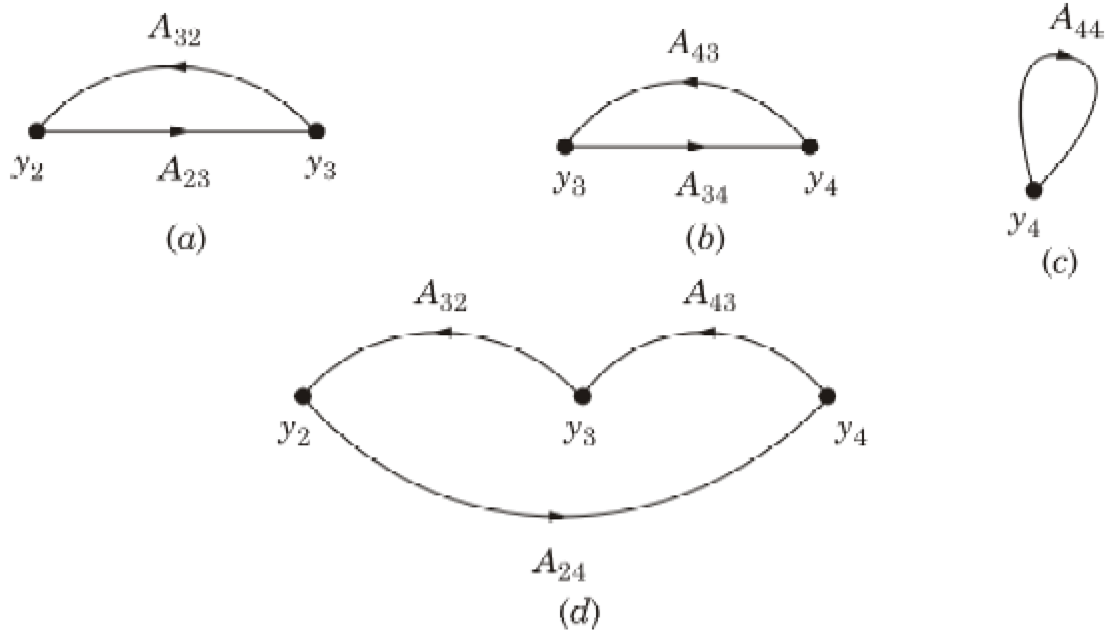


Fig. 2.3 Four Loops in the SFG of Fig. 2.2

Signal Flow Algebra

1. The Addition Rule: The value of variable designated by a node is equal to the sum of all signals entering the node.

$$y_i = \sum_{j=1}^{n} A_{ij} y_j$$

2. The Transmission Rule: The value of the variable designated by a node is transmitted on every branch leaving that node.

$$y_i = A_{ik} y_k \qquad i = 1, 2, \ldots, n \text{ and } k \text{ fixed}$$

The Multiplication Rule: A series (cascaded) connection of $n-1$ branches with transmission functions A_{21}, A_{32}, A_{43}......, $A_{n(n-1)}$ can be replaced by a single branch with a new transmission function branch to the product of the old ones. In other words, the equation

$$y_n = A_{21} \cdot A_{32} \cdot A_{43} \ldots A_{n(n-1)}\, y_1 \qquad \ldots(2.2.3)$$

MASON’S GAIN FORMULA

- Given a SFG or block diagram, the task of solving for the input-output relations by algebraic manipulation could be quite tedious.
- The Mason’s gain formula is used to determine the transfer function of the system from the signal flow graph of the system.
- Let us denote the ratio of the input variable to the output variable by T. For linear feedback control systems, $T = \frac{y_{out}}{y_{in}}$.

Given a SFG with N forward paths and L loops, the gain between the input node y_{in} and output node y_{out} is given by Mason's Gain formula:

$$T = \frac{y_{\text{out}}}{y_{\text{in}}} = \frac{\sum_{K=1}^{N} P_K \Delta_K}{\Delta} \qquad \ldots(2.3)$$

where y_{in} = input-node variable; y_{out} = output node variable

T = gain between y_{in} and y_{out}, $K = 1, 2 \ldots N$

N = total number of forward path between y_{in} and y_{out}.

P_K = gain of the K_{th} forward path between y_{in} and y_{out}.

Δ = 1 – (Sum of all loop gains) + (Sum of all gain products of two nontouching loops) – (Sum of all gain products of three non touching loops) + ..

Δ_K = Δ for that part of the SFG that is nontouching with the K_{th} forward path. Δ_K is called the path factor.

or

Δ_K = 1 – (Sum of all non-touching loop gains with K_{th} forward path) + (Sum of all gain products of two non-touching loops which do not touch the K_{th} forward path) – ...

Two loops, paths, or a loop and a path are said to be nontouching if they have no nodes in common.

Application of Mason's Gain Formula

Let y_{in} be an input and y_{out} be an output node of a SFG shown in Fig 2.2. The gain, y_{out}/y_2, where y_2 is not an input, may be written as

$$\frac{y_{out}}{y_2} = \frac{y_{out}/y_{in}}{y_2/y_{in}} = \frac{\Sigma P_K \Delta_K \mid \text{from } y_{in} \text{ to } y_{out}/\Delta}{\Sigma P_K \Delta_K \mid \text{from } y_{in} \text{ to } y_{out}/\Delta} \quad \ldots(2.4)$$

Since Δ is independent of the inputs and the outputs, Eq. 2.4 is written as

$$\frac{y_{out}}{y_2} = \frac{\Sigma P_K \Delta_K \mid \text{from } y_{in} \text{ to } y_{out}}{\Sigma P_K \Delta_K \mid \text{from } y_{in} \text{ to } y_{out}} \quad \ldots(2.5)$$

Characteristic Equations

- Characteristic equations can be defined with respect to differential equations, transfer functions,or state equations.

