Fuzzy Membership Functions

Fuzzy Operations

Fuzzy Union
Fuzzy Intersection
Fuzzy Complement

Some info for LAB

- Work on an m-file (open m-file for each task, write your programme, save the file (e.g., lab2task1), then execute the file. Now, this file has become a function in MATLAB). (see the first week's slides - <u>Week 1</u>).

- Use **help <*function*>** (e.g., help newfis) if you don't know how to use the function. It gives you information about how to use the function and what parameters it requires

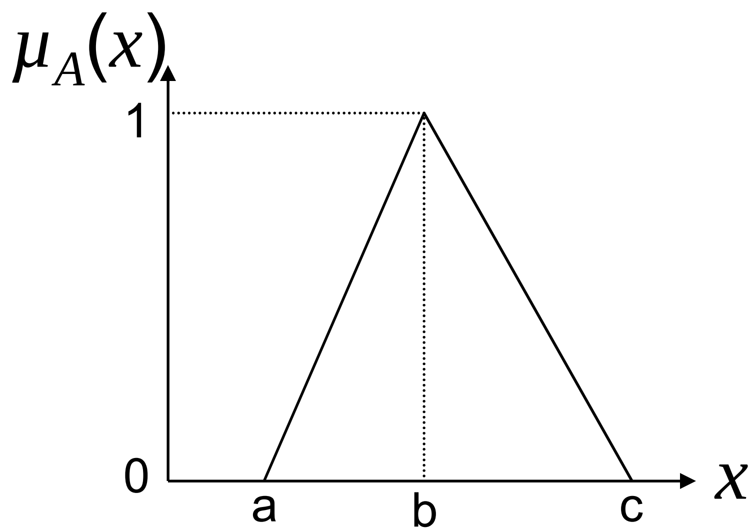
Fuzzy Membership Functions

- One of the key issues in all fuzzy sets is how to determine fuzzy membership functions
- The membership function fully defines the fuzzy set
- A membership function provides a measure of the degree of similarity of an element to a fuzzy set
- Membership functions can take any form, but there are some common examples that appear in real applications

- Membership functions can
  - either be chosen by the user arbitrarily, based on the user's experience (MF chosen by two users could be different depending upon their experiences, perspectives, etc.)
  - Or be designed using machine learning methods (e.g., artificial neural networks, genetic algorithms, etc.)
- There are different shapes of membership functions; triangular, trapezoidal, piecewise-linear, Gaussian, bell-shaped, etc.

- **Triangular membership function**
  - a, b and c represent the *x* coordinates of the three vertices of $\mu_A(x)$ in a fuzzy set A (a: lower boundary and c: upper boundary where membership degree is zero, b: the centre where membership degree is 1)

