Solving Non-Linear Equations
(Root Finding)

Root finding Methods

- What are root finding methods?
- Methods for determining a solution of an equation.
- Essentially finding a root of a function, that is, a zero of the function.

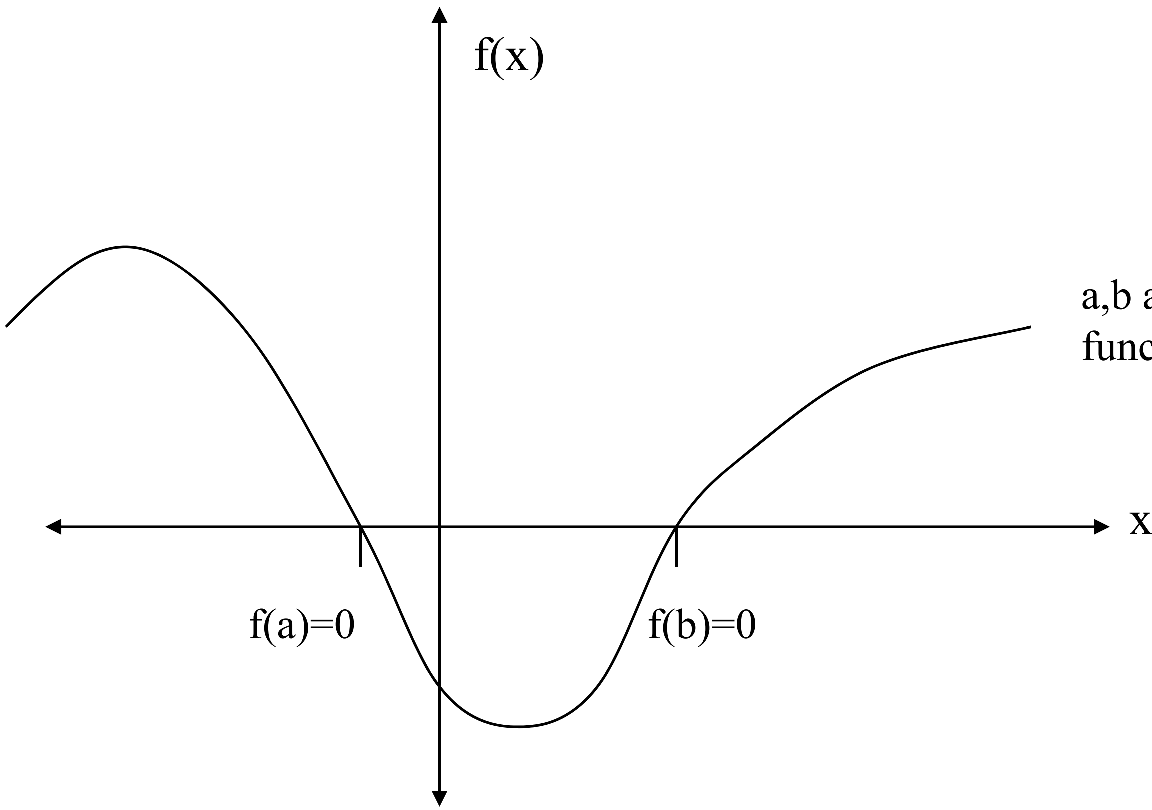
Root finding Methods

- Where are they used?
- Some applications for root finding are: systems of equilibrium, elliptical orbits, the van der Waals equation, and natural frequencies of spring systems.
- The problem of solving non-linear equations or sets of non-linear equations is a common problem in science, applied mathematics.

Cont...

- The problem of solving non-linear equations or sets of non-linear equations is a common problem in science, applied mathematics.
- The goal is to solve $f(x) = 0$, for the function $f(x)$.
- The values of x which make $f(x) = 0$ are the roots of the equation.

Cont...



a,b are roots of the function f(x)

- There are many methods for solving non-linear equations. The methods, which will be highlighted have the following properties:

1. the function f(x) is expected to be continuous. If not the method may fail.
2. the use of the algorithm requires that the solution be bounded.
3. once the solution is bounded, it is refined to specified tolerance.

Cont...

