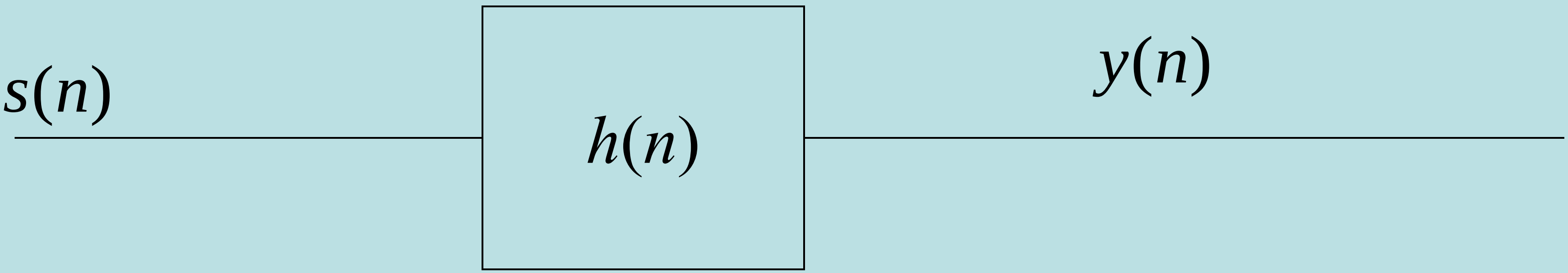
Motivation



$$y(n) = x(n) * h(n)$$

Fast convolution --- compute convolution using FFT

Outline

- Fast convolution of short sequences;
- Fast convolution of long sequences.

Convolution of short sequences

- *Let x*(*n*) be length *L*, (*n=0, 1, … L-1)*;
- *h*(*n*) be length *M, (n=0,1, …, M-1);*
- *y(n) should have L+M-1 samples, given by:*

$$y(n) = x(n) * h(n) = \sum_{m=0}^{N-1} h(m)x(n-m)$$

Where *n=0,1,…, N (N=L+M-1)*

This equation is referred to as linear convolution

Convolution of short sequences

- *Total computations (Assume M<L)*
  - *n=0*, 1 multiplication
  - *n*=1, *2* multiplications and 1 addition;
  - *n=2, 3* multiplications and 2 additions;
  - …
  - if *M-1<=n<=L*-1, *M* multiplications,…
  - ….
  - *n=L+M-2, 1* multiplication and no addition
  - **Hence, ML multiplications for convolving *x*(n) and *H*(n)**

Convolution of short sequences

- Let us see if DFT can be used for computing the convolution.
- As the length of *x(n)*,*h(n)* and *y(n)* are *L*,*M* and *(L+M-1)* respectively, we consider *N*-point DFTs of them, where *N>L+M-1*:

$$X(k) = \sum_{n=0}^{L-1} x(n) W_N^{nk}, \qquad H(k) = \sum_{n=0}^{M-1} h(n) W_N^{nk}$$

$$Y(k) = \sum_{n=0}^{L+M-1} y(n) W_N^{nk}$$

Convolution of short sequences

$$X(k)H(k) = \sum_{n=0}^{L-1} x(n)W_N^{nk} \sum_{m=0}^{M-1} h(m)W_N^{mk}$$

$$= \sum_{n=0}^{L-1}\sum_{m=0}^{M-1} x(n)h(m)W_N^{(n+m)k} \leftarrow let \qquad n+m=l$$

$$= \sum_{l=0}^{L+M-1}\sum_{m=0}^{M-1} x(l-m)h(m)W_N^{lk}$$

$$= \sum_{l=0}^{L+M-1} y(l)W_N^{lk} = Y(k)$$

y(*l*)

Convolution of short sequences

Hence convolution can be computed via DFT's:

- Step 1.

Compute N-point DFT of x(n) and h(n), where $N>L+M-1$

- Step 2.

Compute $Y(k)=X(k)H(k)$

- Step 3.

Compute N-point IDFT of Y(k) to get y(n)

Convolution of short sequences: Is it more efficient to use DFTs?

- Multiplications: (1/2)NlogN for each FFT and IFFT. Hence (3/2) NlogN +N complex multiplications are required; where N>=L+M-1
- The direct convolution involves ML real multiplications;
- Which one is more efficient? FFT is more efficient when L and M are large.
- For example: when L=M ……

Circular convolution

- Note that N must be bigger than L+M-1. Otherwise the result will not be correct. Why?

- Naturally multiplication in frequency domain is equivalent to circular convolution.

- If $N<L+M-1$, the circular convolution will involves overlaps .

Circular convolution

- Circular convolution of x(n) and h(n) is defined as the convolution of h(n) with a periodic signal $x_p(n)$:

$$y_p(n) = x_p(n) * h(n)$$

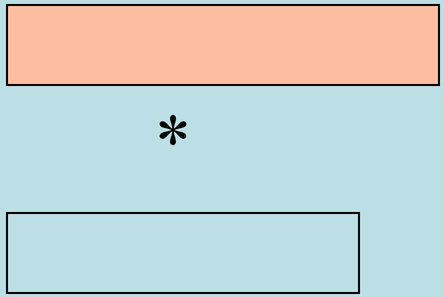
where

$$x_p(n) = x(n \bmod N), \qquad -\infty < n < \infty$$

Circular Convolution

x(n) length N

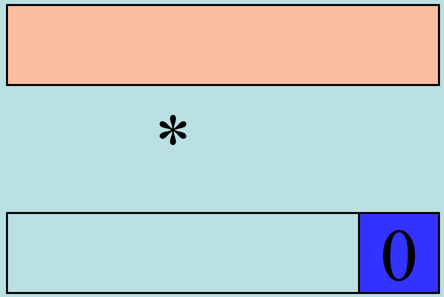
h(n) length M



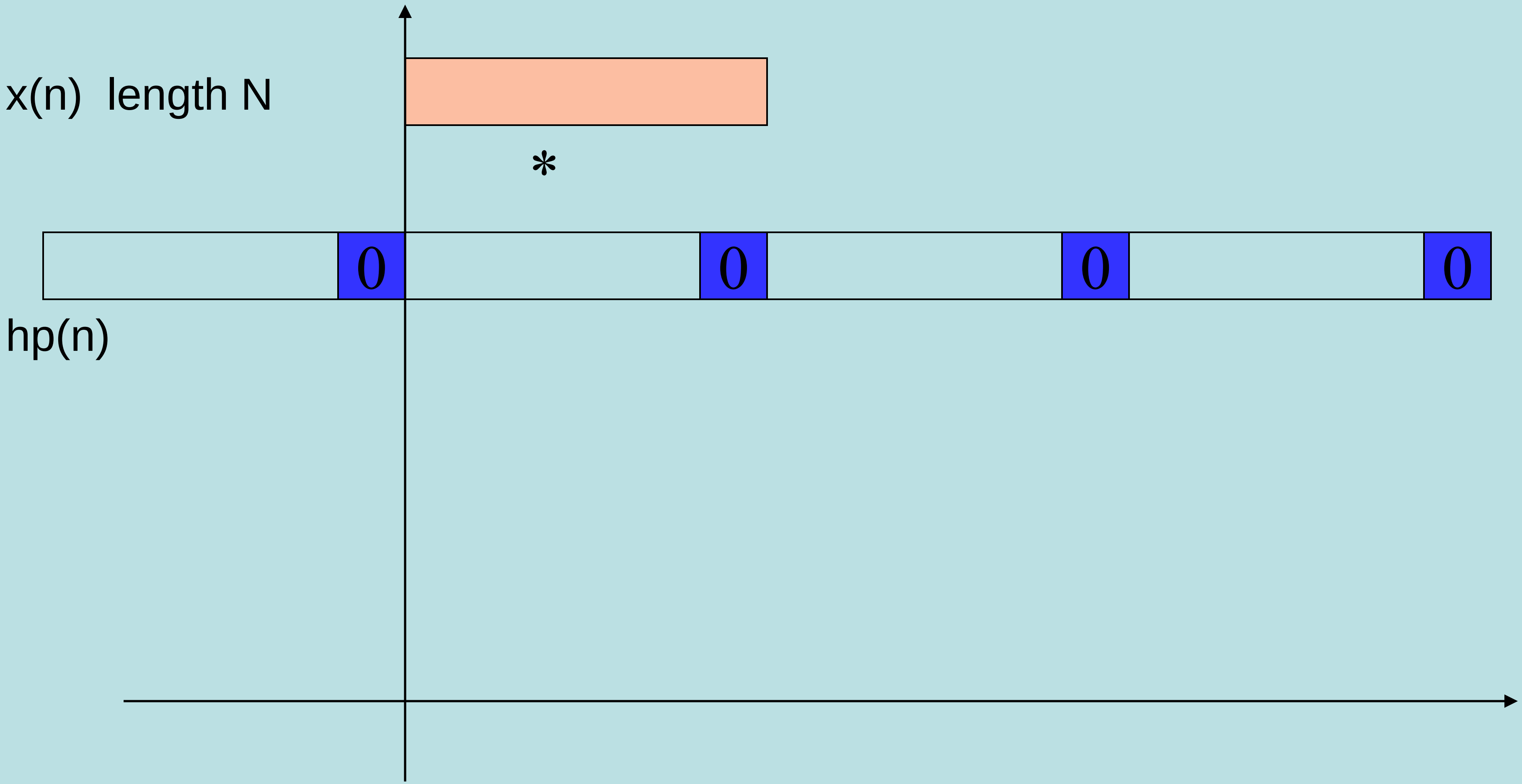
Circular Convolution

x(n) length N

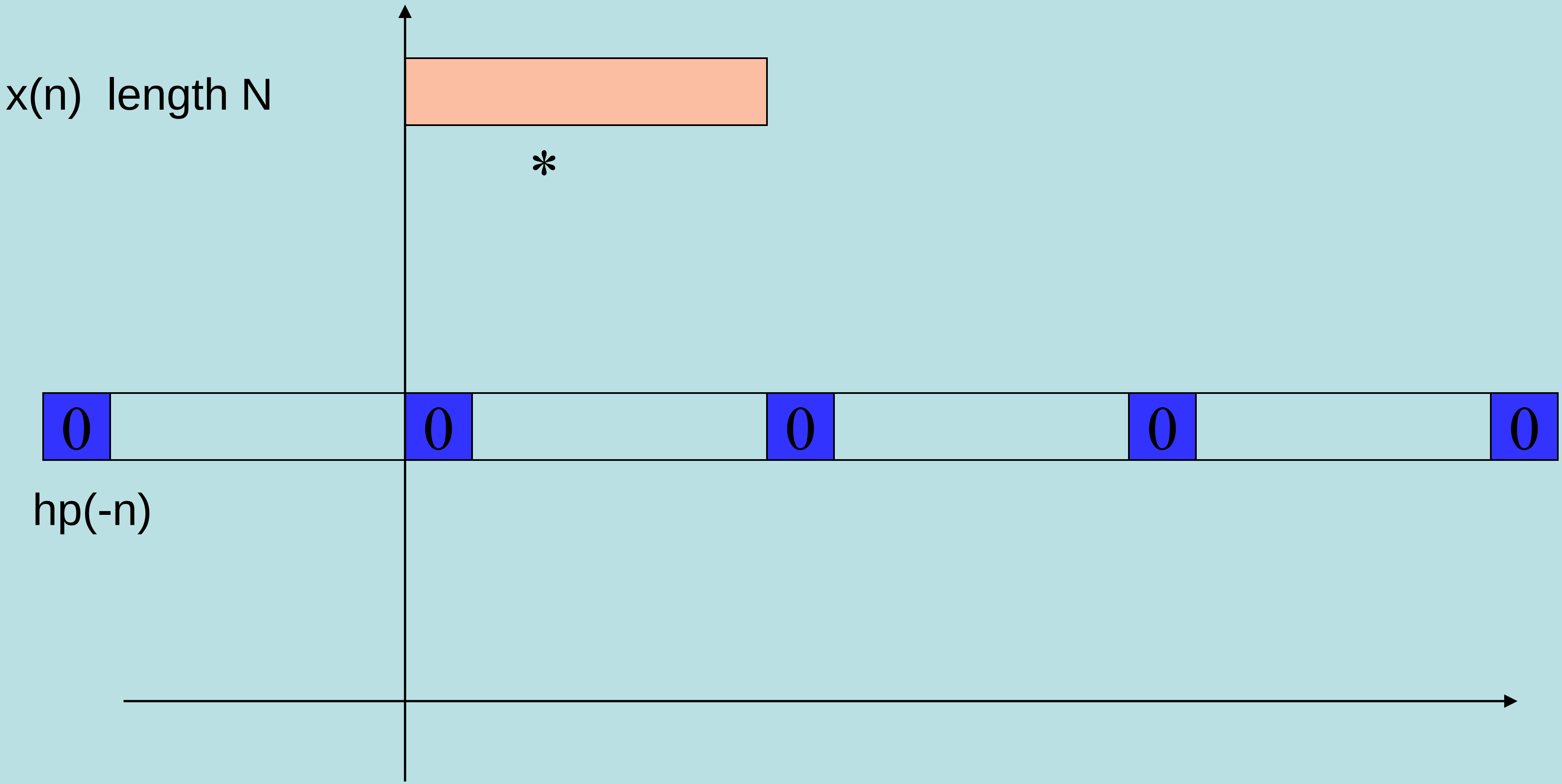
h(n) length M



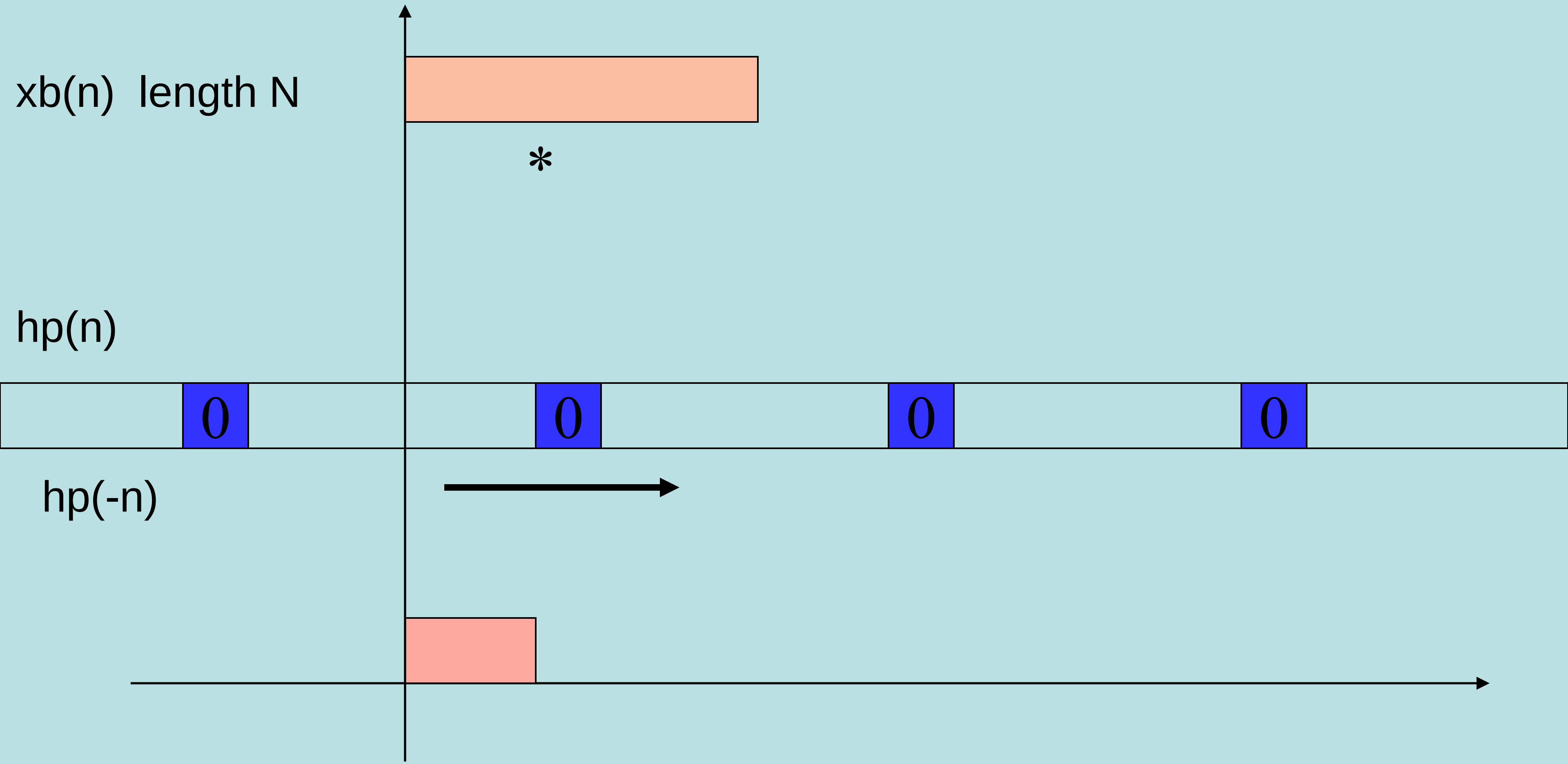
Circular Convolution



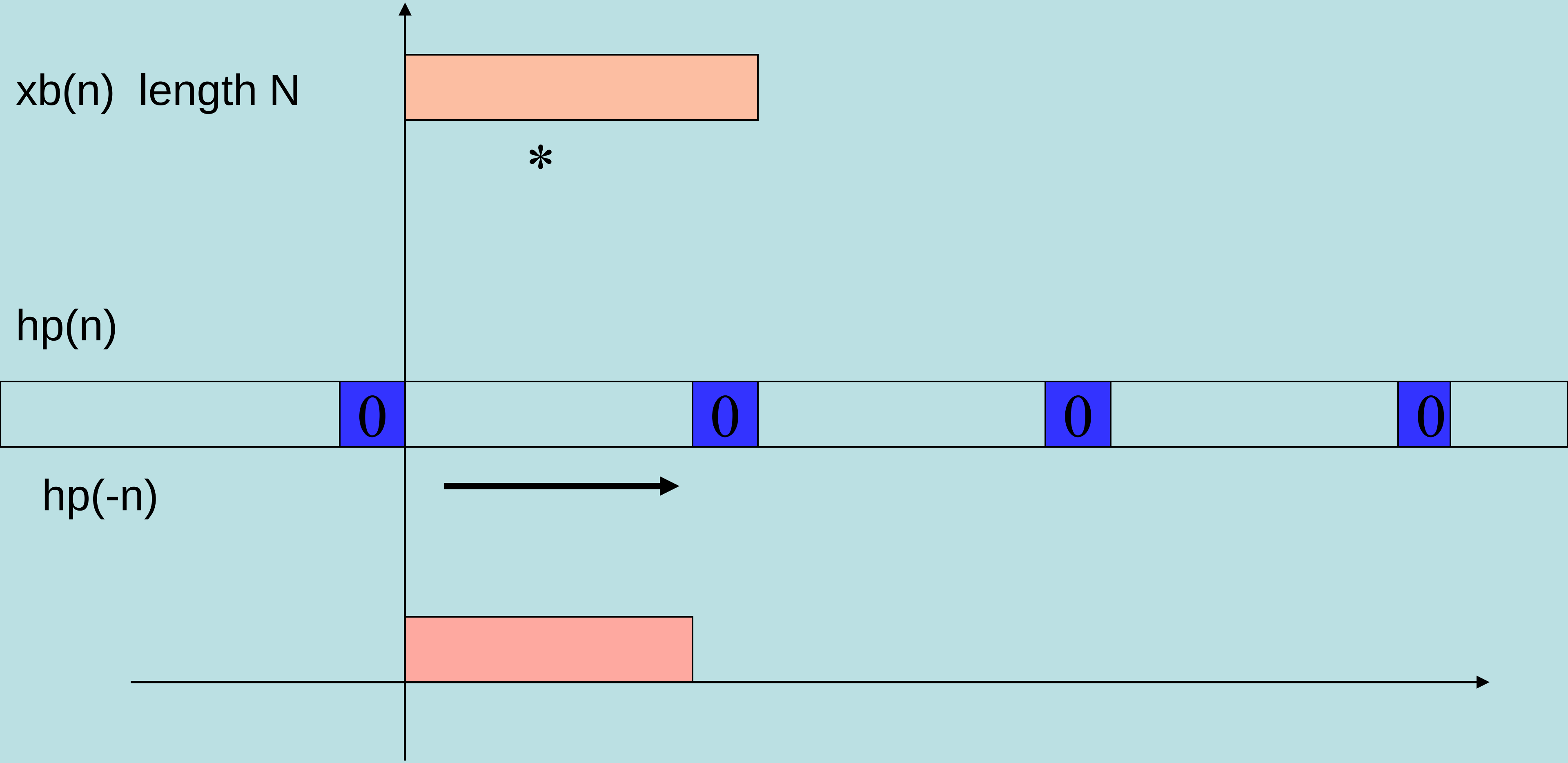
Circular Convolution



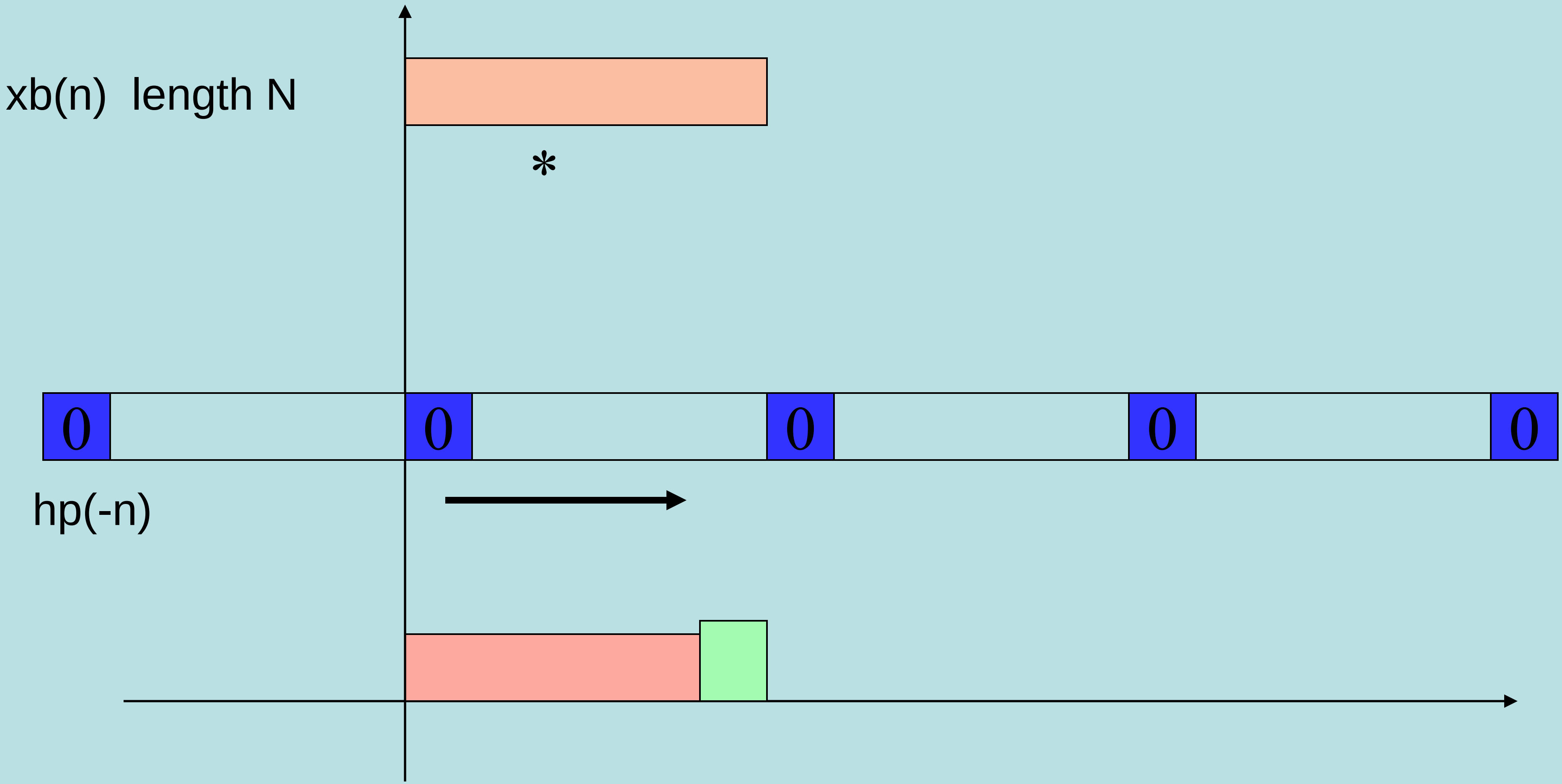
Circular Convolution



Circular Convolution



Circular Convolution



Examples

- Let {x(n)}={1,2,3} and {h(n)}={1,1,1}, then the result should be {y(n)}={1,3,6,5,3}
- With L=M=3, we should choose N=5
- however if we take N=4, the extended signals are
  - {x(n)}={1,2,3,0} and {h(n)}={1,1,1,0}
- The DFT yields
  - X(k)={6,-2-2j,2,-2+2j}
  - H(k)={3,-j,1,j}
  - Y(k)={18,-2+2j,2,-2-2j}
  - Hence y(n)={4,3,6,5}