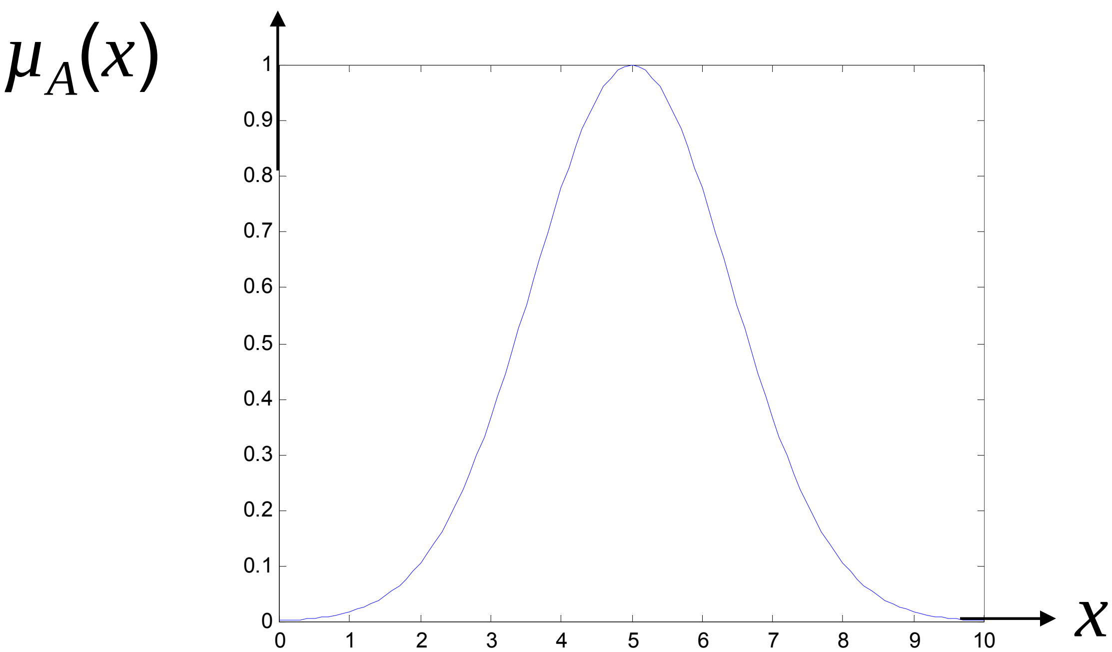
$$\mu_A(x) = \begin{cases} 0 & if\ \ x \le a \\ \dfrac{x-a}{b-a} & if\ \ a \le x \le b \\ \dfrac{c-x}{c-b} & if\ \ b \le x \le c \\ 0 & if\ \ x \ge c \end{cases}$$



- **Gaussian membership function**

$$\mu_A(x,c,s,m) = \exp\left[-\frac{1}{2}\left|\frac{x-c}{s}\right|^m\right]$$

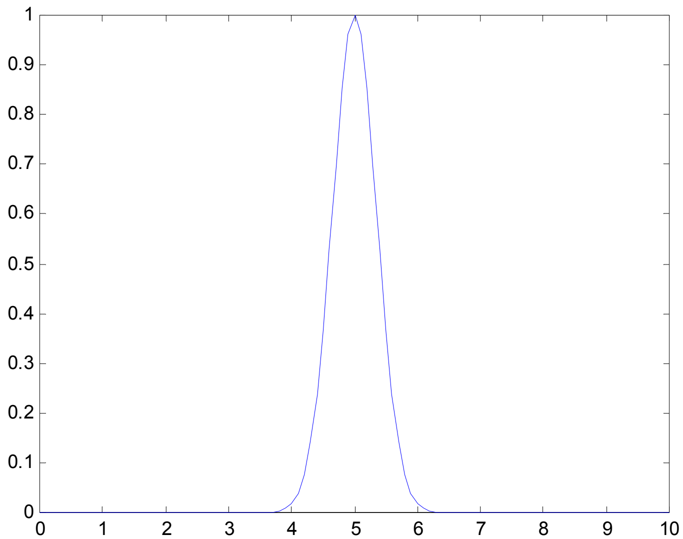
  - c: centre
  - s: width
  - m: fuzzification factor (e.g., m=2)