Characteristic equation from a differential equation: Consider that a linear time invariant system is described by the following equation

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y(t)}{dt^{n-1}} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t)$$

$$= b_m \frac{d^m u(t)}{dt^m} + b_{m-1}\frac{d^{m-1} u(t)}{dt^{m-1}} + \dots + b_1 \frac{du(t)}{dt} + b_0 u(t) \quad \dots(3.53)$$

where $n > m$.

After taking Laplace transform of above eqn. (3.53) we get

$$(s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0)\ Y(s) = (b_m s^m + b_{m-1}s^{m-1} + \dots + b_1 s + b_0)\ U(s) \quad \dots(3.54)$$

The characteristic equation of the system is defined as

$$s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0 = 0$$

which is obtained by setting the homogeneous part of eqn. 3.54 to zero.

Characteristic equation from a transfer function: The transfer function of the system described by eqn. (3.53) is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} \quad \ldots(3.55)$$

The characteristic equation is obtained by equating the denominator polynomial of the transfer function to zero.

Characteristic equation from state equations: From the state variable approach, we can write eqn. (3.51) as

$$G(s) = C\frac{\text{Adj}\,(sI - A)}{|sI - A|}B + D$$

setting the denominator of the transfer function matrix $G(s)$ to zero, we get the characteristic equation

$$|\,sI - A\,| = 0$$

The roots of the characteristic equation are often referred to as the eigen values of the matrix A. Some of the important properties of eigenvalues are given as follows:

1. If the coefficients of A are all real, its eigen values are either real or in complex conjugate pairs.
2. If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigen values of A, then

$$tr(A) = \sum_{i=1}^{n} \lambda_i$$

   That is, the trace of A, which is the sum of all the eigen values of A.
3. If λ_i, $i = 1, 2, ..., n$, is an eigen value of A, then it is an eigen value of A'.
4. If A is nonsingular, with eigen values λ_i, $i = 1, 2, ... n$, then $1/\lambda_i$ are the eigen values of A^{-1}.

Eigen vectors: Any nonzero vector P_i that satisfies the matrix equation

$$(\lambda_i I - A)P_i = 0 \qquad ...(3.56)$$

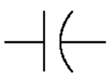
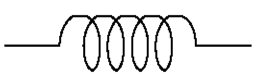
where λ_i, $i = 1, 2, ... n$, denotes the i_{th} eigen values of A which is called the eigen vector of A associated with the eigen value λ_i. If A has distinct eigen values, the eigen vectors can be solved from eqn. (3.56).

Electric Network Transfer Function

- In this section, we formally apply the transfer function to the mathematical modeling of electric circuits including passive networks
- Equivalent circuits for the electric networks that we work with first consist of three passive linear components: resistors, capacitors, and inductors.“
- We now combine electrical components into circuits, decide on the input and output, and find the transfer function. Our guiding principles are Kirchhoff s laws.

Electric Network Transfer Function

Table 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

 Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ = V (volts), $i(t)$ = A (amps), $q(t)$ = Q (coulombs), C = F (farads), R = Ω (ohms), G = ℧ (mhos), L = H (henries).

2.8 Electromechanical System Transfer Functions

- Now, we move to systems that are hybrids of electrical and mechanical variables, the *electromechanical systems*.

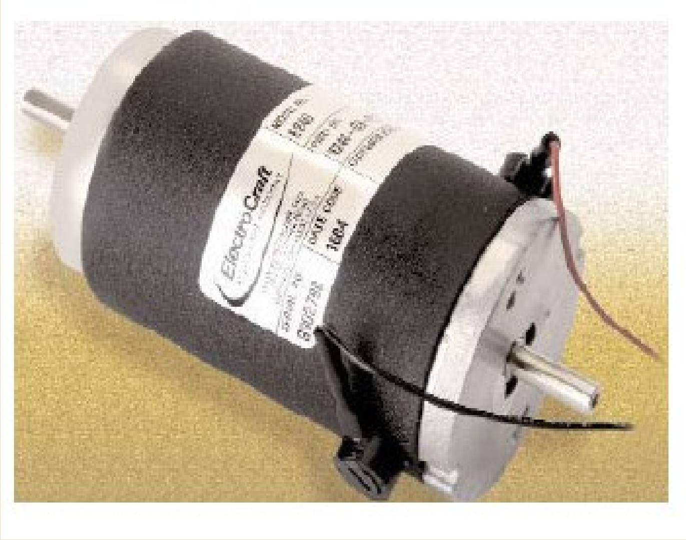
- A motor is an electromechanical component that yields a displacement output for a voltage input, that is, a mechanical output generated by an electrical input.

- We will derive the transfer function for one particular kind of electromechanical system, the armature-controlled dc servomotor.
- Dc motors are extensively used in control systems

Modeling – Electromechanical Systems

What is DC motor?