Examples

$x_p(n)$= {…..3,0 1,2,3,0,1,2,3,0,1,2,3,0,1,2,3,0,1,2,3,0….}

---------------------------------------+-----------------------------→

$x_p(-n)$={….3,2,1,0,3,2,1,0, 3,2,1,0,3,2,1,0,3,2,1,0, ….}

{h(n)}={1,1,1},

y(n)= 4,3,6,5

Examples

If x(n)={1,2,3,0,0} ➔ 5 point DFT

h(n)={1,1,1,0,0} ➔ 5 point DFT

we can get y(n)={1,3,6,5,3}

{1,2,3,0,0,1,2,3,0,0,1,2,3,0,0,1,2,3,0,0,1,2,3,0,0}

{1,1,1}

x_p(n)={…..,2,3,0,0,1,2,3,0,0,1,2,3,0,0,1,2,3,0,0,1,2,3,0…}

---*----------------------------------→

x_p(-n)={…..0,0,3,2,1,0,0,3,2,1,0,0,3,2,1,0,0,3,2,1,0,0,…}

{h(n)}= {1,1,1},

y(n)=1,3,6,5,3

Convolution of long sequences

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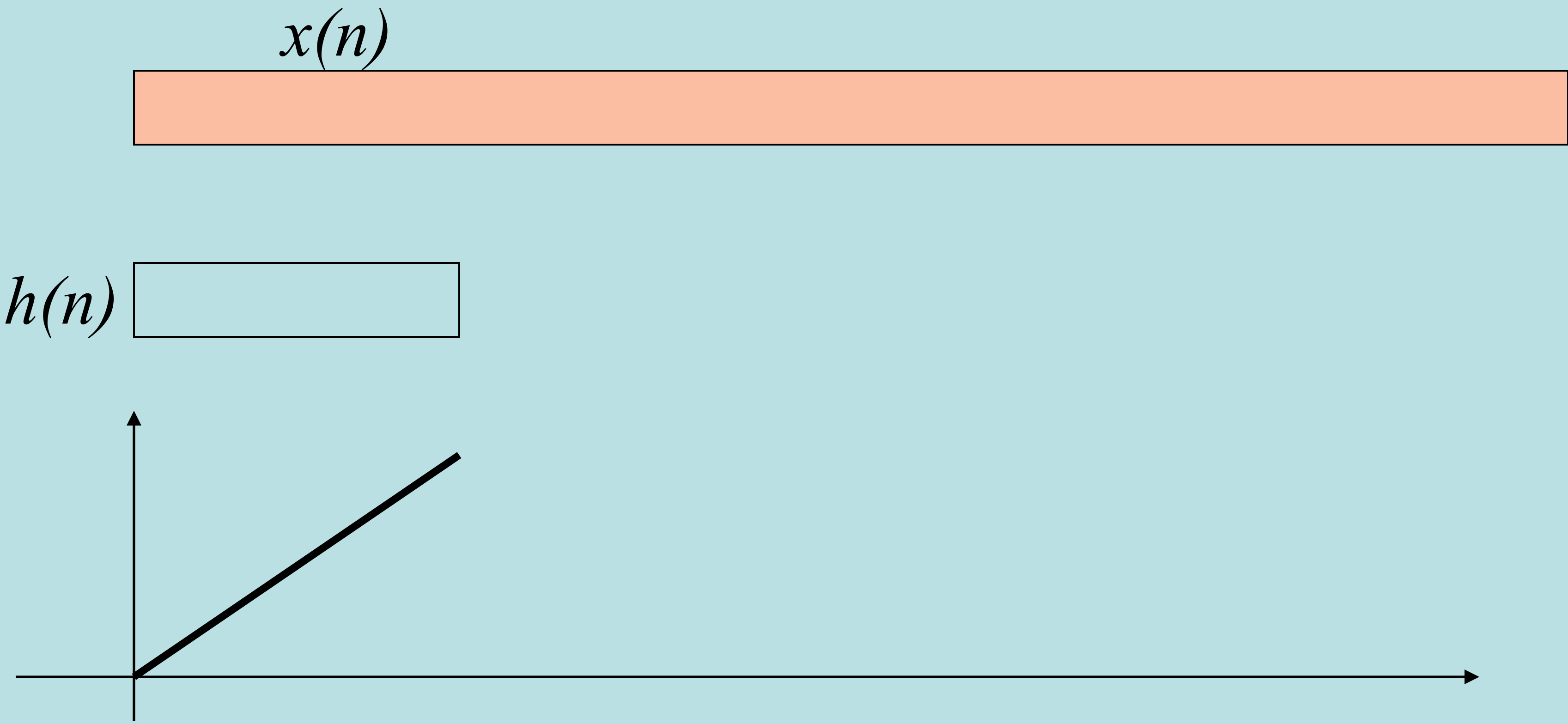
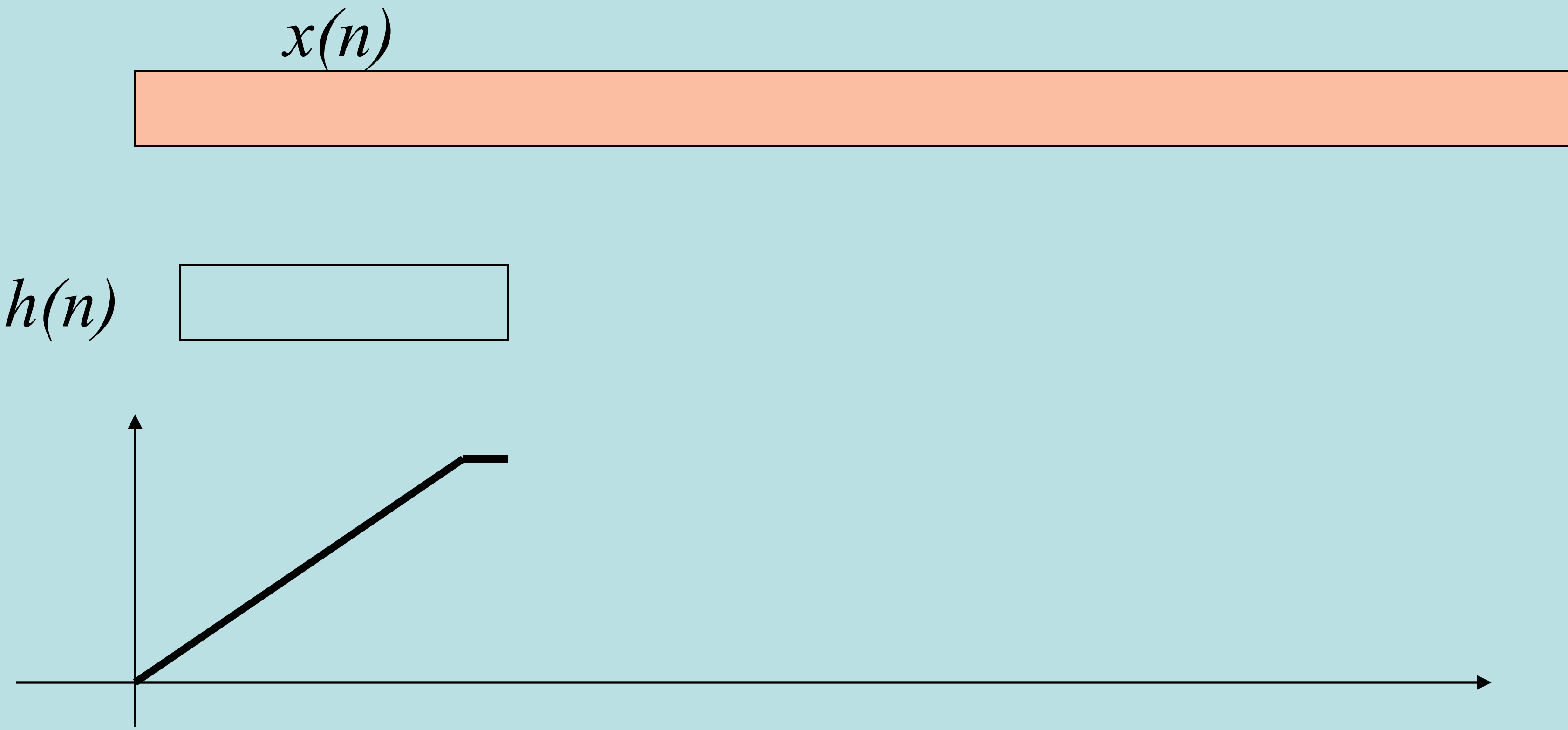
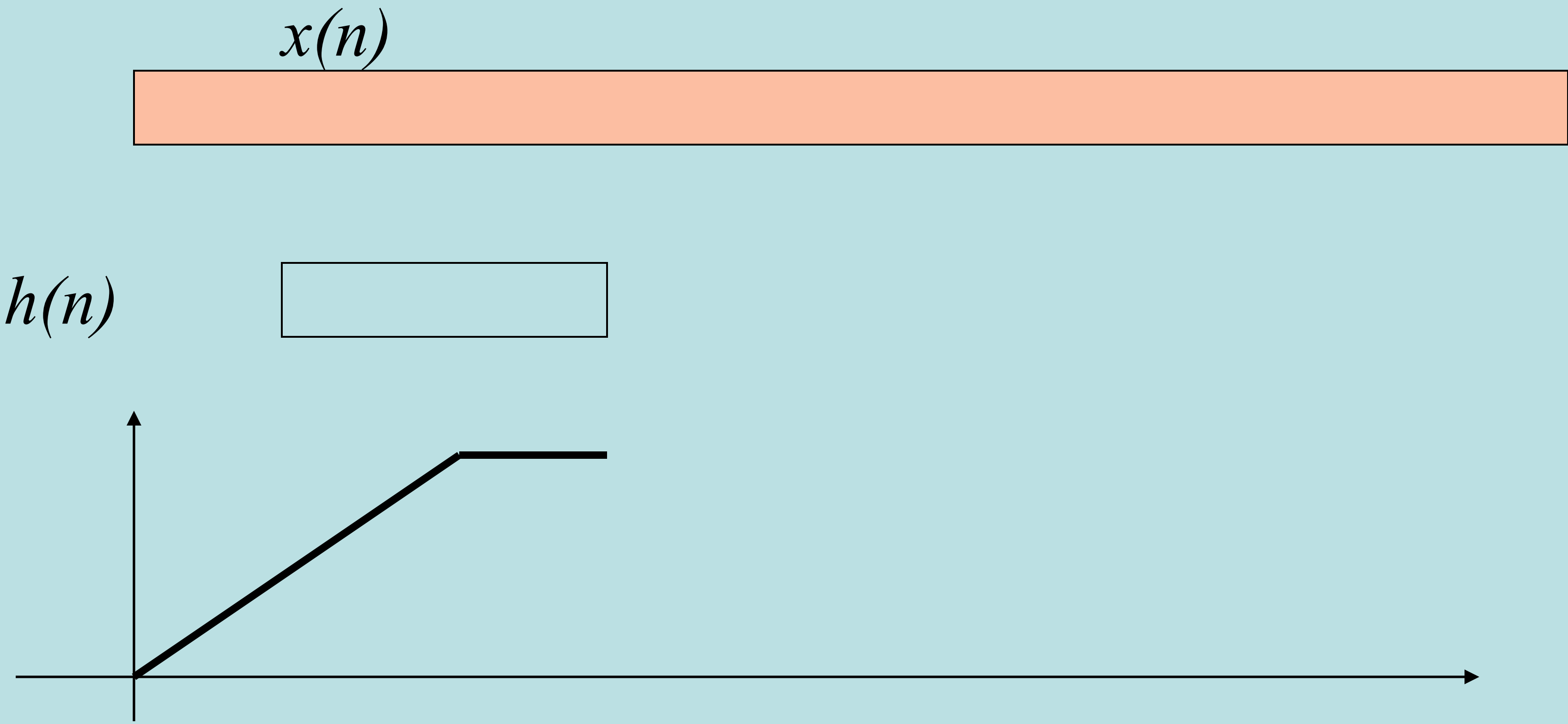