c=5

s=2

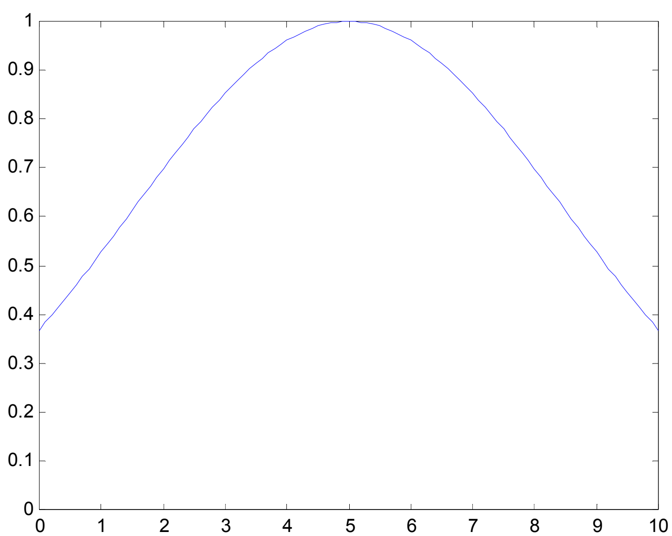
m=2



$c=5$

$s=0.5$

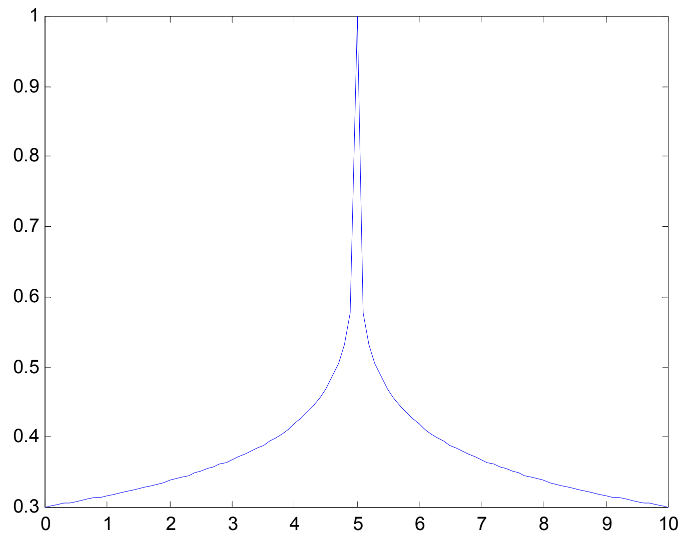
$m=2$



$c=5$

$s=5$

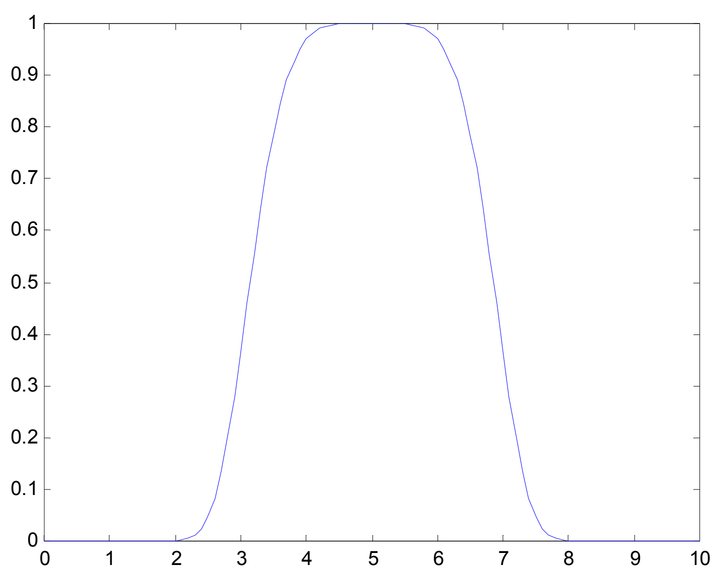
$m=2$



$c=5$

$s=2$

$m=0.2$



$c=5$

$s=5$

$m=5$

Fuzzy Operations (Fuzzy Union, Intersection, and Complement)

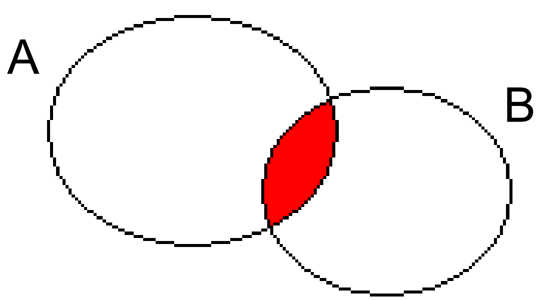
- Fuzzy logic begins by barrowing notions from crisp logic, just as fuzzy set theory borrows from crisp set theory. As in our extension of crisp set theory to fuzzy set theory, our extension of crisp logic to fuzzy logic is made by replacing membership functions of crisp logic with fuzzy membership functions [J.M. Mendel, *Uncertain Rule-Based Fuzzy Logic Systems*, 2001]
- In Fuzzy Logic, intersection, union and complement are defined in terms of their membership functions
- This section concentrates on providing enough of a theoretical base for you to be able to implement computer systems that use fuzzy logic
- Fuzzy intersection and union correspond to ‘AND’ and ‘OR’, respectively, in classic/crisp/Boolean logic
- These two operators will become important later as they are the building blocks for us to be able to compute with fuzzy if-then rules