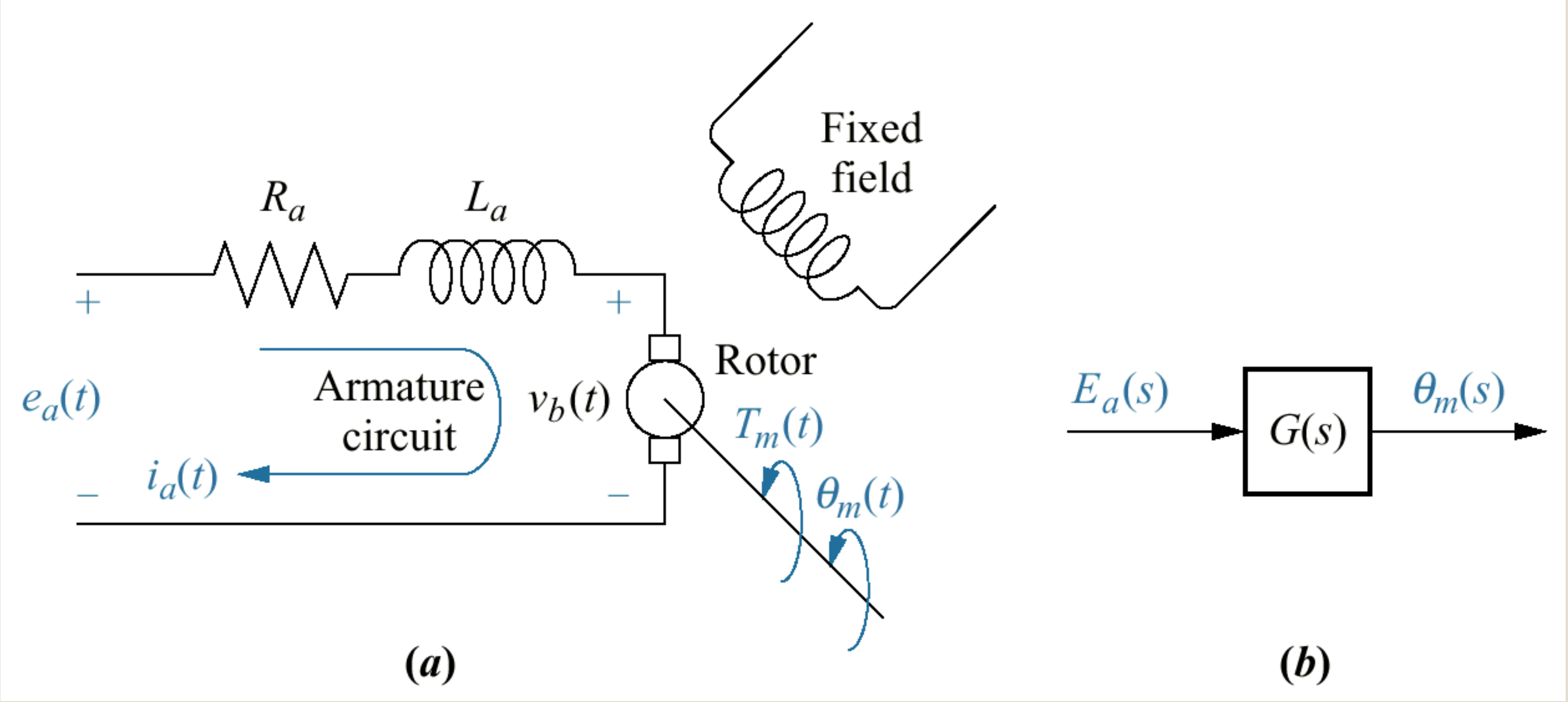
An actuator, converting electrical energy into rotational mechanical energy



Modeling – Why DC motor?

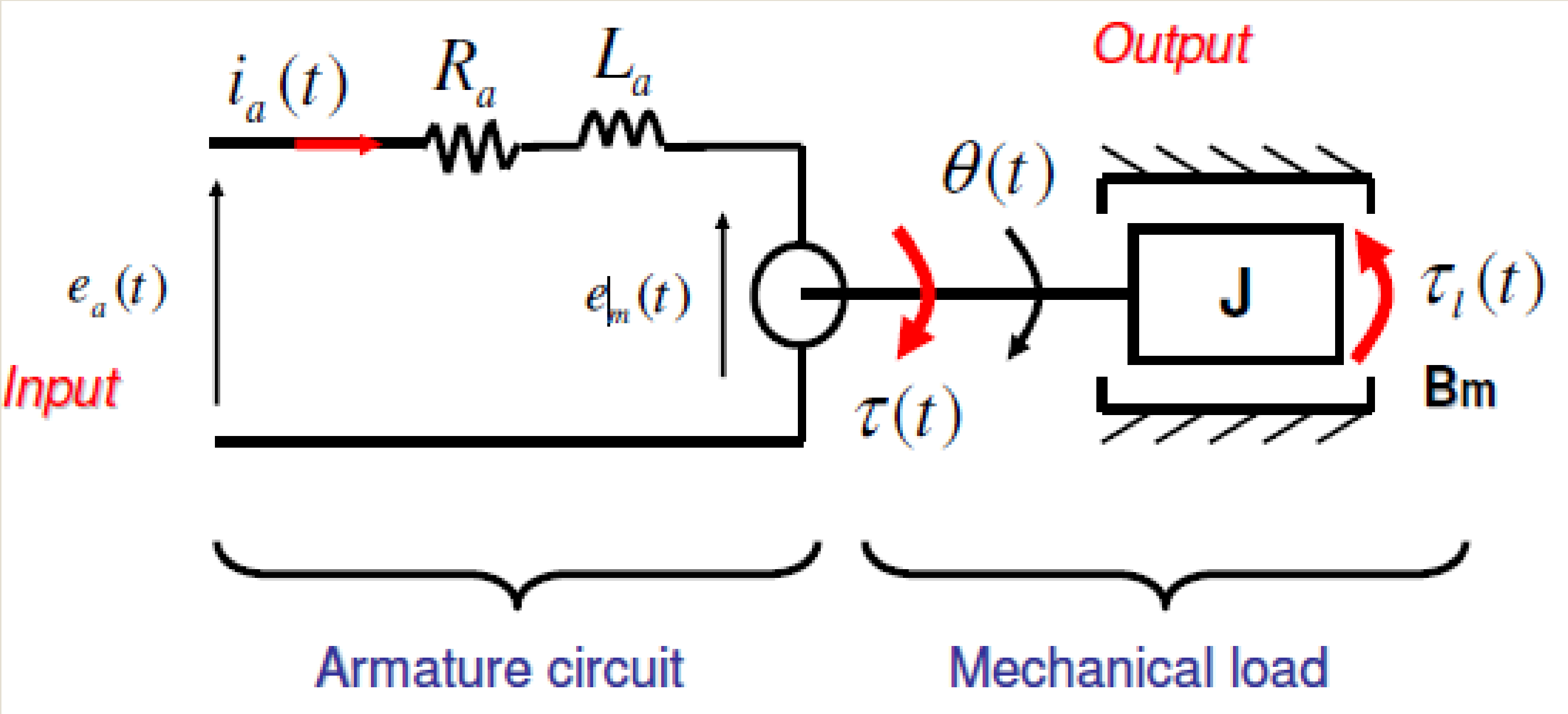
- Advantages:
  - high torque
  - speed controllability
  - portability, etc.
- Widely used in control applications: robot, tape drives, printers, machine tool industries, radar tracking system, etc.
- Used for moving loads when
  - Rapid (microseconds) response is not required
  - Relatively low power is required

