Differential Equations

Differential Equations

Definition

A differential equation is an equation involving derivatives of an unknown function and possibly the function itself as well as the independent variable.

Example

$$y' = \sin(x), \quad (y')^4 - y^2 + 2xy - x^2 = 0, \quad y'' + y^3 + x = 0$$

1st order equations — 2nd order equation

Definition

The order of a differential equation is the highest order of the derivatives of the unknown function appearing in the equation

In the simplest cases, equations may be solved by direct integration.

Examples

$$y' = \sin(x) \Rightarrow y = -\cos(x) + C$$

$$y'' = 6x + e^x \Rightarrow y' = 3x^2 + e^x + C_1 \Rightarrow y = x^3 + e^x + C_1 x + C_2$$

Observe that the set of solutions to the above 1st order equation has 1 parameter, while the solutions to the above 2nd order equation depend on two parameters.

Separable Differential Equations

A separable differential equation can be expressed as the product of a function of x and a function of y.

$$\frac{dy}{dx} = g(x) \cdot h(y) \qquad h(y) \neq 0$$

Example:

$$\frac{dy}{dx} = 2xy^2$$

Multiply both sides by dx and divide both sides by y^2 to separate the variables. (Assume y^2 is never zero.)

$$\frac{dy}{y^2} = 2x\ dx$$

$$y^{-2}dy = 2x\ dx$$

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$$\frac{dy}{y^2} = 2x\ dx$$

$$y^{-2}dy = 2x\ dx$$

$$\int y^{-2}dy = \int 2x\ dx$$

$$-y^{-1} + C_1 = x^2 + C_2$$

$$-\frac{1}{y} = x^2 + C$$

Combined constants of integration

$$-\frac{1}{x^2 + C} = y$$

$$y = -\frac{1}{x^2 + C}$$

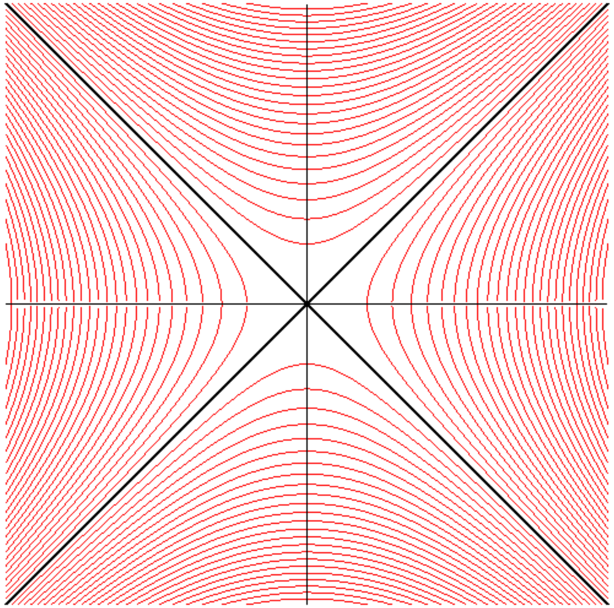
→

Family of solutions (general solution) of a differential equation

Example

$$\frac{dy}{dx} = \frac{x}{y} \qquad \int y\,dy = \int x\,dx$$

$$y^2 = x^2 + C$$



The picture on the right shows some solutions to the above differential equation. The straight lines

$y = x$ and $y = -x$

are special solutions. A unique solution curve goes through any point of the plane different from the origin. The special solutions $y = x$ and $y = -x$ go both through the origin.

Initial conditions

- In many physical problems we need to find the particular solution that satisfies a condition of the form $y(x_0)=y_0$. This is called an **initial condition**, and the problem of finding a solution of the differential equation that satisfies the initial condition is called an **initial-value problem**.
- *Example (cont.):* Find a solution to $y^2 = x^2 + C$ satisfying the initial condition $y(0) = 2$.

$$2^2 = 0^2 + C$$

$$C = 4$$

$$y^2 = x^2 + 4$$

Example:

$$\frac{dy}{dx} = 2x\left(1+y^2\right)e^{x^2}$$ ← Separable differential equation

$$\frac{1}{1+y^2}\,dy = 2x\, e^{x^2}\,dx$$

$$\int \frac{1}{1+y^2}\,dy = \int 2x\, e^{x^2}\,dx$$

$$u = x^2$$
$$du = 2x\,dx$$

$$\int \frac{1}{1+y^2}\,dy = \int e^u\,du$$

$$\tan^{-1} y + C_1 = e^u + C_2$$

$$\tan^{-1} y + C_1 = e^{x^2} + C_2$$

$$\tan^{-1} y = e^{x^2} + C$$ ← Combined constants of integration

→

Example (cont.):

$$\frac{dy}{dx} = 2x\left(1+y^2\right)e^{x^2}$$

$\vdots$

$$\tan^{-1} y = e^{x^2} + C$$

← We now have y as an implicit function of x.

$$\tan\left(\tan^{-1} y\right) = \tan\left(e^{x^2} + C\right)$$

We can find y as an explicit function of x by taking the tangent of both sides.

$$y = \tan\left(e^{x^2} + C\right)$$

→

Law of natural growth or decay

A population of living creatures normally increases at a rate that is proportional to the current level of the population. Other things that increase or decrease at a rate proportional to the amount present include radioactive material and money in an interest-bearing account.

If the rate of change is proportional to the amount present, the change can be modeled by:

$$\frac{dy}{dt} = ky$$

→

$$\frac{dy}{dt} = ky$$

Rate of change is proportional to the amount present.

$$\frac{1}{y}dy = k\ dt$$

Divide both sides by y.

$$\int \frac{1}{y}dy = \int k\ dt$$

Integrate both sides.

$$\ln|y| = kt + C$$

$$e^{\ln|y|} = e^{kt+C}$$

Exponentiate both sides.

$$|y| = e^{C} \cdot e^{kt}$$

$$y = \pm e^{C} e^{kt}$$

$$y = Ae^{kt}$$

→

Logistic Growth Model

Real-life populations do not increase forever. There is some limiting factor such as food or living space.

There is a maximum population, or carrying capacity, M.

A more realistic model is the logistic growth model where growth rate is proportional to both the size of the population (y) and the amount by which y falls short of the maximal size (M-y). Then we have the equation:

$$\frac{dy}{dt} = ky(M-y)$$

The solution to this differential equation (derived in the textbook):

$$y = \frac{y_0 M}{y_0 + (M - y_0)e^{-kMt}}, \quad \text{where } y_0 = y(0)$$

Mixing Problems

A tank contains 1000 L of brine with 15 kg of dissolved salt. Pure water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate.

How much salt is in the tank

(a) after *t* minutes;

(b) after 20 minutes?

This problem can be solved by modeling it as a differential equation.

(the solution on the board)

Mixing Problems

Problem 45.

A vat with 500 gallons of beer contains 4% alcohol (by volume). Beer with 6% alcohol is pumped into the vat at a rate of 5 gal/min and the mixture is pumped out at the same rate. What is the percentage of alcohol after an hour?