Classic/Crisp/Boolean Logic

- Logical AND (∩)

Truth Table

A	B	A ∩ B
0	0	0
0	1	0
1	0	0
1	1	1

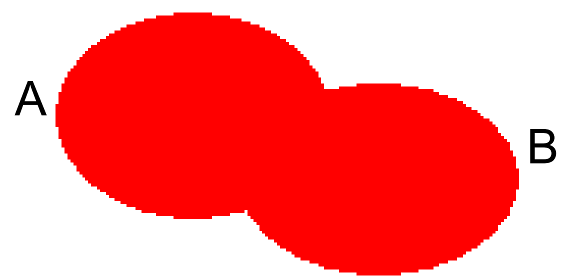


Crisp Intersection

- Logical OR (**U**)

Truth Table

A	B	A **U** B
0	0	0
0	1	1
1	0	1
1	1	1



Crisp Union

Fuzzy Union

- The union (OR) is calculated using t-conorms
- t-conorm operator is a function s(.,.)
- Its features are
  - s(1,1) = 1, s(a,0) = s(0,a) = a (boundary)
  - s(a,b) ≤ s(c,d) if a ≤ c and b ≤ d (monotonicity)
  - s(a,b) = s(b,a) (commutativity)
  - s(a,s(b,c)) = s(s(a,b),c) (associativity)
- The most commonly used method for fuzzy union is to take the maximum. That is, given two fuzzy sets *A* and *B* with membership functions $\mu_A(x)$ and $\mu_B(x)$

$$\mu_{AUB}(x) = \max(\mu_A(x), \mu_B(x))$$

Fuzzy Intersection

- The intersection (AND) is calculated using t-norms.
- t-norm operator is a function t(.,.)
- Its features
  - t(0,0) = 0, t(a,1) = t(1,a) = a (boundary)
  - t(a,b) ≤ t(c,d) if a ≤ c and b ≤ d (monotonicity)
  - t(a,b) = t(b,a) (commutativity)
  - t(a, t(b,c)) = t(t(a,b),c) (associativity)
- The most commonly adopted t-norm is the minimum. That is, given two fuzzy sets *A* and *B* with membership functions $\mu_A(x)$ and $\mu_B(x)$

$$\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$$

Fuzzy Complement

- To be able to develop fuzzy systems we also have to deal with NOT or complement.
- This is the same in fuzzy logic as for Boolean logic
- For a fuzzy set A, $\overline{A}$ denotes the fuzzy complement of A
- Membership function for fuzzy complement is

$$\mu_{\overline{A}}(x) = 1 - \mu_A(x)$$

Example 1:

Suppose we have the following (discrete) fuzzy sets:

$$A = 0.4/1+0.6/2+0.7/3+0.8/4$$

$$B = 0.3/1+0.65/2+0.4/3+0.1/4$$

The union of the fuzzy sets *A* and *B*

$= 0.4/1+0.65/2+0.7/3+0.8/4$

The intersection of the fuzzy sets *A* and *B*

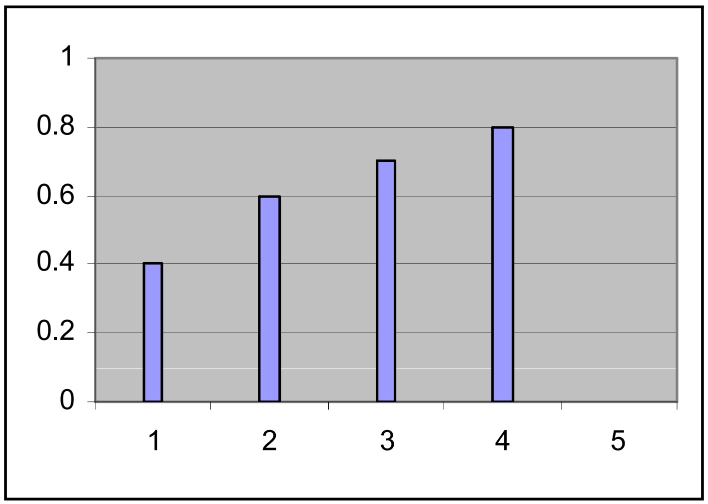
$= 0.3/1+0.6/2+0.4/3+0.1/4$

The complement of the fuzzy set A

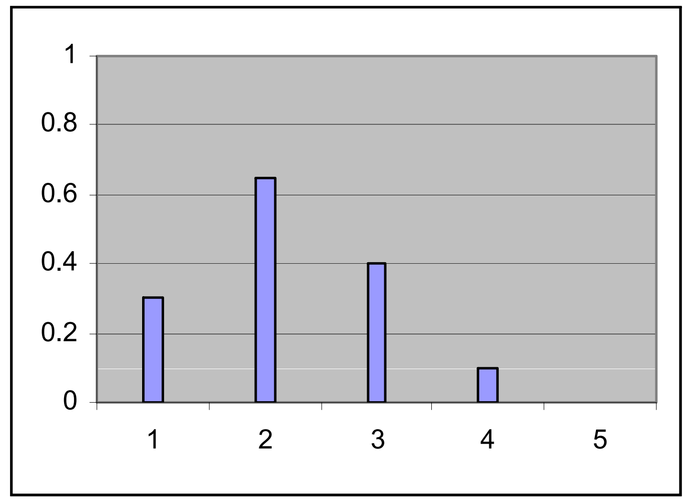
$= 0.6/1+0.4/2+0.3/3+0.2/4$

Example 1: (cont.)

Let's show the fuzzy sets A and B graphically



A



B

Example 2 (2003 exam question)

Given two fuzzy sets A and B

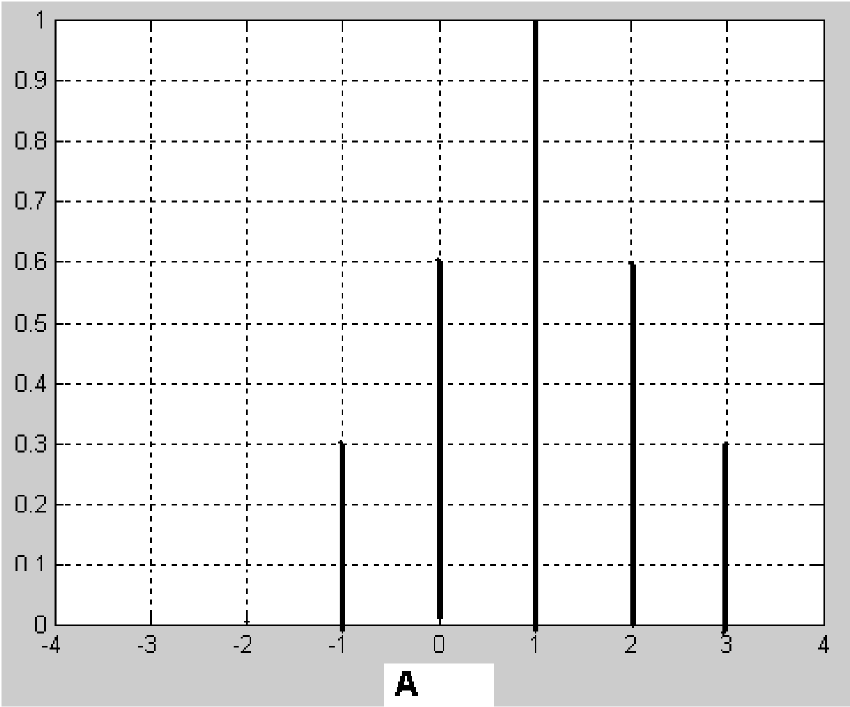
a. Represent A and B fuzzy sets graphically
b. Calculate the of union of the set A and set B
c. Calculate the intersection of the set A and set B
d. Calculate the complement of the union of A and B

