Solution of linear system of equations

- Circuit analysis (Mesh and node equations)
- Numerical solution of differential equations (Finite Difference Method)
- Numerical solution of integral equations (Finite Element Method, Method of Moments)

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
\quad \Rightarrow \quad
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

Consistency (Solvability)

- The linear system of equations $Ax=b$ has a solution, or said to be consistent IFF

$$\text{Rank}\{A\}=\text{Rank}\{A|b\}$$

- A system is inconsistent when

$$\text{Rank}\{A\}<\text{Rank}\{A|b\}$$

Rank{A} is the maximum number of linearly independent columns or rows of A. Rank can be found by using ERO (Elementary Row Oparations) or ECO (Elementary column operations).

ERO$\Rightarrow$# of rows with at least one nonzero entry
ECO$\Rightarrow$# of columns with at least one nonzero entry

Solution Techniques

- Direct solution methods
  - Finds a solution in a finite number of operations by transforming the system into an <u>equivalent system</u> that is ‘easier’ to solve.
  - Diagonal, upper or lower triangular systems are easier to solve
  - Number of operations is a function of system size n.
- Iterative solution methods
  - Computes succesive approximations of the solution vector for a given A and b, starting from an initial point x_0.
  - Total number of operations is uncertain, may not converge.

Direct solution Methods

- Gaussian Elimination
  - By using ERO, matrix A is transformed into an upper triangular matrix (all elements below diagonal 0)
  - Back substitution is used to solve the upper-triangular system

$$\begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ii} & \cdots & a_{in} \\ \vdots & & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix} \underset{\text{ERO}}{\Rightarrow} \begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & \tilde{a}_{ii} & \cdots & \tilde{a}_{in} \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \tilde{a}_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ \tilde{b}_i \\ \vdots \\ \tilde{b}_n \end{bmatrix} \Big\uparrow \text{Back substitution}$$

First step of elimination

Pivotal element

$$\begin{bmatrix} \boxed{a_{11}^{(1)}} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} \\ a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & \cdots & a_{3n}^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1}^{(1)} & a_{n2}^{(1)} & a_{n3}^{(1)} & \cdots & a_{nn}^{(1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(1)} \\ b_3^{(1)} \\ \vdots \\ b_n^{(1)} \end{bmatrix}$$

$$\begin{aligned} m_{2,1} &= a_{21}^{(1)} / a_{11}^{(1)} \\ m_{3,1} &= a_{31}^{(1)} / a_{11}^{(1)} \\ &\vdots \\ m_{n,1} &= a_{n1}^{(1)} / a_{11}^{(1)} \end{aligned} \qquad \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(2)} & a_{n3}^{(2)} & \cdots & a_{nn}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(2)} \\ b_3^{(2)} \\ \vdots \\ b_n^{(2)} \end{bmatrix}$$

Second step of elimination

Pivotal element

$$\begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & \boxed{a_{22}^{(2)}} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(2)} & a_{n3}^{(2)} & \cdots & a_{nn}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(2)} \\ b_3^{(2)} \\ \vdots \\ b_n^{(2)} \end{bmatrix}$$

$$m_{3,2} = a_{32}^{(2)} / a_{22}^{(2)}$$

$$\vdots$$

$$m_{n,2} = a_{n2}^{(2)} / a_{22}^{(2)}$$

$$\begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \cdots & a_{3n}^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{n3}^{(3)} & \cdots & a_{nn}^{(3)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(2)} \\ b_3^{(3)} \\ \vdots \\ b_n^{(3)} \end{bmatrix}$$

Gaussion elimination algorithm

Define number of steps as p (pivotal row)