DC Motor



Fixed field
R_a
L_a
+
+
Rotor
$e_a(t)$
Armature circuit
$v_b(t)$
$T_m(t)$
$i_a(t)$
–
–
$\theta_m(t)$
(a)
$E_a(s)$
$G(s)$
$\theta_m(s)$
(b)

Modeling – Model of DC Motor



Dc Motor

$i_a(t)$ = armature current R_a = armature resistance

$E_i(t)$ = back emf $T_L(t)$ = load torque

$T_m(t)$ = motor torque $\theta_m(t)$ = rotor displacement

K_i — torque constant L_a = armature inductance

$e_a(t)$ = applied voltage K_b = back-emf constant

ω_m magnetic flux in the air gap $\theta_m(t)$ — rotor angular velocity

J_m = rotor inertia B_m = viscous-friction coefficient

The Mathematical Model Of Dc Motor

The relationship between the armature current, *ia(t),* the applied armature voltage, *ea(t),* and the back emf, *vb(t),* is found by writing a loop equation around the Laplace transformed armature circuit

$$R_a I_a(s) + L_a s I_a(s) + V_b(s) = E_a(s)$$

The torque developed by the motor is proportional to the armature current; thus

$$T_m(s) = K_t I_a(s)$$

where *Tm* is the torque developed by the motor, and *Kt* is a constant of proportionality, called the motor torque constant, which depends on the motor and magnetic field characteristics.

The Mathematical Model Of Dc Motor

Mechanical System

$$T_m(s) = (J_m s^2 + B_m s)\theta_m(s)$$

Since the current-carrying armature is rotating in a magnetic field, its voltage is proportional to speed. Thus,

$$v_b(t) = K_b \frac{d\theta_m(t)}{dt}$$

Taking Laplace Transform

$$V_b(s) = K_b s \theta_m(s)$$

The Mathematical Model Of Dc Motor

We have

Electrical System

$$R_a I_a(s) + L_a s I_a(s) + V_b(s) = E_a(s)$$

$$V_b(s) = K_b s \theta_m(s)$$

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Mechanical System

$$T_m(s) = (J_m s^2 + B_m s)\theta_m(s)$$

$$T_m(s) = K_t I_a(s)$$

The Mathematical Model Of Dc Motor

To find T.F

$$\frac{(R_a + L_a s)T_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s)$$

$$\frac{(R_a + L_a s)(J_m s^2 + B_m s)\theta_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s)$$

If we assume that the armature inductance, *La,* is small compared to the armature resistance, *Ra,* which is usual for a dc motor, above Eq. Becomes