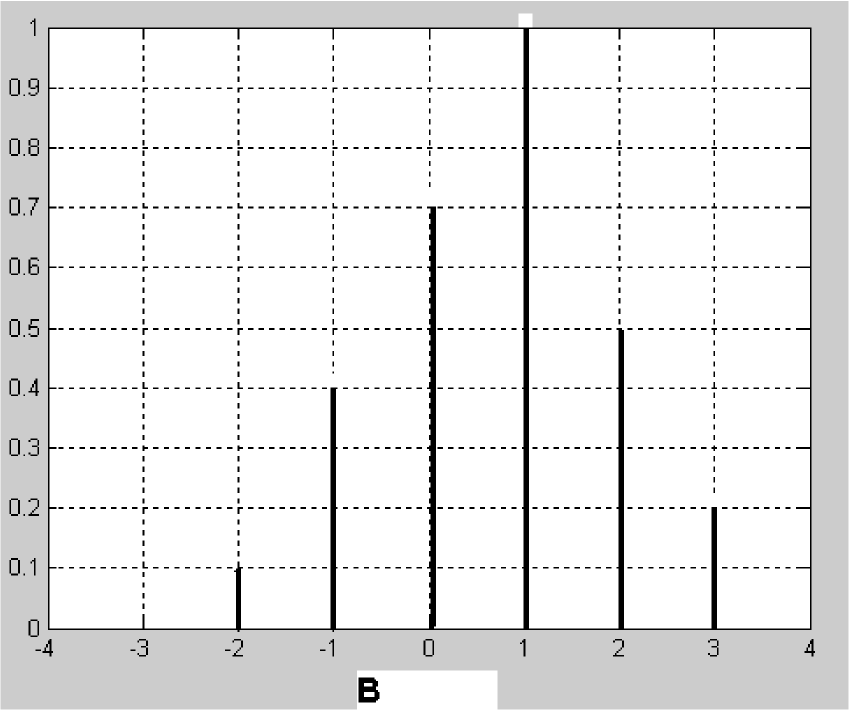
$$A = 0.0/-2 + 0.3/-1 + 0.6/0 + 1.0/1 + 0.6/2 + 0.3/3 + 0.0/4$$

$$B = 0.1/-2 + 0.4/-1 + 0.7/0 + 1.0/1 + 0.5/2 + 0.2/3 + 0.0/4$$

Example 2 (cont)

a





Example 2 (cont)

b

$$\text{Union} = \max(A, B) = 0.1/-2 + 0.4/-1 + 0.7/0 + 1.0/1 + 0.6/2 + 0.3/3 + 0.0/4$$

c

$$\text{Intersection} = \min(A, B) = 0.0/-2 + 0.3/-1 + 0.6/0 + 1.0/1 + 0.5/2 + 0.2/3 + 0.0/4$$

d

$$\text{Complement of (b)} = 1 - \max(A, B) = 0.9/-2 + 0.6/-1 + 0.3/0 + 0.0/1 + 0.4/2 + 0.7/3 + 1.0/4$$

Example 3: Graphical representation of the Fuzzy operations (taken from J.M. Mendel, *Uncertain Rule-Based Fuzzy Logic Systems*, 2001)

Consider the fuzzy sets A = damping ratio x considerably larger than 0.5, and B = damping ratio x approximately equal to 0.707. Note that damping ratio is a positive real number, i.e., its universe of discourse, X, is the positive real numbers $0 \le x \le 1$

Consequently, $A = \{(x, \mu_A(x)) | x \varepsilon X\}$ and $B = \{(x, \mu_B(x)) | x \varepsilon X\}$