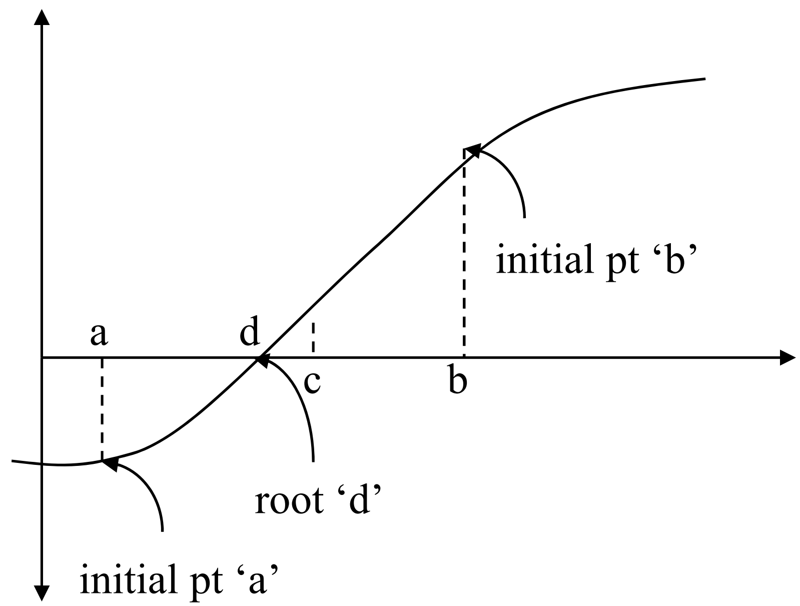
Four such methods are:

- Interval Halving (Bisection method)
- Regula Falsi (False position)
- Secant method
- Fixed point iteration
- Newton’s method

Bisection Method

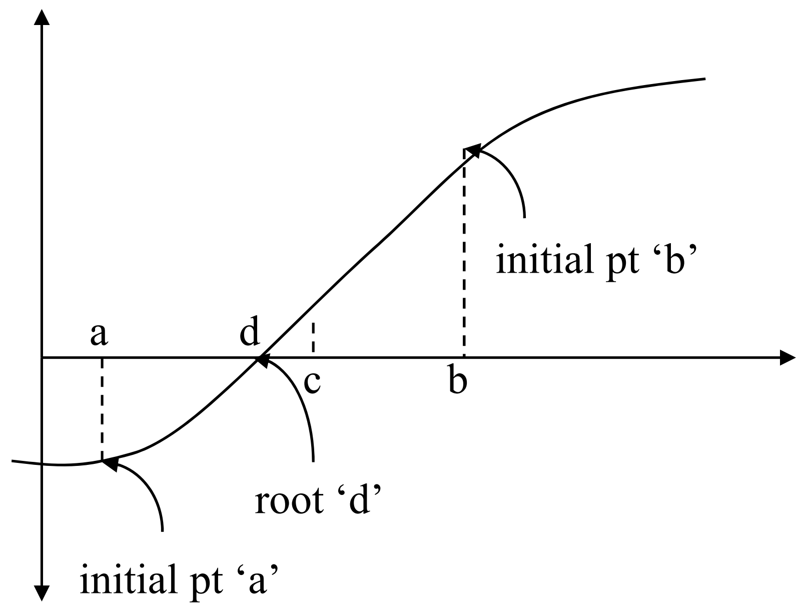
Bisection Method

- It is the simplest root-finding algorithm.
- requires previous knowledge of two initial guesses, *a* and *b*, so that *f*(*a*) and *f*(*b*) have opposite signs.



