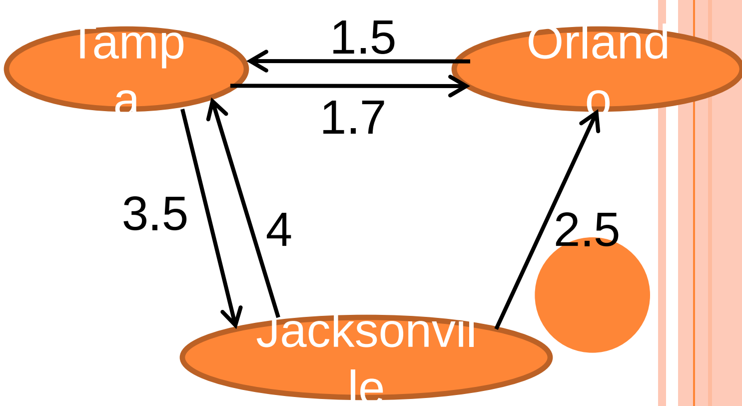
# Floyd-Warshall Algorithm

# Floyd-Warshall Algorithm

- A weighted, directed graph is a collection vertices connected by weighted edges (where the weight is some real number).
  - One of the most common examples of a graph in the real world is a road map.
    - Each location is a vertex and each road connecting locations is an edge.
    - We can think of the distance traveled on a road from one location to another as the weight of that edge.

| | Tampa | Orlando | Jaxville |
|---|---|---|---|
| **Tampa** | 0 | 1.7 | 3.5 |
| **Orlando** | 1.5 | 0 | ∞ |
| **Jax** | 4 | 2.5 | 0 |

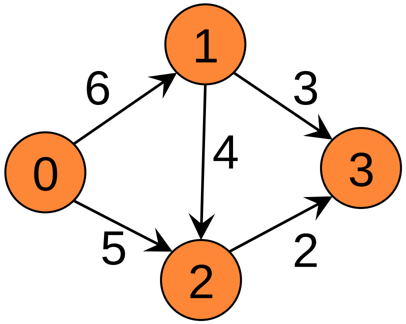


# Storing a Weighted, Directed Graph

- Adjacency Matrix:
  - Let **D** be an edge-weighted graph in adjacency-matrix form
  - **D(*i,j*)** is the weight of edge (i, j), or ∞ if there is no such edge.
- Update matrix **D**, with the shortest path ***through immediate vertices.***

**D =**

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 6 | 5 | ∞ |
| 1 | ∞ | 0 | 4 | 3 |
| 2 | ∞ | ∞ | 0 | 2 |
| 3 | ∞ | ∞ | ∞ | 0 |



# Floyd-Warshall Algorithm

- Given a weighted graph, we want to know the shortest path from one vertex in the graph to another.
  - The Floyd-Warshall algorithm determines the shortest path between all pairs of vertices in a graph.

  - What is the difference between Floyd-Warshall and Dijkstra's??

# Floyd-Warshall Algorithm

- If V is the number of vertices, Dijkstra's runs in $\Theta(V^2)$
  - We could just call Dijkstra |V| times, passing a different source vertex each time.
  - $\Theta(V \times V^2) = \Theta(V^3)$
  - (Which is the same runtime as the Floyd-Warshall Algorithm)

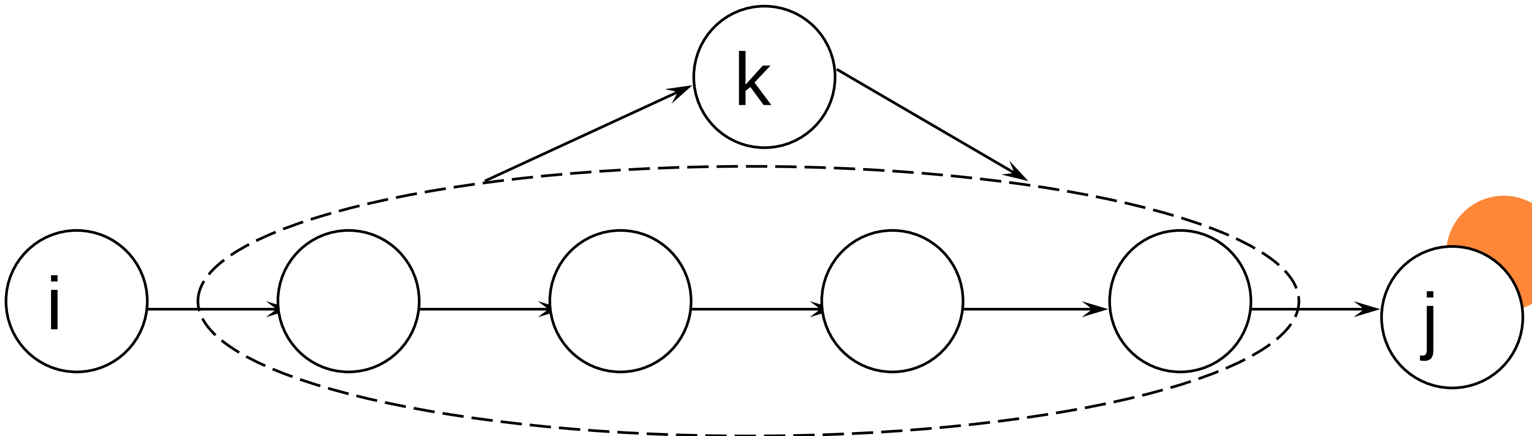
- ***BUT***, Dijkstra's doesn't work with negative-weight edges.