For p=1,n-1

For r=p+1 to n

$$m_{r,p} = a_{rp}^{(p)} / a_{pp}^{(p)}$$

$$a_{rp}^{(p)} = 0$$

$$b_r^{(p+1)} = b_r^{(p)} - m_{r,p} \times b_p^{(p)}$$

For c=p+1 to n

$$a_{rc}^{(p+1)} = a_{rc}^{(p)} - m_{r,p} \times a_{pc}^{(p)}$$

Back substitution algorithm

$$\begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \cdots & a_{3n}^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_{n-1\,n-1}^{(n)} & a_{n-1\,n}^{(n)} \\ 0 & 0 & 0 & 0 & a_{nn}^{(n)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(2)} \\ b_3^{(3)} \\ \vdots \\ b_{n-1}^{(n-1)} \\ b_n^{(n)} \end{bmatrix}$$

$$x_n = \frac{b_n^{(n)}}{a_{nn}^{(n)}} \qquad x_{n-1} = \frac{1}{a_{n-1n-1}^{(n-1)}}\left[b_{n-1}^{(n-1)} - a_{n-1n}^{n-1} x_n\right]$$

$$x_i = \frac{1}{a_{ii}^{(i)}}\left[b_i^{(i)} - \sum_{k=i+1}^{n} a_{ik}^{(i)} x_k\right] \qquad i = n-1, n-2, \ldots, 1$$

Operation count

- Number of arithmetic operations required by the algorithm to complete its task.
- Generally only multiplications and divisions are counted
- Elimination process $\frac{n^3}{3}+\frac{n^2}{2}-\frac{5n}{6}$
- Back substitution $\frac{n^2+n}{2}$
- Total $\frac{n^3}{3}+n^2-\frac{n}{3}$

Dominates
Not efficient for different RHS vectors

LU Decomposition

$A=LU$

$Ax=b \Rightarrow LUx=b$

Define $Ux=y$

$Ly=b$ Solve y by forward substitution

ERO's must be performed on b as well as A

The information about the ERO's are stored in L

Indeed y is obtained by applying ERO's to b vector

$Ux=y$ Solve x by backward substitution

LU Decomposition by Gaussian elimination

There are infinitely many different ways to decompose A.
Most popular one: U=Gaussian eliminated matrix
L=Multipliers used for elimination

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ m_{2,1} & 1 & 0 & \cdots & 0 & 0 \\ m_{3,1} & m_{3,2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & 0 \\ m_{n-1,1} & m_{n-1,2} & m_{n-1,3} & \cdots & 1 & \vdots \\ m_{n,1} & m_{n,2} & m_{n,3} & m_{n,4} & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \cdots & a_{3n}^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_{n-1n-1}^{(n)} & a_{n-1n}^{(n)} \\ 0 & 0 & 0 & 0 & a_{nn}^{(n)} \end{bmatrix}$$

Compact storage: The diagonal entries of L matrix are all 1's, they don't need to be stored. LU is stored in a single matrix.

Operation count

- A=LU Decomposition $\frac{n^3}{3} - \frac{n}{3}$ Done only once
- Ly=b forward substitution $\frac{n^2 - n}{2}$
- Ux=y backward substitution $\frac{n^2 + n}{2}$
- Total $\frac{n^3}{3} + n^2 - \frac{n}{3}$
- For different RHS vectors, the system can be efficiently solved.

Pivoting

- Computer uses *finite-precision* arithmetic
- A small error is introduced in each arithmetic operation, *error propagates*
- When the pivotal element is very small, the multipliers will be large.
- Adding numbers of widely differening magnitude can lead to *loss of significance*.
- To reduce error, row interchanges are made to *maximise* the magnitude of *the pivotal element*

Example: Without Pivoting

4-digit arithmetic
$$\begin{bmatrix} 1.133 & 5.281 \\ 24.14 & -1.210 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.414 \\ 22.93 \end{bmatrix}$$

$$m_{21} = \frac{24.14}{1.133} = 21.31 \qquad \begin{bmatrix} 1.133 & 5.281 \\ 0.000 & -113.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.414 \\ -113.8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.9956 \\ 1.001 \end{bmatrix}$$

Loss of significance

Example: With Pivoting

$$\begin{bmatrix} 24.14 & -1.210 \\ 1.133 & 5.281 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 22.93 \\ 6.414 \end{bmatrix}$$

$$m_{21} = \frac{1.133}{24.14} = 0.04693 \qquad \begin{bmatrix} 24.14 & -1.210 \\ 0.000 & 5.338 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 22.93 \\ 5.338 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.000 \\ 1.000 \end{bmatrix}$$