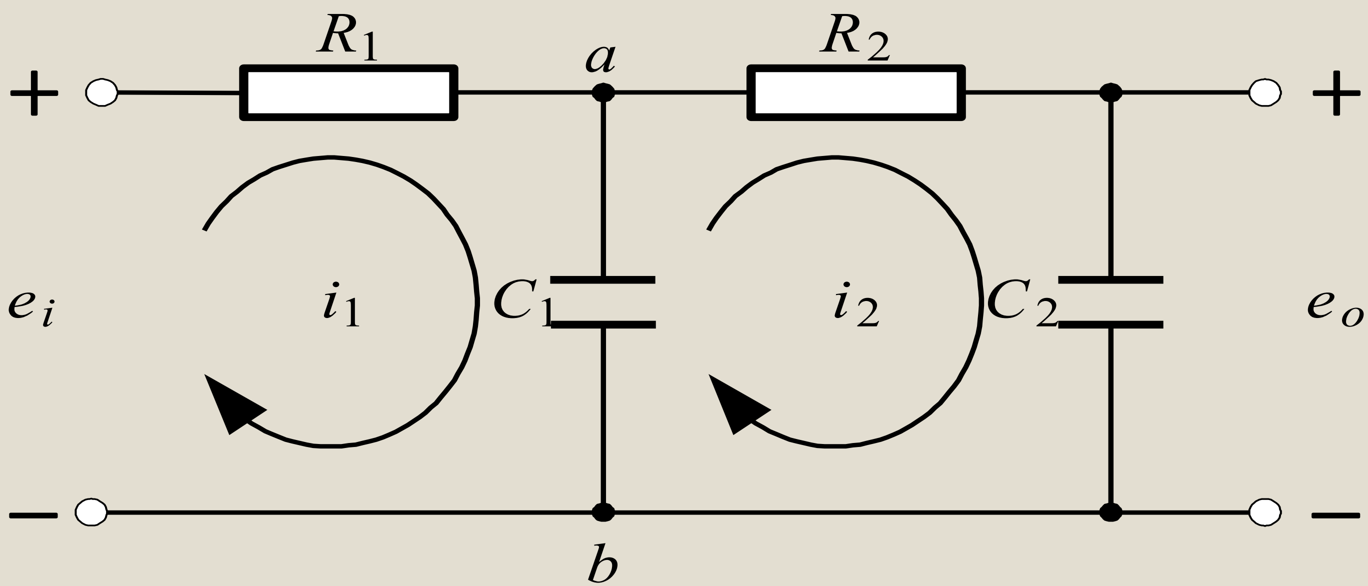
$$\left[\frac{R_a}{K_t}(J_m s + B_m) + K_b\right]s\theta_m(s) = E_a(s)$$

the desired transfer function of DC Motor:

$$\frac{\theta_m(s)}{E_a(s)} = \frac{K_t/(R_a J_m)}{s\left[s + \frac{1}{J_m}\left(B_m + \frac{K_t K_b}{R_a}\right)\right]}$$

2. Transfer functions of cascaded elements

Example. Find $E_o(s)/E_i(s)$.



Note that in this circuit, the second portion (R_2C_2) produces a *loading effect* on the first stage (R_1C_1 portion); that is , we cannot obtain the transfer function as we did for transfer functions in cascade.

$$\begin{cases} e_i = i_1 R_1 + u_{c_1} \\ C_1 \dfrac{du_{c_1}}{dt} = i_1 - i_2 \Rightarrow u_{c_1} = \dfrac{1}{C_1}\int (i_1 - i_2)dt \end{cases}$$

and

$$\begin{cases} e_o + i_2 R_2 = \underbrace{\dfrac{1}{C_1}\int (i_1 - i_2)dt}_{u_{c1}} \\ e_o = \dfrac{1}{C_2}\int i_2 dt \end{cases}$$

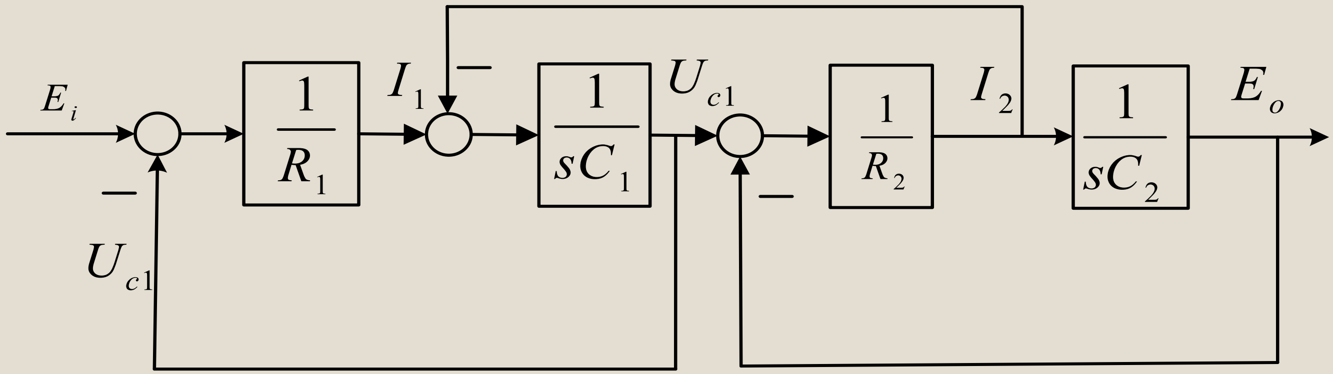
Taking the Laplace transforms of the above equations, we obtain