# Floyd Warshall Algorithm

- Let's go over the premise of how Floyd-Warshall algorithm works…
  - Let the vertices in a graph be numbered from 1 … n.
  - Consider the subset {1,2,…, k} of these n vertices.

  - Imagine finding the shortest path from vertex i to vertex j that uses vertices in the set {1,2,…,k} only.

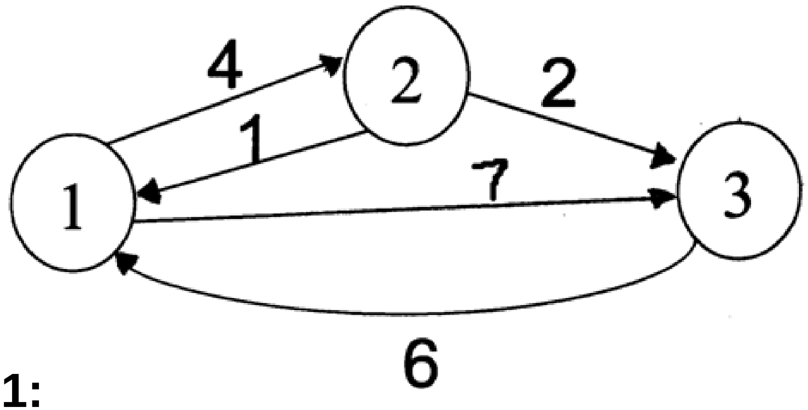
  - There are two situations:
    1) k is an intermediate vertex on the shortest path.
    2) k is not an intermediate vertex on the shortest path.



# Floyd Warshall Algorithm - Example

$$D^{(0)} = \begin{bmatrix} 0 & 4 & 7 \\ 1 & 0 & 2 \\ 6 & \infty & 0 \end{bmatrix}$$

Original weights.




$$D^{(1)} = \begin{bmatrix} 0 & 4 & 7 \\ 1 & 0 & 2 \\ 6 & 10 & 0 \end{bmatrix}$$

**Consider Vertex 1:**

D(3,2) = D(3,1) + D(1,2)

$$D^{(2)} = \begin{bmatrix} 0 & 4 & 6 \\ 1 & 0 & 2 \\ 6 & 10 & 0 \end{bmatrix}$$

**Consider Vertex 2:**

D(1,3) = D(1,2) + D(2,3)

$$D^{(3)} = \begin{bmatrix} 0 & 4 & 6 \\ 1 & 0 & 2 \\ 6 & 10 & 0 \end{bmatrix}$$

**Consider Vertex 3:**

Nothing changes.

# Floyd Warshall Algorithm

- Looking at this example, we can come up with the following algorithm:
  - Let D store the matrix with the initial graph edge information initially, and update D with the calculated shortest paths.

```
For k=1 to n {
  For i=1 to n {
              For j=1 to n
                    D[i,j] = min(D[i,j],D[i,k]+D[k,j])
  }
}
```

  - The final D matrix will store all the shortest paths.

# Floyd Warshall Algorithm

- Example on the board…

# Floyd Warshall – Path Reconstruction

- The path matrix will store the last vertex visited on the path from i to j.
  - So path[i][j] = k means that in the shortest path from vertex i to vertex j, the LAST vertex on that path before you get to vertex j is k.