Pivoting procedures

Eliminated part

$$
\begin{bmatrix}
a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1i}^{(1)} & \cdots & a_{1j}^{(1)} & \cdots & a_{1n}^{(1)} \\
0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2i}^{(2)} & \cdots & a_{2j}^{(2)} & \cdots & a_{2n}^{(2)} \\
0 & 0 & a_{33}^{(3)} & \cdots & a_{3i}^{(3)} & \cdots & a_{3j}^{(3)} & \cdots & a_{3n}^{(3)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{ii}^{(i)} & \cdots & a_{ij}^{(i)} & \cdots & a_{in}^{(i)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{ji}^{(i)} & \cdots & a_{jj}^{(i)} & \cdots & a_{jn}^{(i)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{ni}^{(i)} & \cdots & a_{nj}^{(i)} & \cdots & a_{nn}^{(i)}
\end{bmatrix}
$$

Pivotal row

Pivotal column

Row pivoting

- Most commonly used *partial pivoting* procedure
- Search the pivotal column
- Find the largest element in magnitude
- Then switch this row with the pivotal row

Row pivoting

$$\begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1i}^{(1)} & \cdots & a_{1j}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2i}^{(2)} & \cdots & a_{2j}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \cdots & a_{3i}^{(3)} & \cdots & a_{3j}^{(3)} & \cdots & a_{3n}^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{ii}^{(i)} & \cdots & a_{ij}^{(i)} & \cdots & a_{in}^{(i)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{ji}^{(i)} & \cdots & a_{jj}^{(i)} & \cdots & a_{jn}^{(i)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{ni}^{(i)} & \cdots & a_{nj}^{(i)} & \cdots & a_{nn}^{(i)} \end{bmatrix}$$

Interchange these rows

Largest in magnitude

Column pivoting

$$
\begin{bmatrix}
a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1i}^{(1)} & \cdots & a_{1j}^{(1)} & \cdots & a_{1n}^{(1)} \\
0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2i}^{(2)} & \cdots & a_{2j}^{(2)} & \cdots & a_{2n}^{(2)} \\
0 & 0 & a_{33}^{(3)} & \cdots & a_{3i}^{(3)} & \cdots & a_{3j}^{(3)} & \cdots & a_{3n}^{(3)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{ii}^{(i)} & \cdots & a_{ij}^{(i)} & \cdots & a_{in}^{(i)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{ji}^{(i)} & \cdots & a_{jj}^{(i)} & \cdots & a_{jn}^{(i)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{ni}^{(i)} & \cdots & a_{nj}^{(i)} & \cdots & a_{nn}^{(i)}
\end{bmatrix}
$$

Largest in magnitude

Interchange these columns

Complete pivoting

$$
\begin{bmatrix}
a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1i}^{(1)} & \cdots & a_{1j}^{(1)} & \cdots & a_{1n}^{(1)} \\
0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2i}^{(2)} & \cdots & a_{2j}^{(2)} & \cdots & a_{2n}^{(2)} \\
0 & 0 & a_{33}^{(3)} & \cdots & a_{3i}^{(3)} & \cdots & a_{3j}^{(3)} & \cdots & a_{3n}^{(3)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{ii}^{(i)} & \cdots & a_{ij}^{(i)} & \cdots & a_{in}^{(i)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{ji}^{(i)} & \cdots & a_{jj}^{(i)} & \cdots & a_{jn}^{(i)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{ni}^{(i)} & \cdots & a_{nj}^{(i)} & \cdots & a_{nn}^{(i)}
\end{bmatrix}
$$

Interchange these rows

Largest in magnitude

Interchange these columns

Row Pivoting in LU Decomposition

- When two rows of A are interchanged, those rows of b should also be interchanged.
- Use a pivot vector. Initial pivot vector is integers from 1 to n.
- When two rows (i and j) of A are interchanged, apply that to pivot vector.

$$p = \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ i \\ \vdots \\ j \\ \vdots \\ n \end{bmatrix} \qquad p = \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ j \\ \vdots \\ i \\ \vdots \\ n \end{bmatrix}$$

Example

$$A = \begin{bmatrix} \boxed{\begin{matrix} 0 \\ -4 \\ 1 \end{matrix}} & \begin{matrix} 3 \\ -2 \\ 4 \end{matrix} & \begin{matrix} 2 \\ 1 \\ -2 \end{matrix} \end{bmatrix} \quad b = \begin{bmatrix} 12 \\ -5 \\ 3 \end{bmatrix} \quad p = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Column search: Maximum magnitude second row
Interchange 1^{st} and 2^{nd} rows

$$A' = \begin{bmatrix} -4 & -2 & 1 \\ 0 & 3 & 2 \\ 1 & 4 & -2 \end{bmatrix} \quad p = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Example continued...