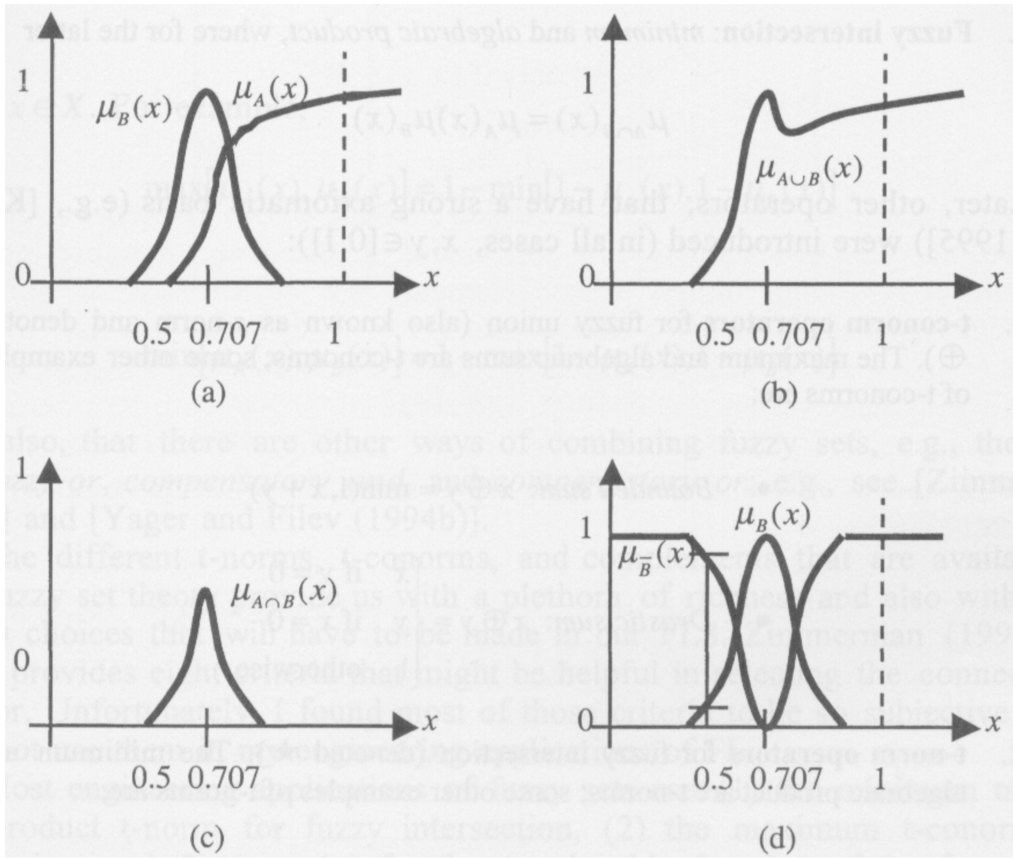
where, for example, $\mu_A(x)$ and $\mu_B(x)$ are specified, as:

$$\mu_A(x) = \begin{cases} 0 & if\ 0 \le x \le 0.5 \\ \dfrac{1}{1+(x-0.5)^{-2}} & if\ 0.5 < x \le 1 \end{cases} \qquad \mu_B(x) = \frac{1}{1+(x-0.707)^4} \qquad 0 \le x \le 1$$

Example 3: (cont.)

Figure (a): $\mu_A(x)$, $\mu_B(x)$ *Figure (b)*: $\mu_{A\cup B}(x)$

Figure (c): $\mu_{A\cap B}(x)$ *Figure (d)*: $\mu_B(x)$, $\mu_{\overline{B}}(x)$



Example 3: (cont.)

- This example demonstrates that for fuzzy sets, the Law of Excluded Middle and Concentration are broken, i.e., *for fuzzy sets A and B*:

$$A \cup \bar{A} \neq X \;\text{ and }\; A \cap \bar{A} \neq \emptyset$$

- In fact, one of the ways to describe the difference between crisp set theory and fuzzy set theory is to explain that these two laws do not hold in fuzzy set theory