$$I_1 = \frac{1}{R_1}[E_i - U_{c1}]$$

$$U_{c1} = \frac{1}{C_1 s}[I_1(s) - I_2(s)]$$

$$I_2(s) = \frac{1}{R_2}[U_{c1} - E_o(s)]$$

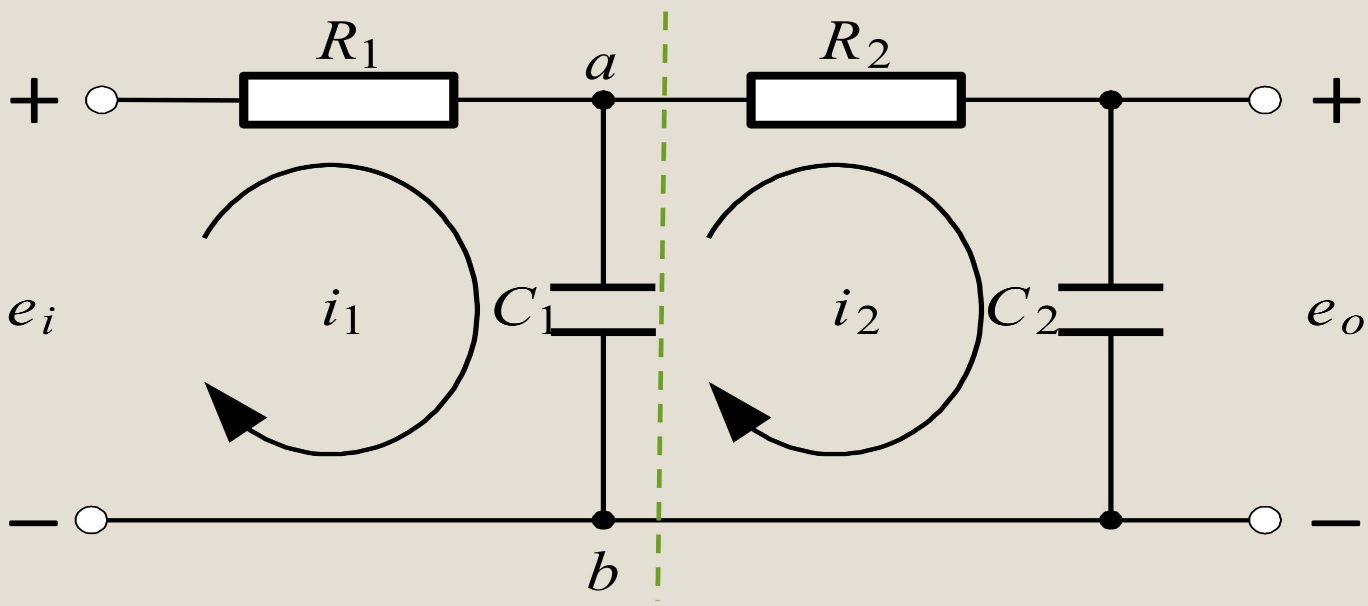
$$E_0(s) = \frac{1}{C_2 s} I_2(s)$$



from which we obtain

$$\frac{E_0(s)}{E_i(s)} = \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2)s + 1}$$

The above analysis shows that, if two *RC* circuits connected in cascade so that the output from the first circuit is the input to the second, the overall transfer function is not the product of $1/(R_1C_1s+1)$ and $1/(R_2C_2s+1)$ due to the loading effect (a certain amount of power is withdrawn).



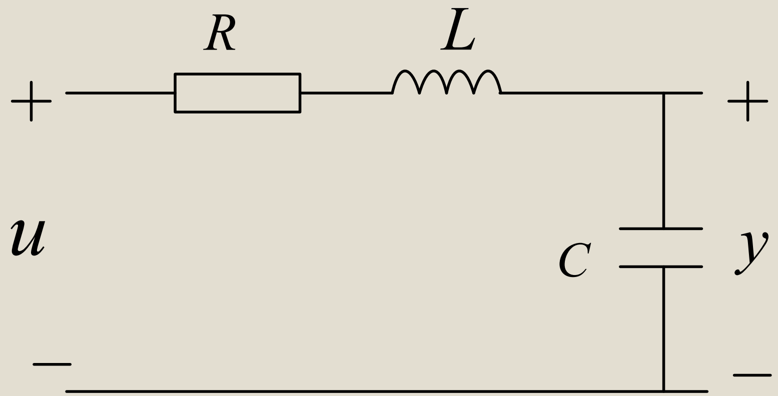
3. Complex impedances

Resistance *R*: *R*

Capacitance: 1/*Cs*

Inductance: *Ls*

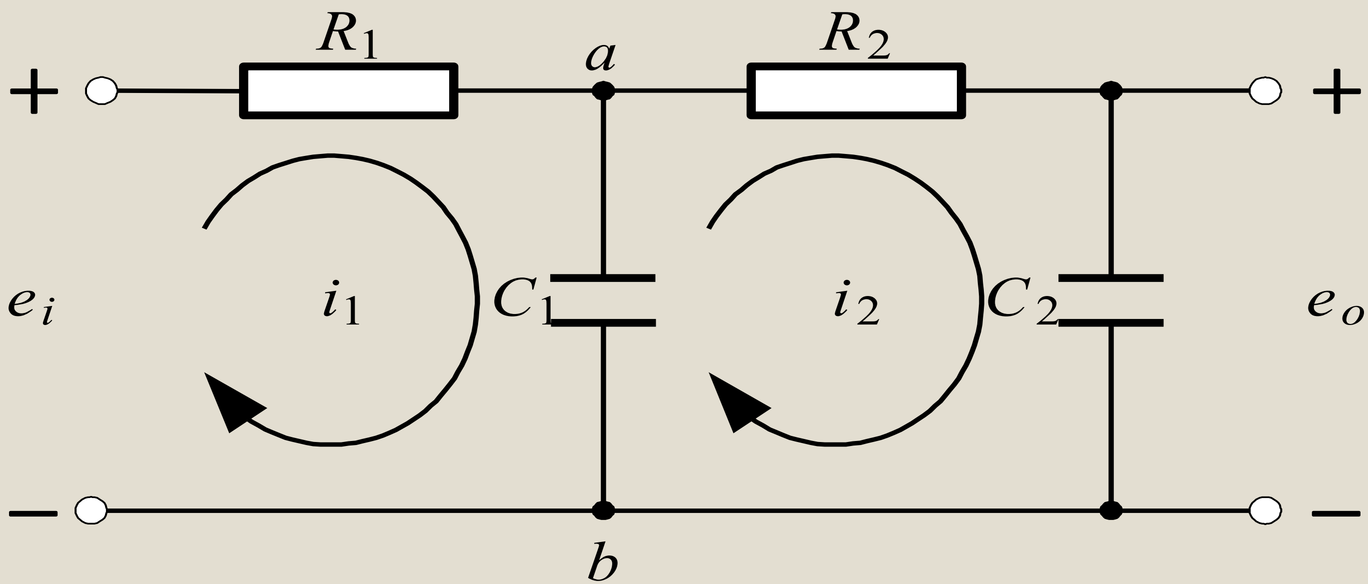
Example. Find *Y*(*s*)/*U*(*s*)