$$A' = \begin{bmatrix} -4 & -2 & 1 \\ 0 & 3 & 2 \\ 1 & 4 & -2 \end{bmatrix} \quad p = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Eliminate a_{21} and a_{31} by using a_{11} as pivotal element

A=LU in compact form (in a single matrix)

$$A' = \begin{bmatrix} -4 & -2 & 1 \\ \boxed{\begin{matrix} 0 \\ -0.25 \end{matrix}} & \begin{matrix} 3 \\ 3.5 \end{matrix} & \begin{matrix} 2 \\ -1.75 \end{matrix} \end{bmatrix} \quad p = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Multipliers (L matrix)

Example continued...

$$A' = \begin{bmatrix} -4 & -2 & 1 \\ 0 & \boxed{3} & 2 \\ -0.25 & \boxed{3.5} & -1.75 \end{bmatrix} \quad p = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Column search: Maximum magnitude at the third row
Interchange 2nd and 3rd rows

$$A' = \begin{bmatrix} -4 & -2 & 1 \\ -0.25 & 3.5 & -1.75 \\ 0 & 3 & 2 \end{bmatrix} \quad p = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

Example continued...

$$A' = \begin{bmatrix} -4 & -2 & 1 \\ -0.25 & 3.5 & -1.75 \\ 0 & 3 & 2 \end{bmatrix} \quad p = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

Eliminate a_{32} by using a_{22} as pivotal element

$$A' = \begin{bmatrix} -4 & -2 & 1 \\ -0.25 & 3.5 & -1.75 \\ 0 & 3/3.5 & 3.5 \end{bmatrix} \quad p = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

Multipliers (L matrix)

Example continued...

$$A' = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0 & 3/3.5 & 1 \end{bmatrix} \begin{bmatrix} -4 & -2 & 1 \\ 0 & 3.5 & -1.75 \\ 0 & 0 & 3.5 \end{bmatrix} \quad p = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$p = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \quad b = \begin{bmatrix} 12 \\ -5 \\ 3 \end{bmatrix} \Rightarrow b' = \begin{bmatrix} -5 \\ 3 \\ 12 \end{bmatrix}$$

A'x=b'   LUx=b'

Ux=y

Ly=b'

Example continued…

Ly=b'

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0 & 3/3.5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 12 \end{bmatrix} \xrightarrow{\text{Forward substitution}} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.75 \\ 10.5 \end{bmatrix}$$

Ux=y

$$\begin{bmatrix} -4 & -2 & 1 \\ 0 & 3.5 & -1.75 \\ 0 & 0 & 3.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.75 \\ 10.5 \end{bmatrix} \xrightarrow{\text{Backward substitution}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Gauss-Jordan elimination

- The elements above the diagonal are made zero at the same time that zeros are created below the diagonal

$$\left[\begin{array}{cccc|c} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} & b_1^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} & b_2^{(1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1}^{(1)} & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} & b_n^{(1)} \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} & b_2^{(2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{n2}^{(2)} & \cdots & a_{nn}^{(2)} & b_n^{(2)} \end{array}\right]$$

$$\left[\begin{array}{cccc|c} a_{11}^{(1)} & 0 & \cdots & a_{nn}^{(2)} & b_1^{(2)} \\ 0 & a_{22}^{(2)} & \cdots & a_{nn}^{(2)} & b_2^{(2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn}^{(3)} & b_n^{(3)} \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} a_{11}^{(1)} & 0 & \cdots & 0 & b_1^{(n-1)} \\ 0 & a_{22}^{(2)} & \cdots & 0 & b_2^{(n-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn}^{(n)} & b_n^{(n)} \end{array}\right]$$

Gauss-Jordan Elimination

- Almost 50% more arithmetic operations than Gaussian elimination
- Gauss-Jordan (GJ) Elimination is prefered when the inverse of a matrix is required.

$$\left[A \;\vdots\; I \right]$$

- Apply GJ elimination to convert A into an identity matrix.

$$\left[I \;\vdots\; A^{-1} \right]$$