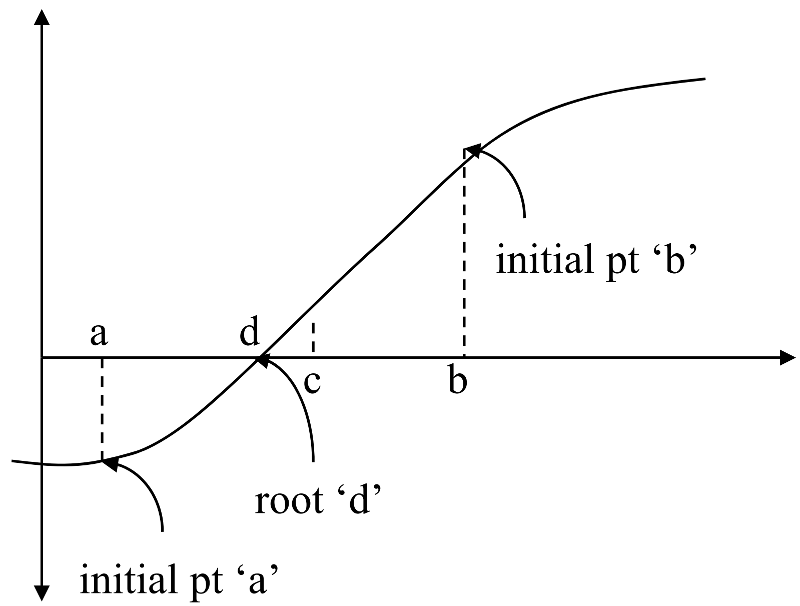
Bisection Method

- Two estimates are chosen so that the solution is bracketed. ie $f(a)$ and $f(b)$ have opposite signs.
- In the diagram this $f(a) < 0$ and $f(b) > 0$.
- **The root d lies between a and b!**



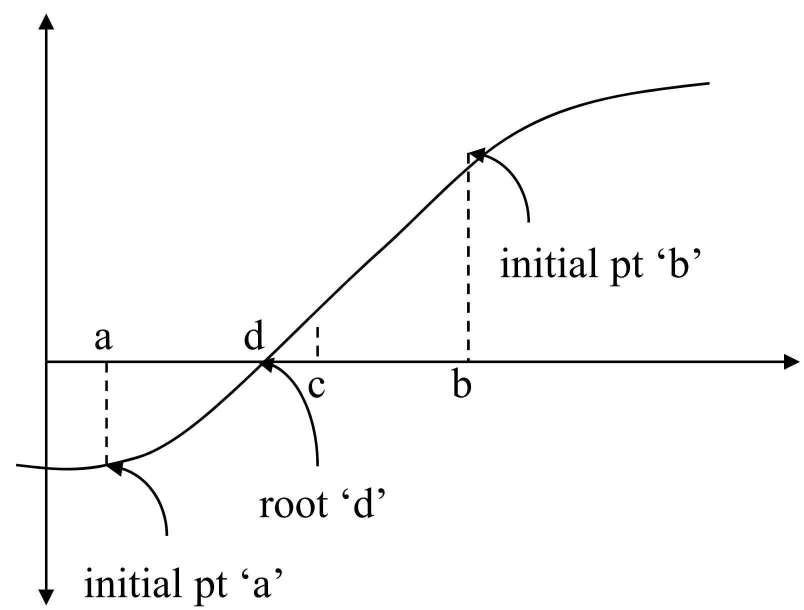
Bisection Method

- **The root d must always be bracketed (be between a and b)!**
- Iterative method.



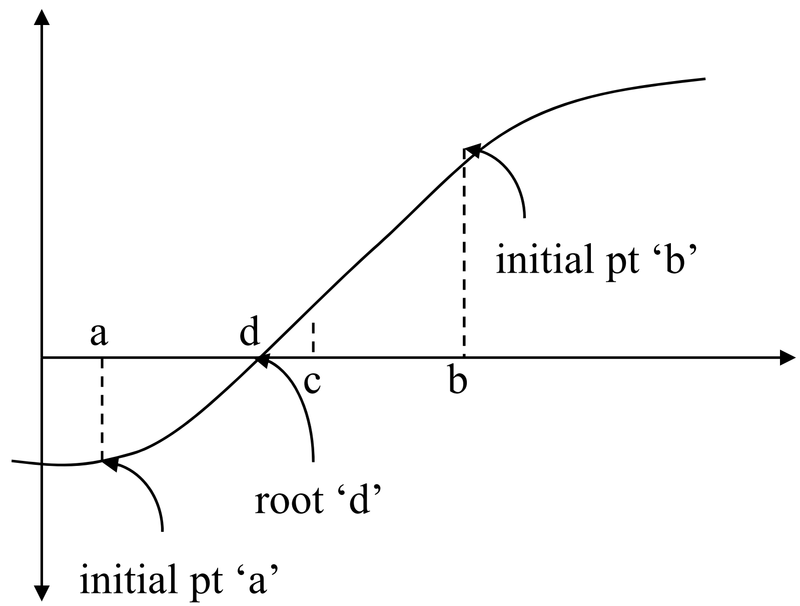
Bisection Method

- The interval between a and b is halved by calculating the average of a and b.
- **The new point c = (a+b)/2.**
- This produces are two possible intervals: $a < x < c$ and $c < x < b$.