Solution:

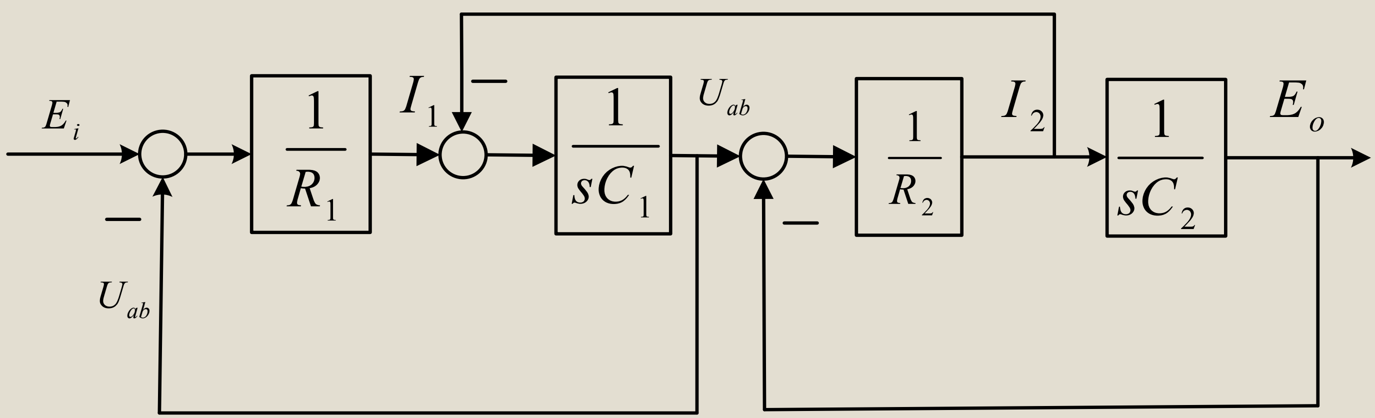
$$\frac{Y(s)}{U(s)} = \frac{1/Cs}{R + Ls + 1/Cs}$$

Example. Find $E_o(s)/E_i(s)$.

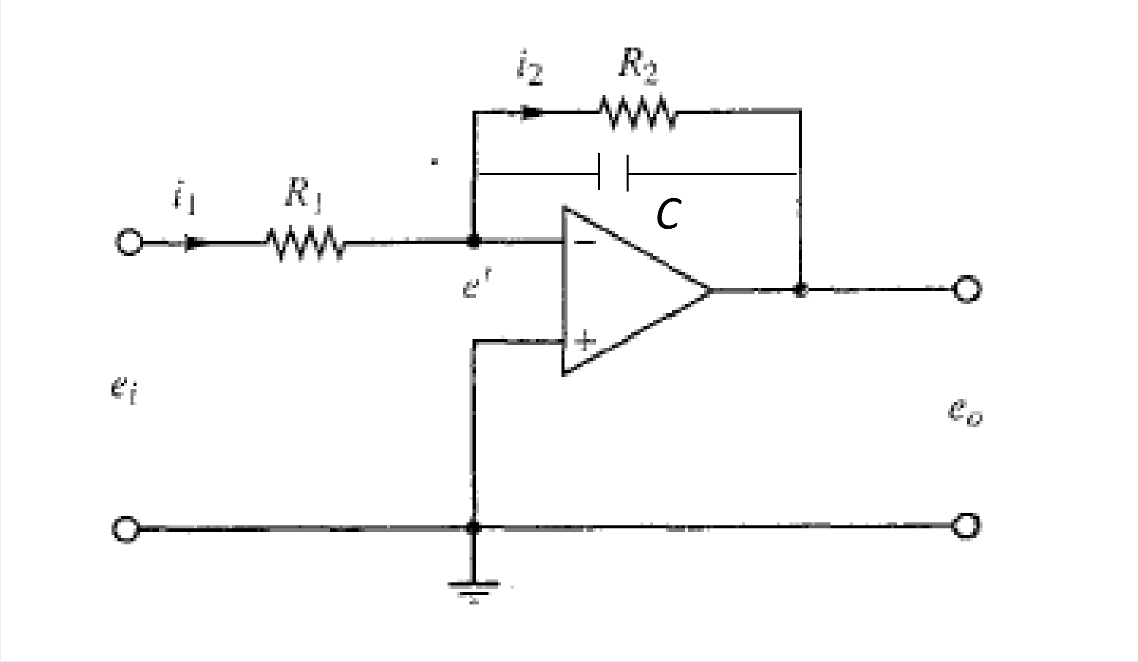


Solution: Utilizing complex impedance approach, we have (from left to right)

$$
\begin{cases}
I_1(s) = \dfrac{E_i(s) - U_{ab}(s)}{R_1} = \dfrac{1}{R_1}[E_i(s) - U_{ab}(s)] \\
U_{ab}(s) = \dfrac{1}{sC_1}[I_1(s) - I_2(s)] \\
I_2(s) = \dfrac{1}{R_2}[U_{ab}(s) - E_0(s)] \\
E_0(s) = \dfrac{1}{sC_2} I_2(s)
\end{cases}
$$