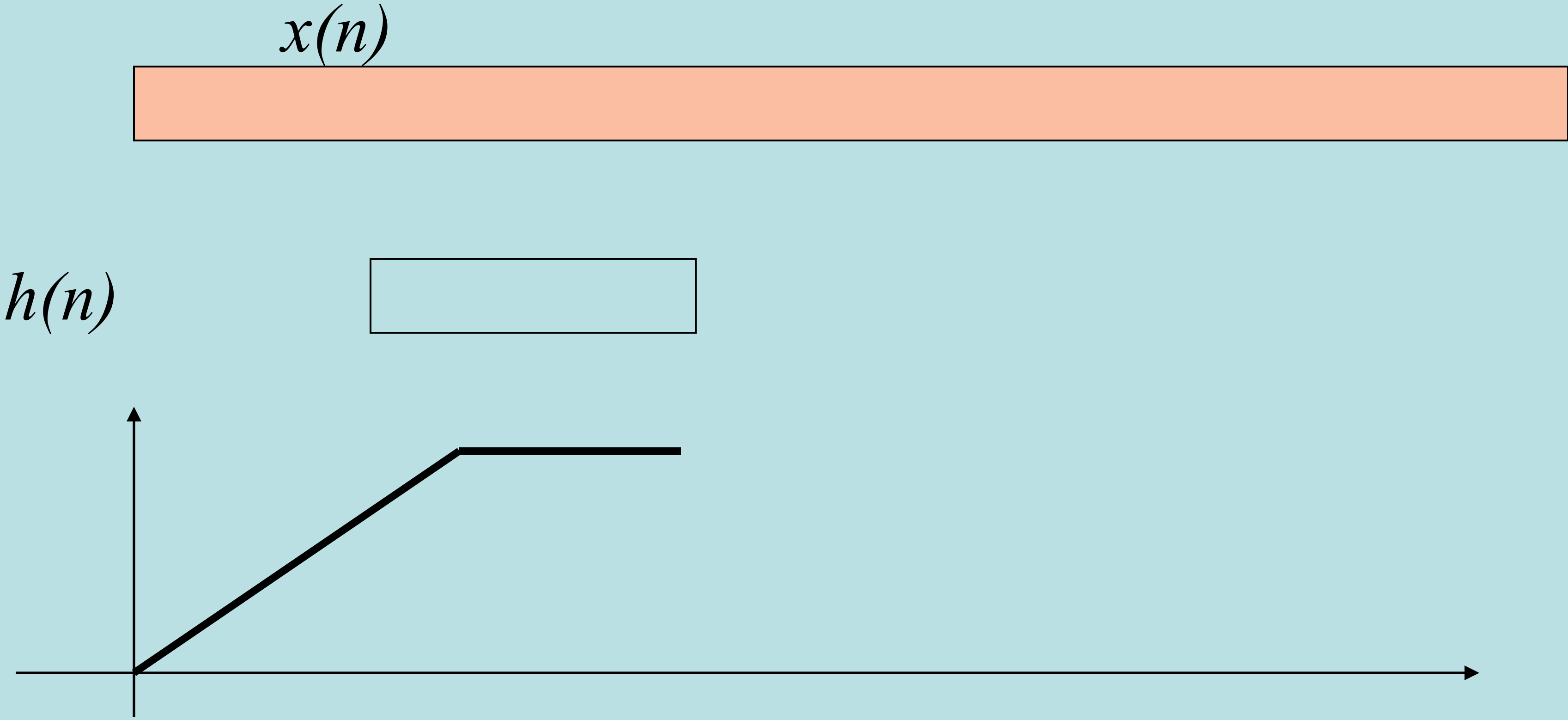
Convolution of long sequences



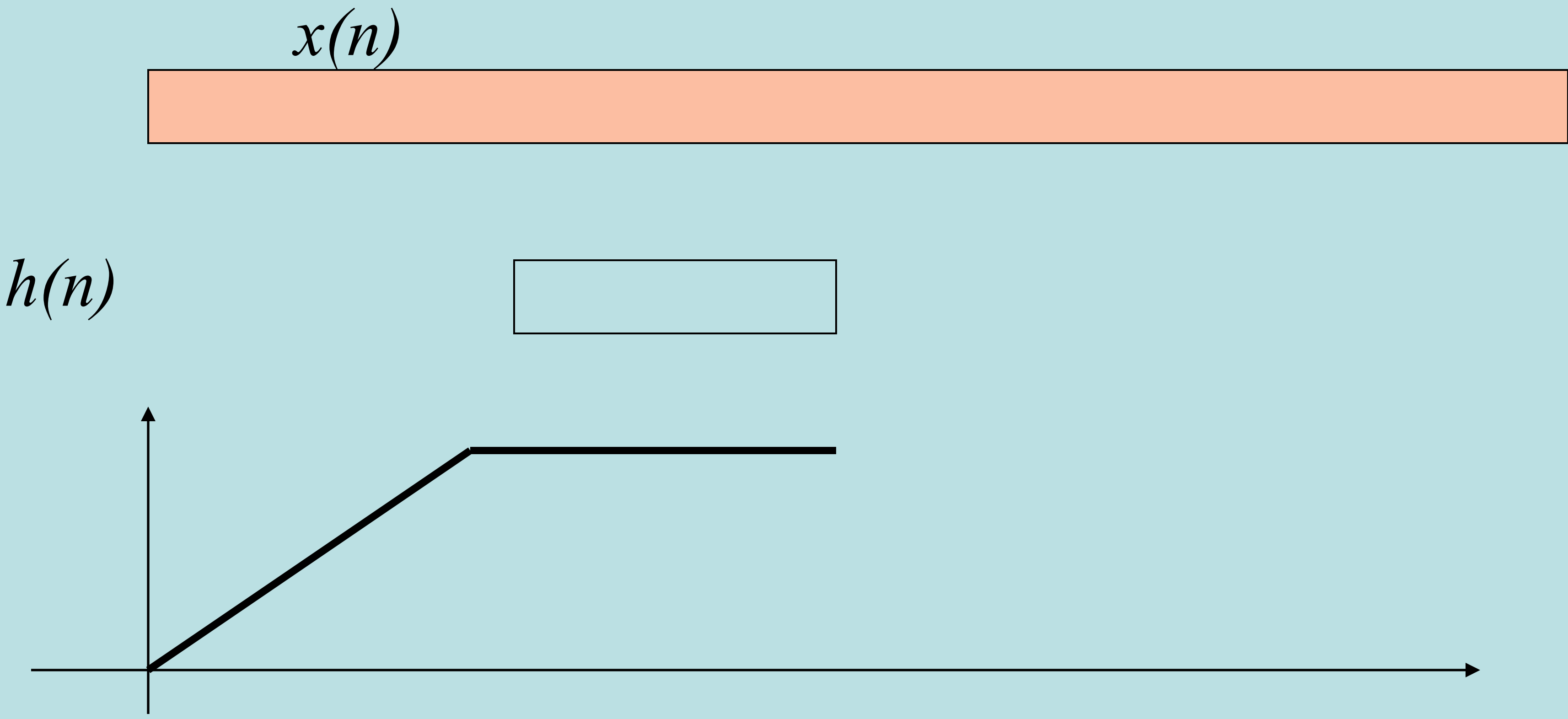
Convolution of long sequences



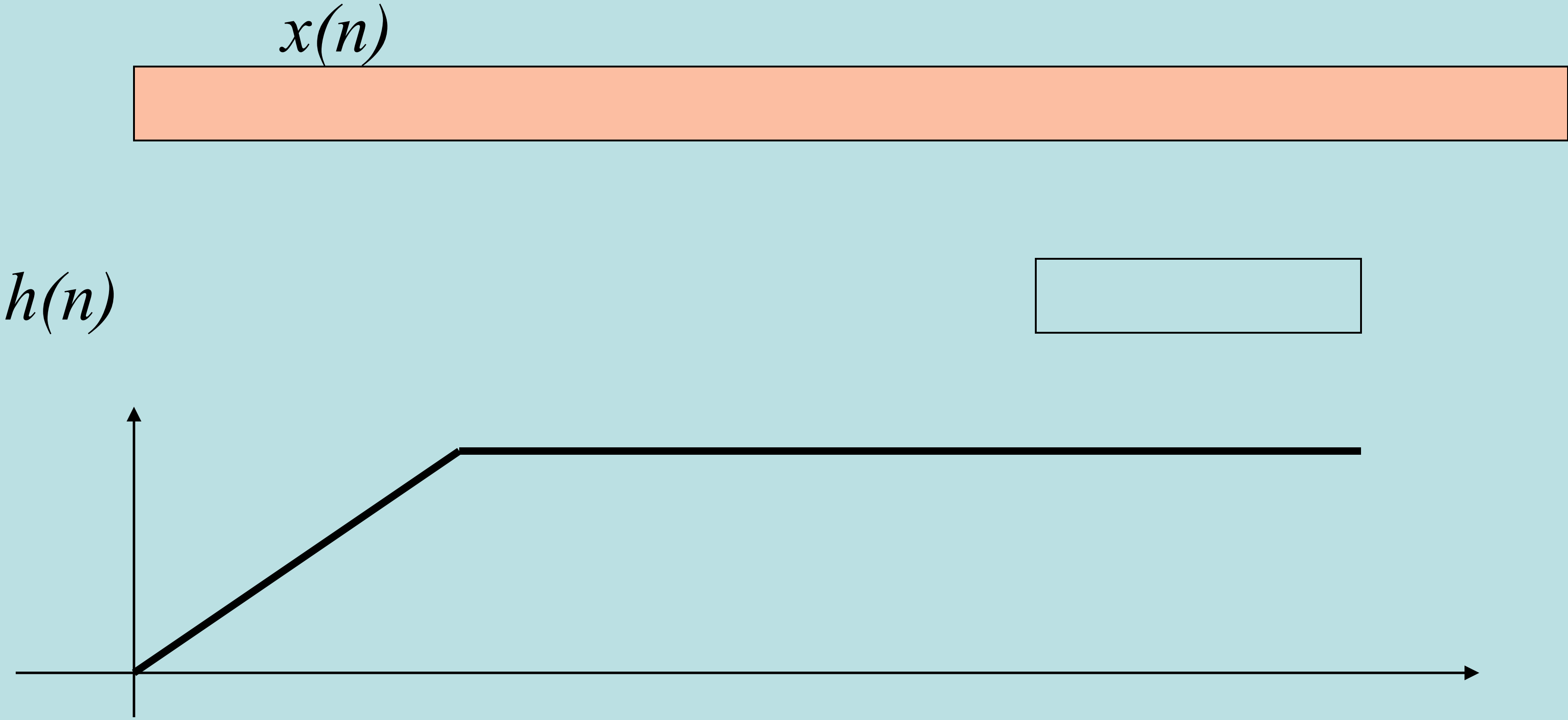
Convolution of long sequences



Convolution of long sequences



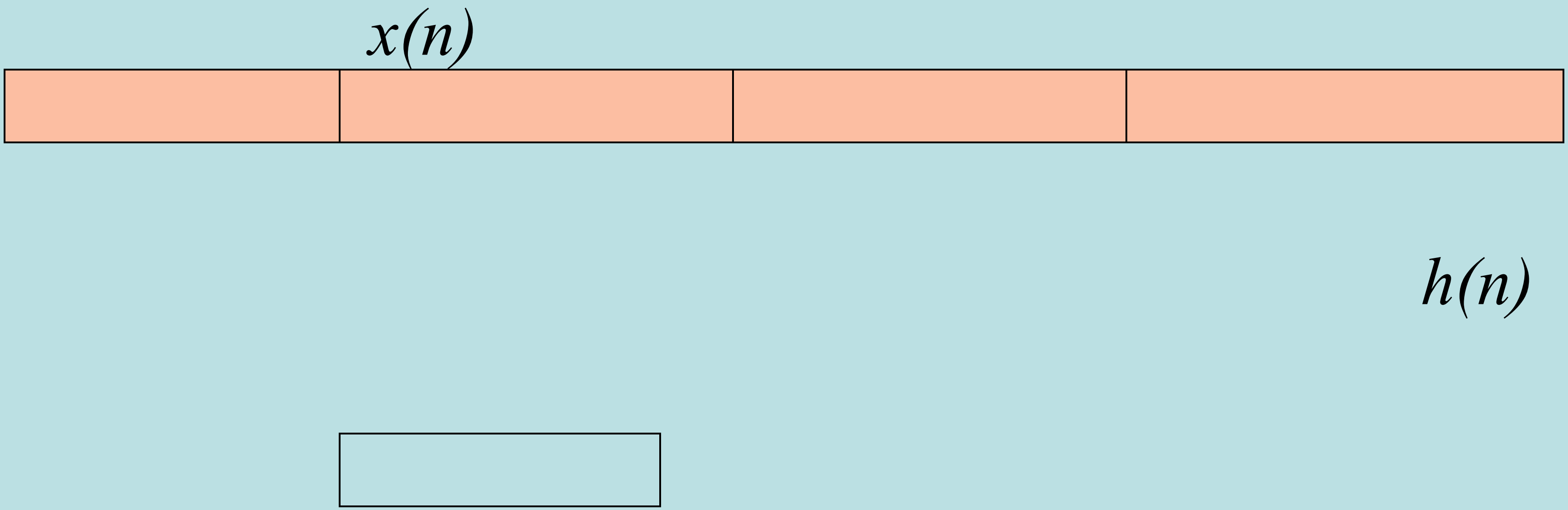
Convolution of long sequences



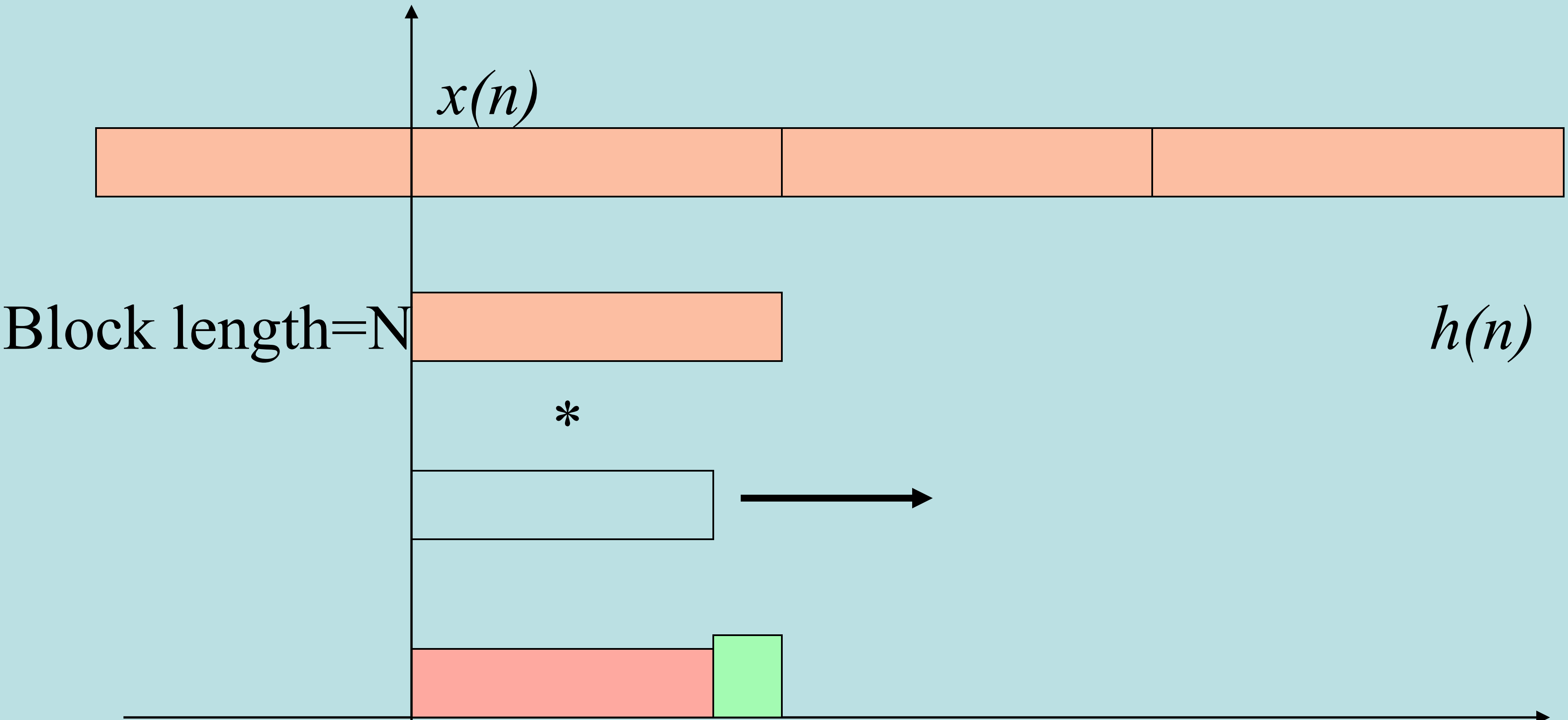
Convolution of Long Sequences --- Block Based approach

- x(n) are divided into blocks;
- convolutions are performed for each block and *h(n)* --- short time convolution;