- Based on this definition, we must initialize the path matrix as follows:
  - path[i][j] = i if i!=j and there exists an edge from i to j
  - = NIL otherwise

- The reasoning is as follows:
  - If you want to reconstruct the path at this point of the algorithm when you aren't allowed to visit intermediate vertices, the previous vertex visited MUST be the source vertex i.
  - NIL is used to indicate the absence of a path.

# Floyd Warshall – Path Reconstruction

- Before you run Floyd's, you initialize your distance matrix ***D*** and path matrix ***P*** to indicate the use of no immediate vertices.
  - (Thus, you are only allowed to traverse direct paths between vertices.)

- Then, at each step of Floyd's, you essentially find out whether or not using vertex k will *improve* an estimate between the distances between vertex i and vertex j.

- <u>If it ***does improve*** the estimate here's what you need to record:</u>

1) record the new shortest path weight between i and j
2) record the fact that the shortest path between i and j goes through k

# Floyd Warshall – Path Reconstruction

- If it ***does improve*** the estimate here's what you need to record:
  1) record the new shortest path weight between i and j
     - **We don't need to change our path and we do not update the path matrix**
  2) record the fact that the shortest path between i and j goes through k
     - **We want to store the last vertex from the shortest path from vertex k to vertex j. *This will NOT necessarily be k, but rather, it will be path[k][j].***

This gives us the following update to our algorithm:

```
if (D[i][k]+D[k][j] < D[i][j]) { // Update is necessary to use k as intermediate vertex
    D[i][j] = D[i][k]+D[k][j];
    path[i][j] = path[k][j];
}
```

## Path Reconstruction

- Example on the board…

# PATH RECONSTRUCTION

- Now, the once this path matrix is computed, we have all the information necessary to reconstruct the path.
  - Consider the following path matrix (indexed from 1 to 5 instead of 0 to 4):

| | | | | |
|---|---|---|---|---|
| **NIL** | **3** | **4** | **5** | **1** |
| **4** | **NIL** | **4** | **2** | **1** |
| **4** | **3** | **NIL** | **2** | **1** |
| **4** | **3** | **4** | **NIL** | **1** |
| **4** | **3** | **4** | **5** | **NIL** |

- Reconstruct the path from vertex1 to vertex 2:
  - First look at path[1][2] = 3. This signifies that on the path from 1 to 2, 3 is the last vertex visited before 2.
    - Thus, the path is now, 1…3->2.
  - Now, look at path[1][3], this stores a 4. Thus, we find the last vertex visited on the path from 1 to 3 is 4.
  - So, our path now looks like 1…4->3->2. So, we must now look at path[1][4]. This stores a 5,
  - thus, we know our path is 1…5->4->3->2. When we finally look at path[1][5], we find 1,
  - which means our path really is 1->5->4->3->2.

# TRANSITIVE CLOSURE

- Computing a transitive closure of a graph gives you complete information about which vertices are connected to which other vertices.

- Input:
  - Un-weighted graph $G$: $W[i][j] = 1$, if $(i,j) \in E$, $W[i][j] = 0$ otherwise.
- Output:
  - $T[i][j] = 1$, if there is a path from $i$ to $j$ in $G$, $T[i][j] = 0$ otherwise.
- Algorithm:
  - Just run Floyd-Warshall with weights 1, and make $T[i][j] = 1$, whenever $D[i][j] < \infty$.
  - More efficient: use only Boolean operators

# TRANSITIVE CLOSURE

```
Transitive-Closure(W[1..n][1..n])
T ← W    // T(0)
for k ←1 to n do // compute T(k)
    for i ←1 to n do
        for j ←1 to n do
            T[i][j] ← T[i][j] ∨ (T[i][k] ∧ T[k][j])
return T
```

- This is the SAME as the other Floyd-Warshall Algorithm, except for when we find a non-infinity estimate, we simply add an edge to the transitive closure graph.
- Every round we build off the previous paths reached.
  - After iterating through all vertices being intermediate vertices, we have tried to connect all pairs of vertices i and j through all intermediate vertices k.

# Transitive Closure

- Example on the board…

# References


- Slides adapted from Arup Guha's Computer Science II Lecture notes: http://www.cs.ucf.edu/~dmarino/ucf/cop3503/lectures/
- Additional material from the textbook:

  Data Structures and Algorithm Analysis in Java (Second Edition) by Mark Allen Weiss
- Additional images:

  www.wikipedia.com

  xkcd.com