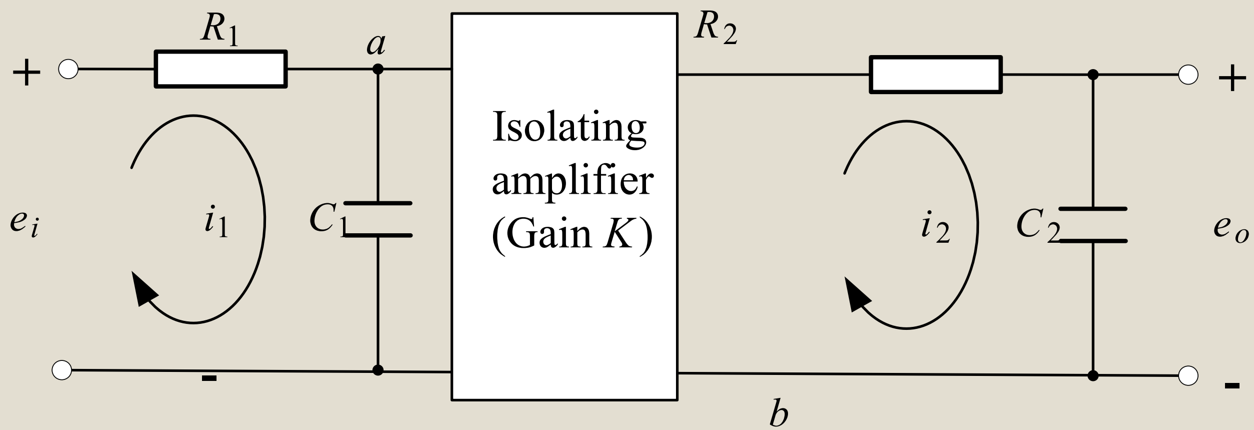
Example. Find $E_o(s)/E_i(s)$.



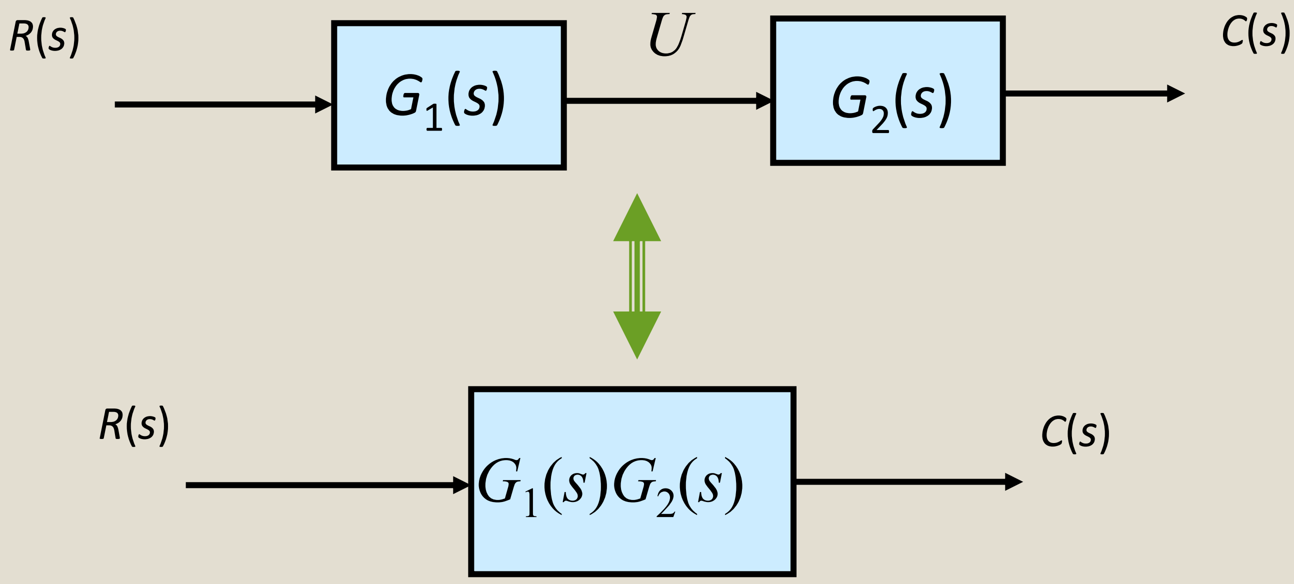
4. Transfer functions of nonloading cascaded elements

Again, consider the two simple *RC* circuits. Now, the circuits are isolated by an amplifier as shown below and therefore, have negligible loading effects, and the transfer function

$$\frac{E_0(s)}{E_i(s)} = \frac{K}{(R_1C_1s+1)(R_2C_2s+1)}$$



In general, the transfer function of a system consisting of two or more nonloading cascaded elements can be obtained by eliminating the intermediate inputs and outputs:



$$C(s) = G_2(s)U(s) = G_2(s)G_1(s)R(s)$$