- construct the output by combining the results of block convolution;
- Issues : how to construct the blocks? How to construct the output?
- Two approaches: overlap-save and overlap-add

Convolution of long sequences

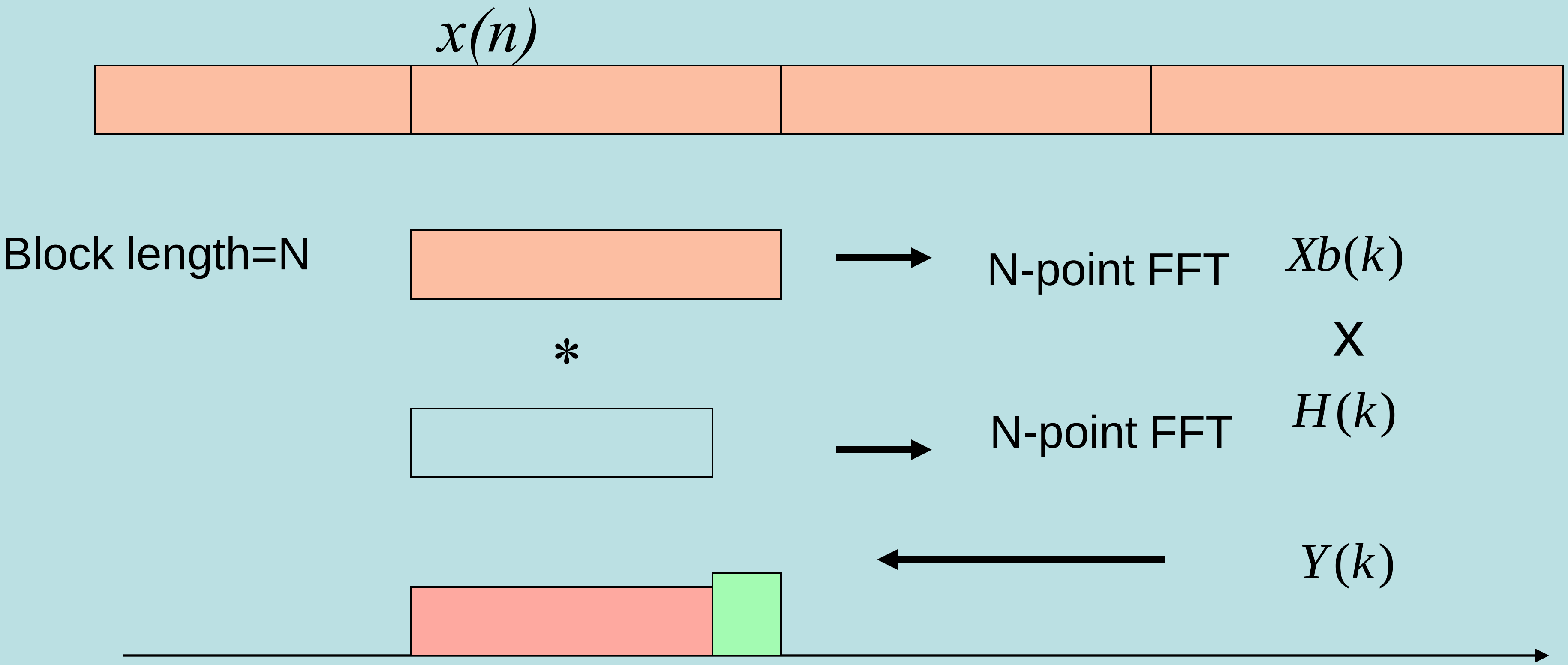


Overlap-save approach



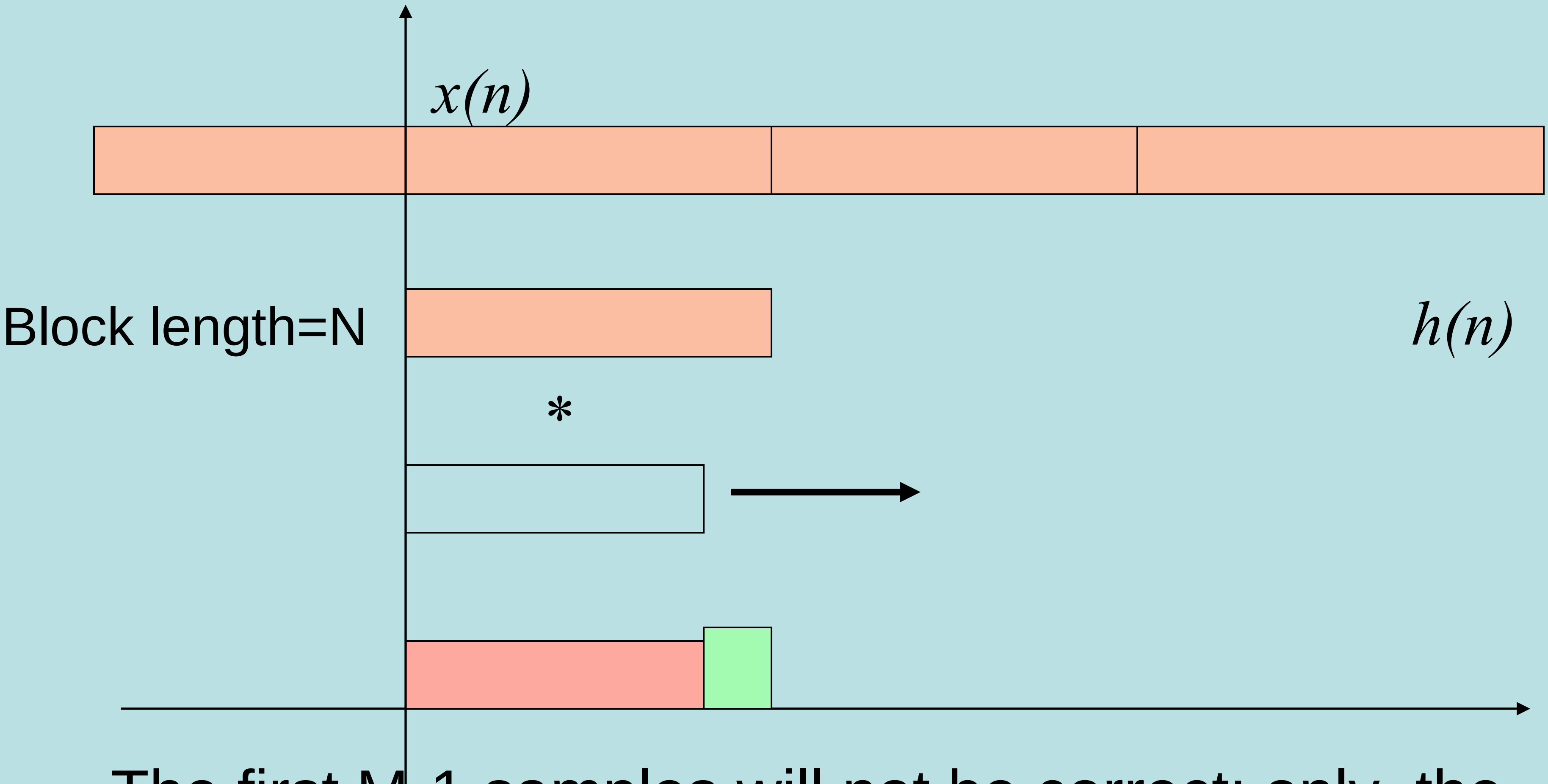
If N-point circular convolution is performed, the first M-1 samples of the result will not be correct; only the last N-M+1 samples are correct;

Overlap-save approach



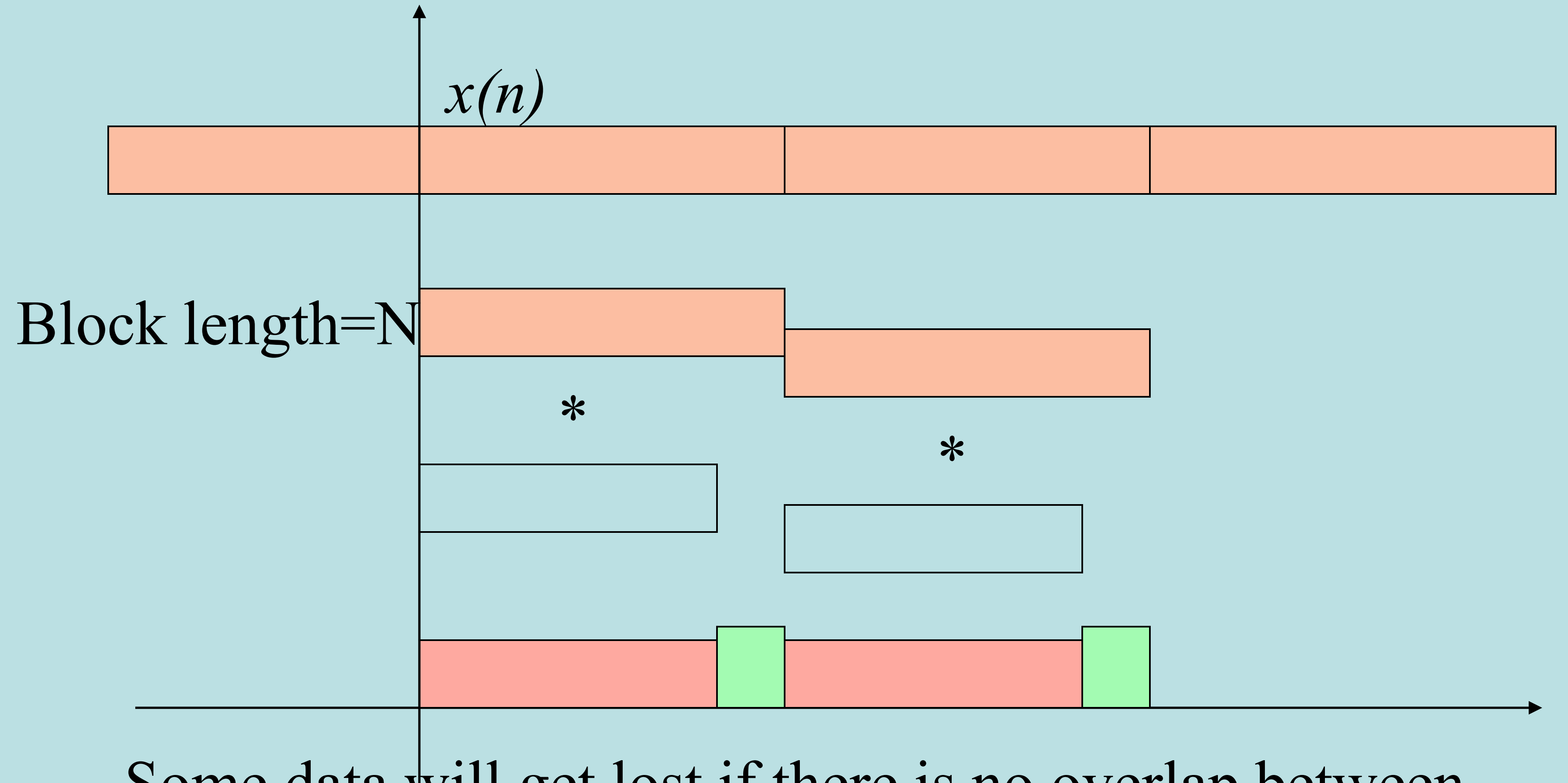
The first (M-1) samples will not be correct; only the (N-M+1) samples are correct;

Overlap-save approach

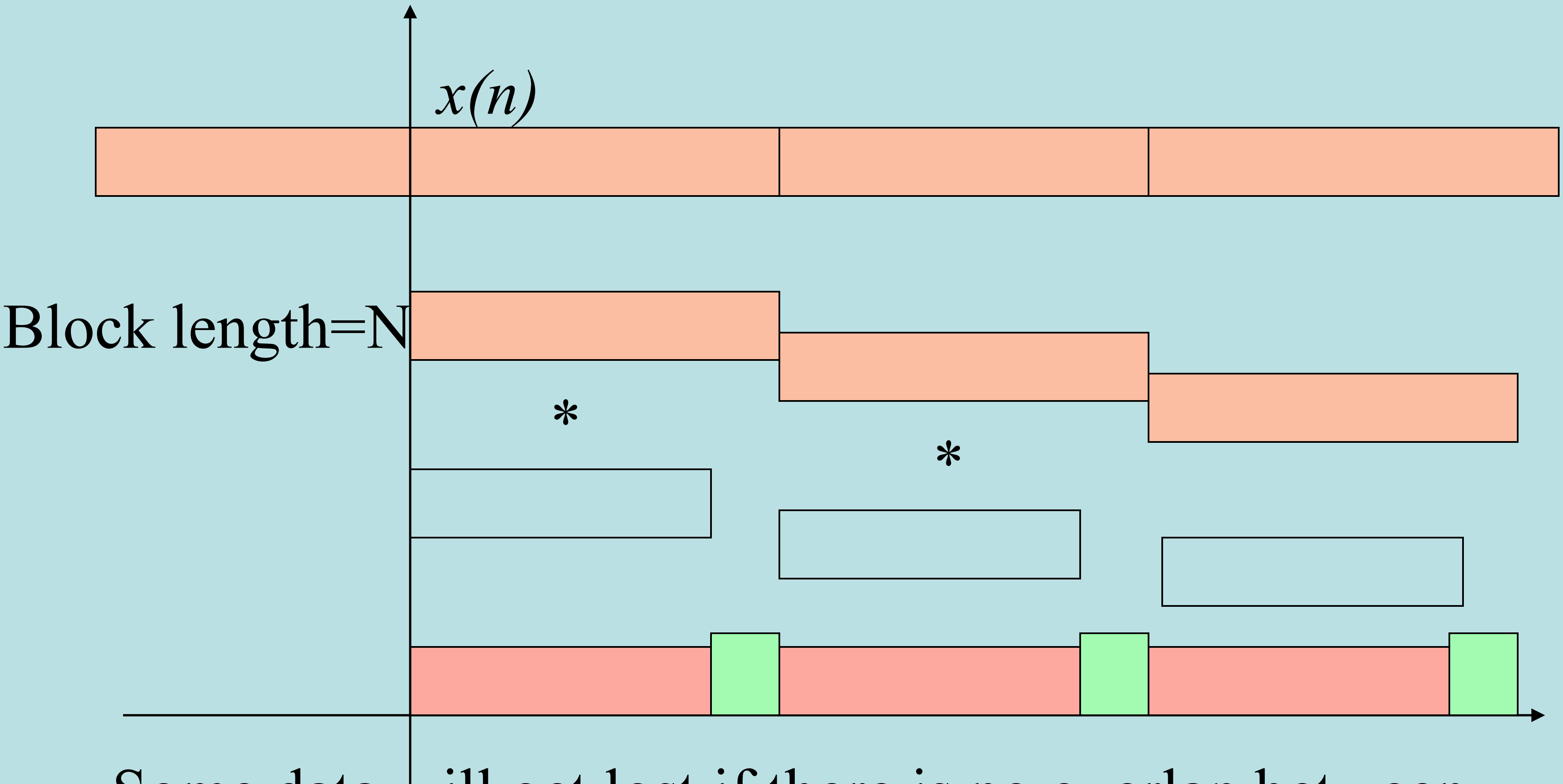


The first M-1 samples will not be correct; only the N-M+1 samples are correct;

Overlap-save approach

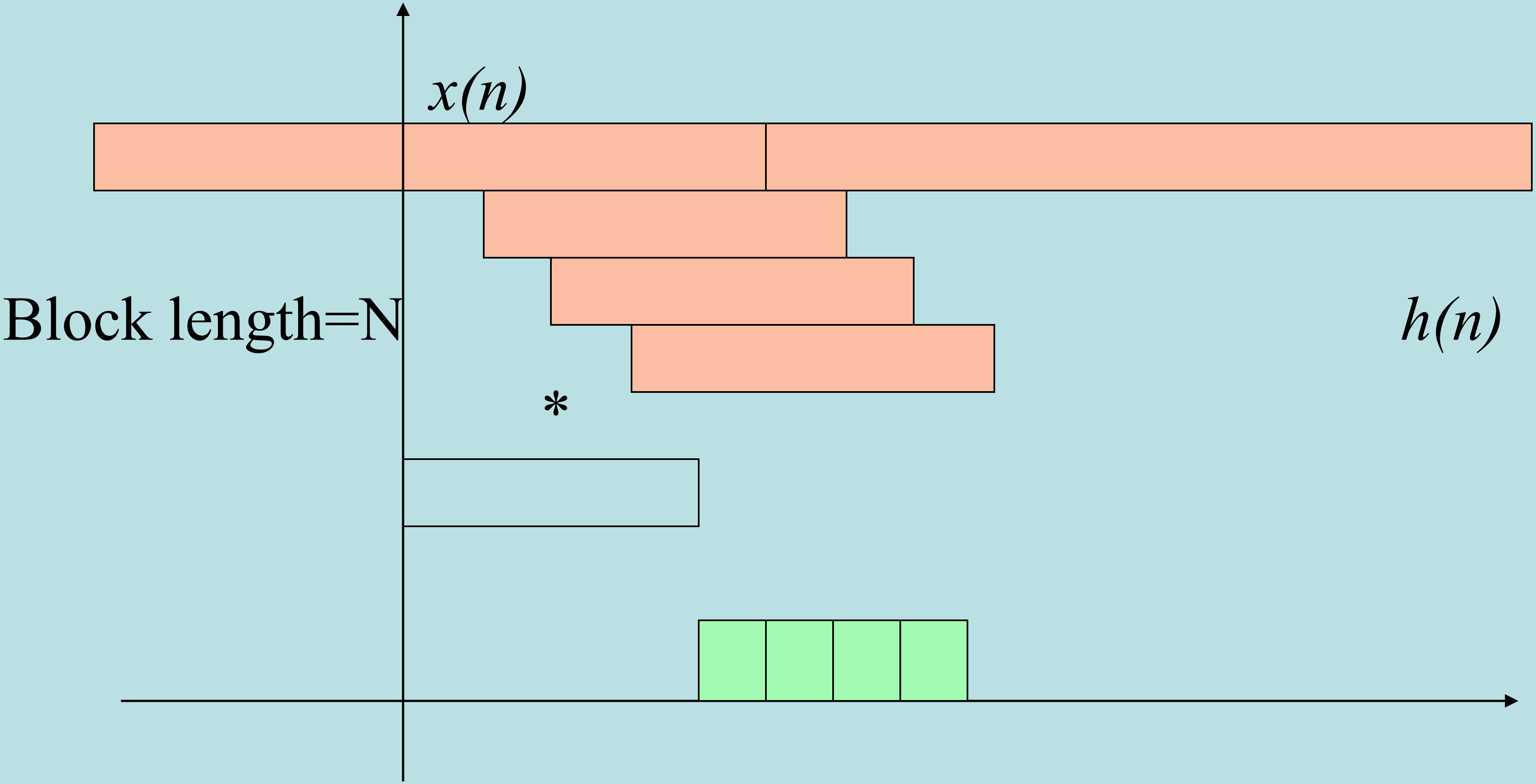


Some data will get lost if there is no overlap between the blocks



Some data will get lost if there is no overlap between the blocks

Overlap-save approach



Overlap-save approach

- The above process is called overlap-save methods:
  - Take N signal samples as a block;
  - do N-point DFT of the block, and N-point DFT of h(n) ($N>M$ the length of h(n));
  - Multiple $X_b(k)$ and $H(k)$;
  - Do the IDFT of $Y(k)$
  - Discard the first $(M-1)$ samples of $y(n)$;

Overlap-save approach

- Get the next block by getting *N-M+1* new samples, and discard (N-M+1) oldest samples
- Repeat the above convolution process.

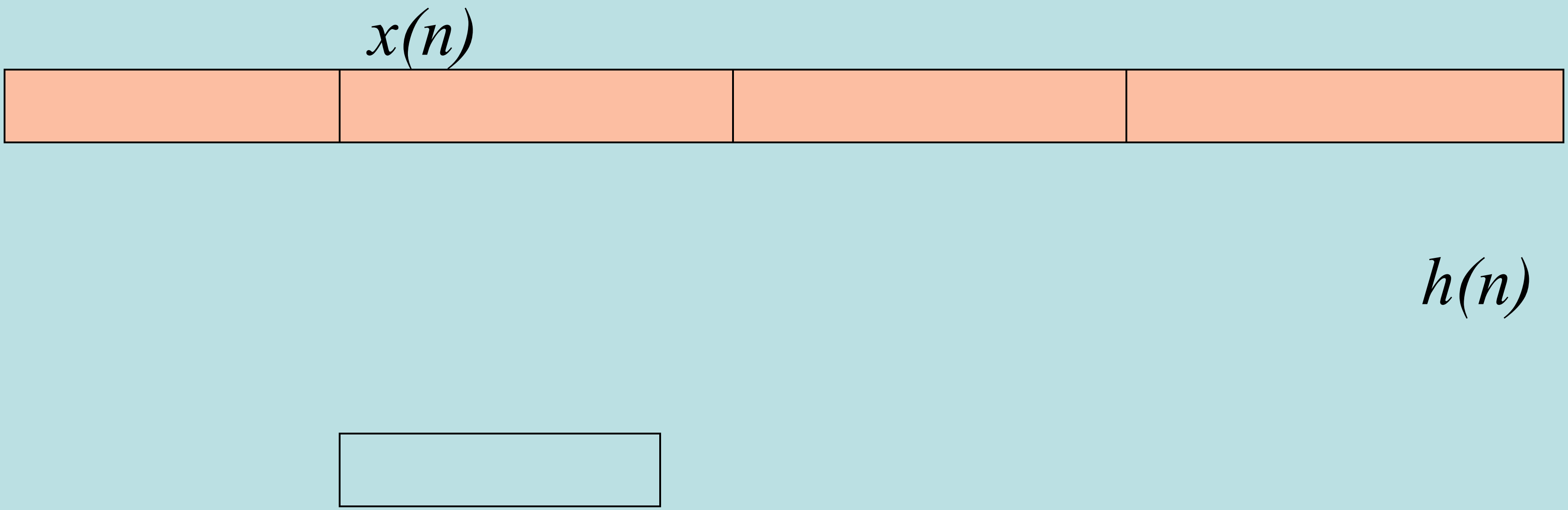
Overlap-save approach--- an example

- Convolve a 50-pint sequence h(n) with a long sequence x(n):
  - 1. Let N=64;
  - 2. taking 64 samples from x(n), perform circular convolution using 64-point FFT. Discard the first 49 samples and keep the last 64-50+1=15 samples;
  - Move the block by getting 15 samples from x(n), repeat step 2 and keep the next 15 samples of the result….
  - Combine all the 15 samples together to get the convolution results

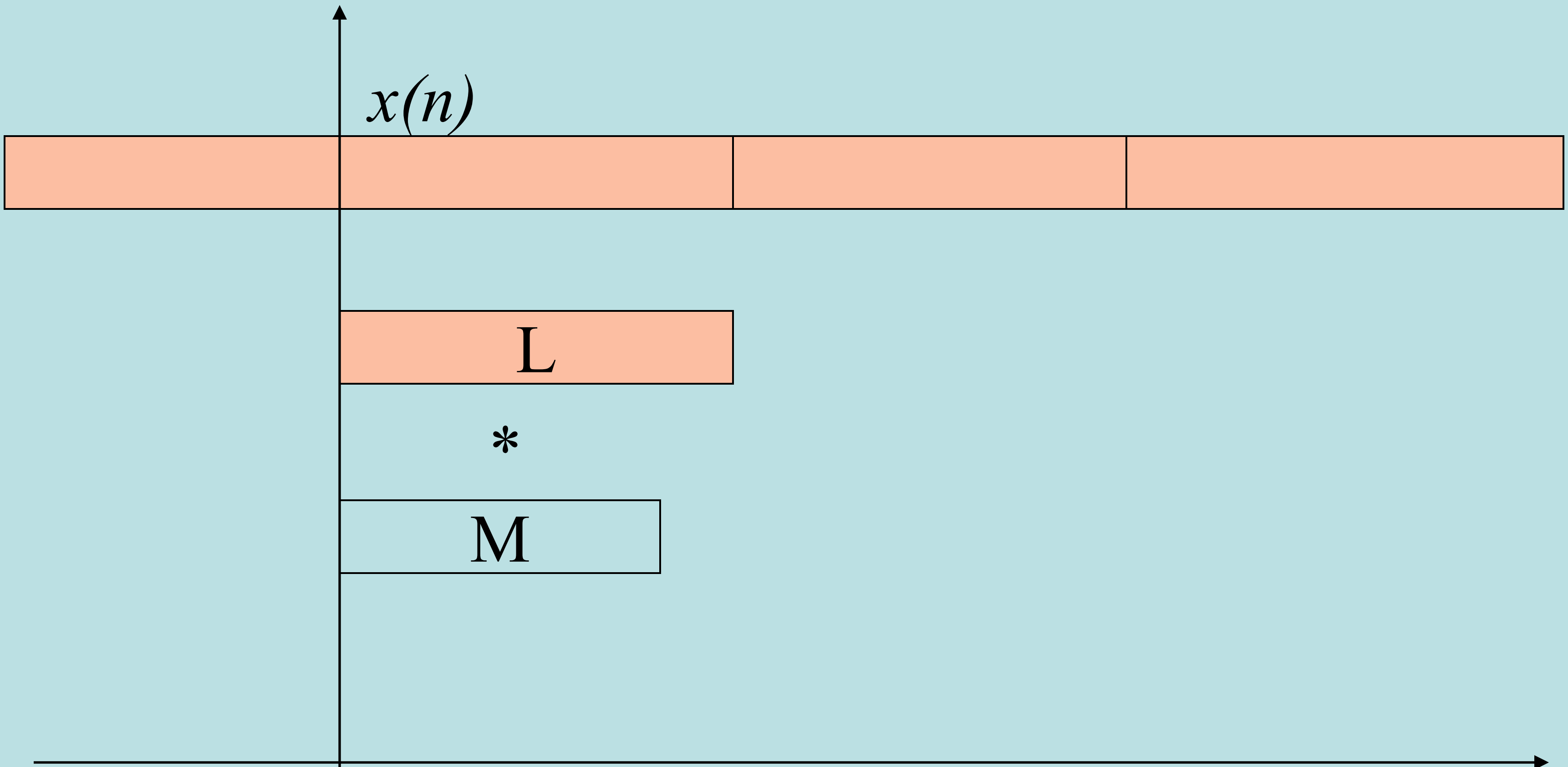
Convolution of Long Sequences --- Overlap-Add Method

- Here we try to use linear convolution instead of circular convolution:
  - Take a block $x_b(n)$ of length L;
  - H(n) is of length M;
  - Take the N-point DFT of them, where $N=L+M-1$
  - Calculate Y(k)=X(k)H(k), k=0, 1, …, N-1
  - Calculate IDFT of Y(k) yield y(n), n=0, 1, …, N-1

Convolution of long sequences

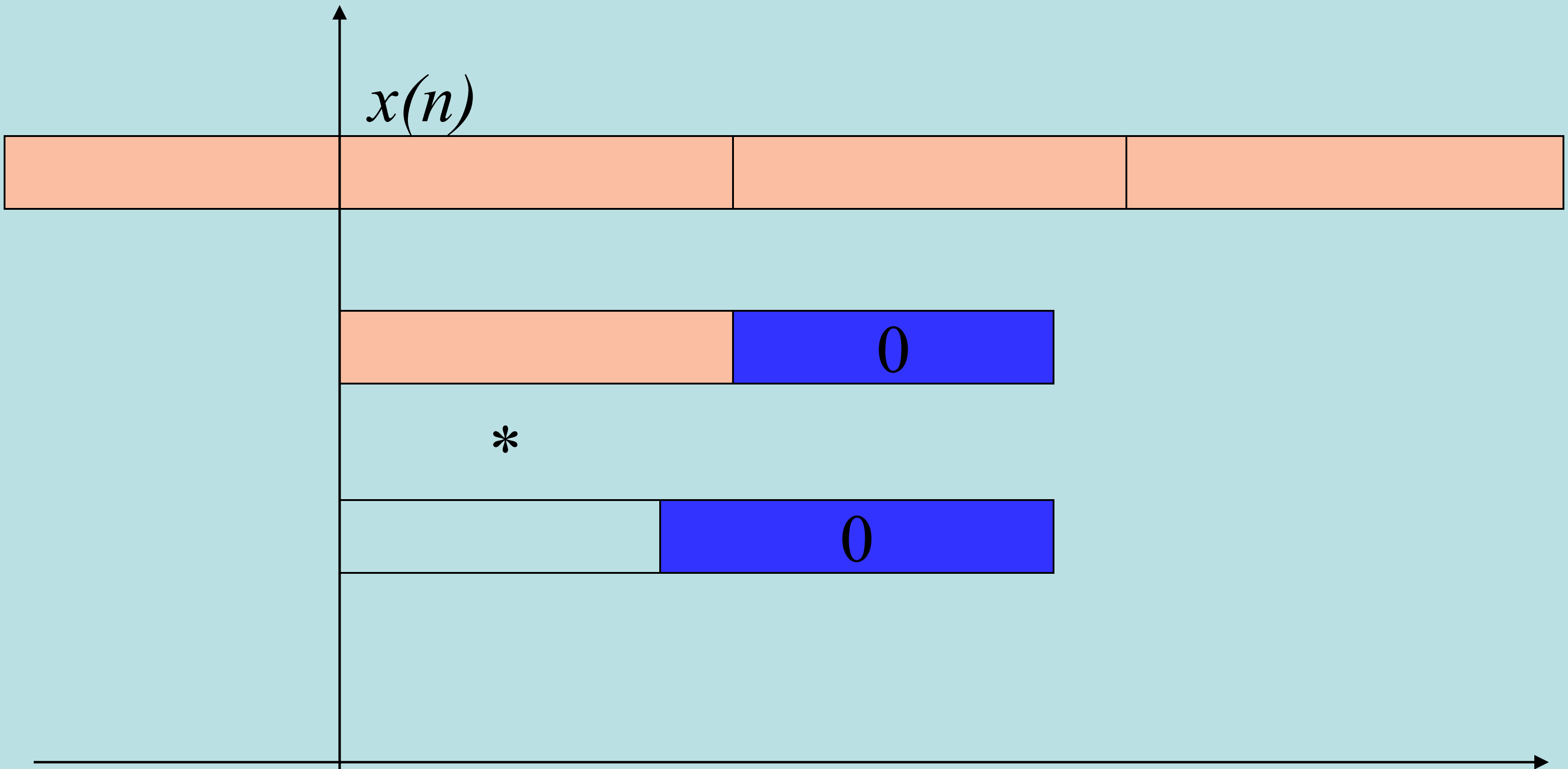


Convolution of long sequences



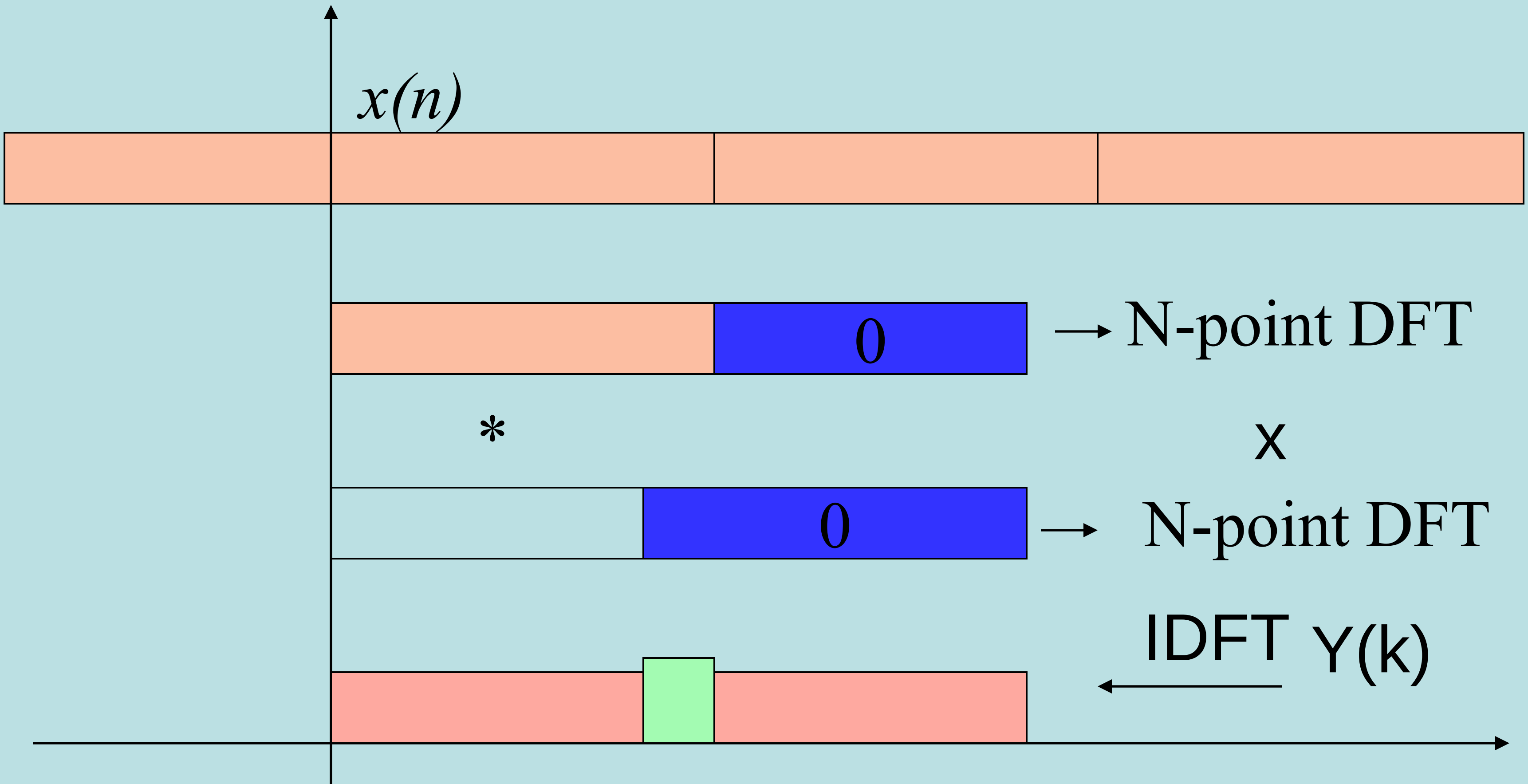
The first M and the last M samples will not be correct; only the N-M samples are correct;

Convolution of long sequences



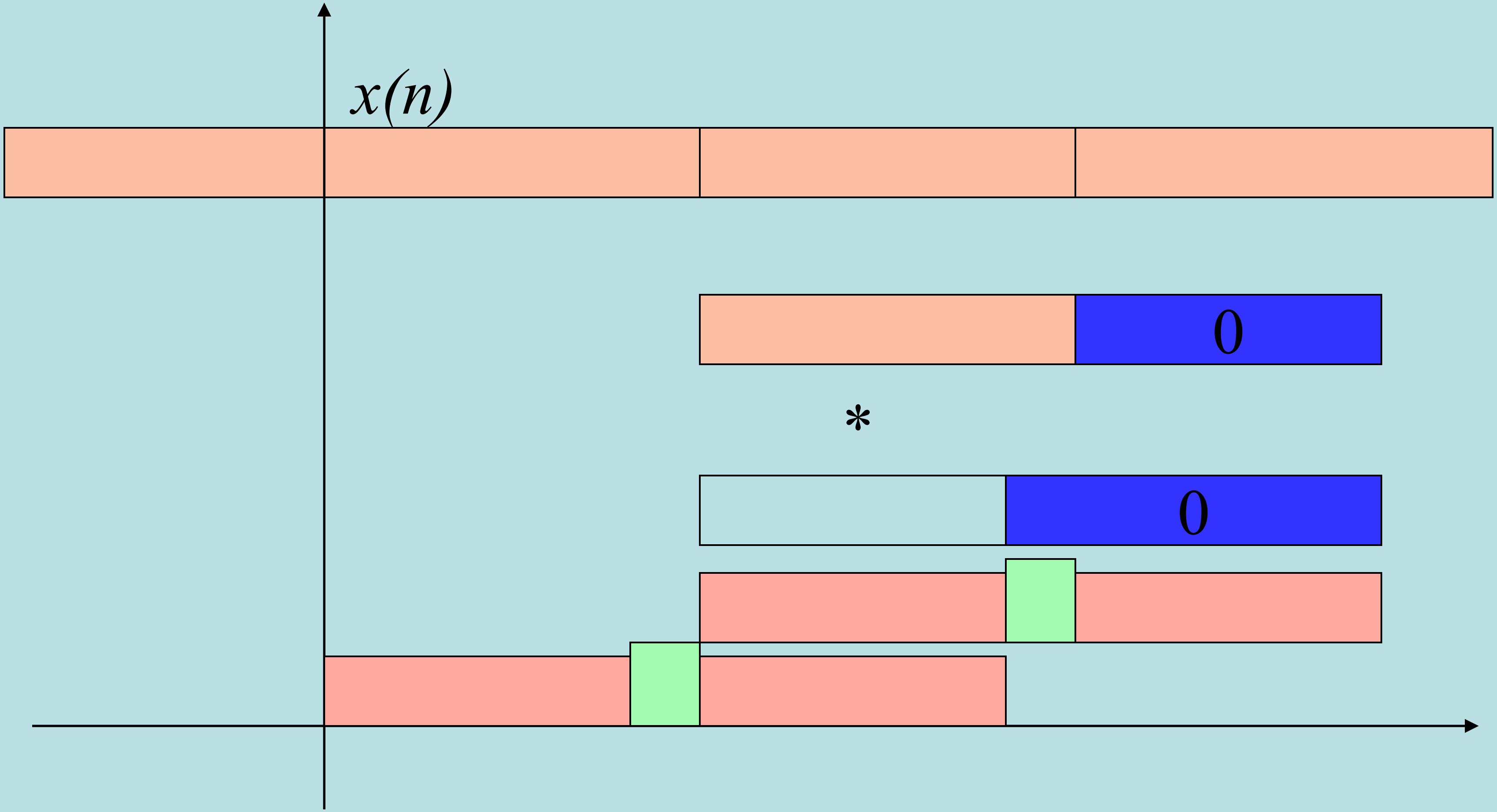
The first M-1 and the last M-1 samples will not be correct; only the N-M samples are correct;

Convolution of long sequences

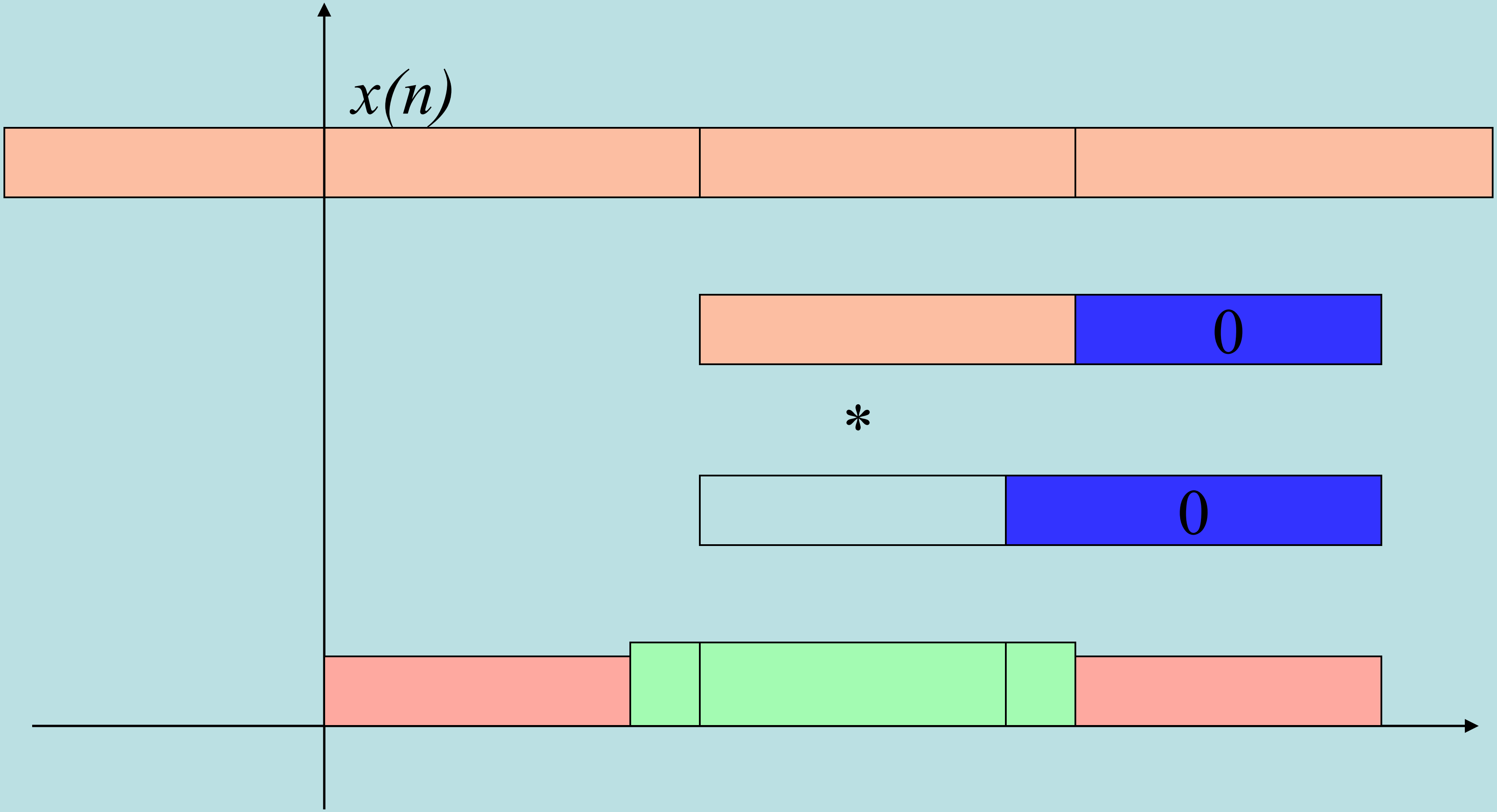


The first M-1 and the last M-1 samples will not be correct; only the N-M+1 samples are correct;

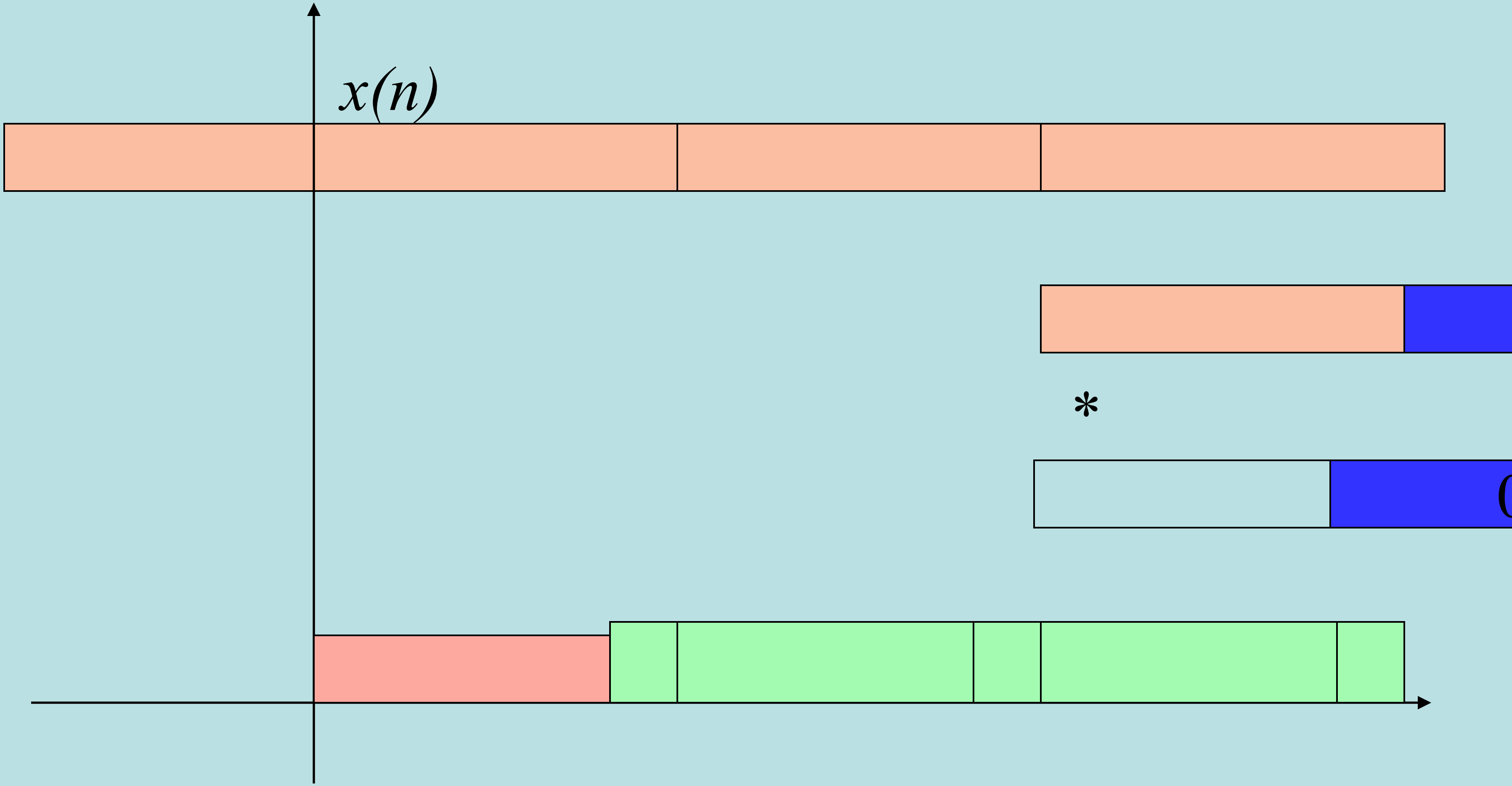
Convolution of long sequences



Convolution of long sequences



Convolution of long sequences



Convolution of Long Sequences --- Overlap-Add Method

- Construct the mth block $x_b(n)$ as:

  $\{x(mL), x(mL+1), \ldots x(mL+L-1), 0, \ldots, 0\} \rightarrow$ Length N
- Take the N-point DFTs of $x_b(n)$ and $h(n)$;
- Multiplication $Y_m(k)=X_b(k)H(k)$
- IDFT: $y(n)=\mathrm{IDFT}(Y(k))$
- Repeat the operation for next block

  $\{x((m+1)L), x((m+1)L+1), \ldots x((m+1)L+L-1), 0, \ldots, 0\}$

  ……….

Convolution of Long Sequences --- Overlap-Add Method

- The last (M-1) points for the first y(n) are overlapped and added to the first (M-1) points of the second y(n);
- The last (M-1) points for the second y(n) are overlapped and added to the first (M-1) points of the third y(n);
- ……
- The above process will result in the convolution of h(n) and x(n)

Summary

- Fast convolution of short sequences
  - Linear convolution
  - Circular convolution
  - When they can be equal?
- Fast convolution of short sequences
  - Overlap-saving (block overlapping, discard some results)
  - Overlap-adding(block separate, overlap and add some results)