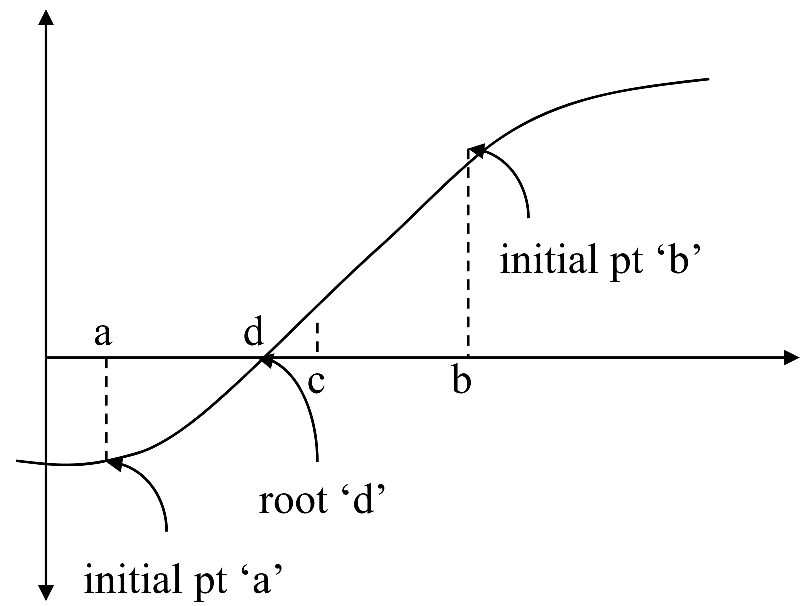
Bisection Method

- This produces are two possible intervals: $a < x < c$ and $c < x < b$.
- If $f(c) > 0$, then $x = d$ must be to the left of c :interval $a < x < c$.
- If $f(c) < 0$, then $x = d$ must be to the right of c :interval $c < x < b$.



Bisection Method

- If f(c) > 0, let $a_{new} = a$ and $b_{new} = c$ and repeat process.
- If f(c) < 0, let $a_{new} = c$ and $b_{new} = b$ and repeat process.
- **This reassignment ensures the root is always bracketed!!**



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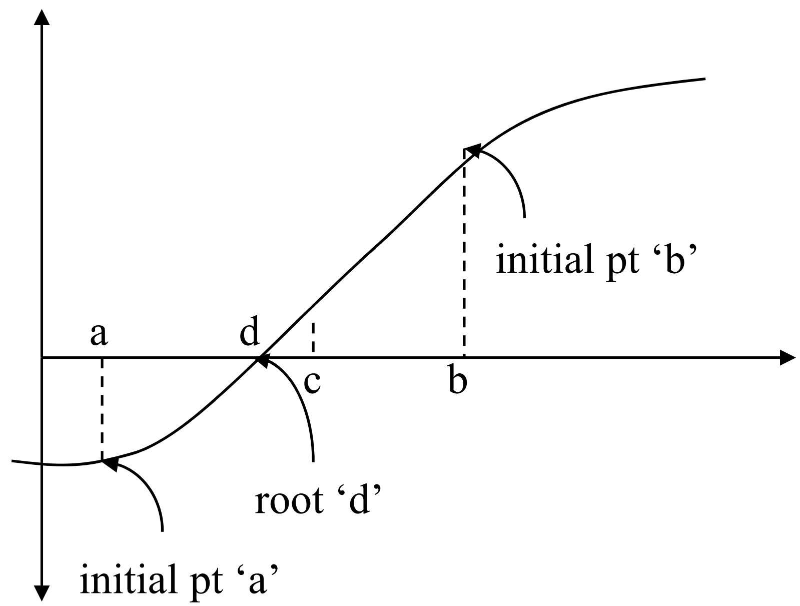


Bisection Method

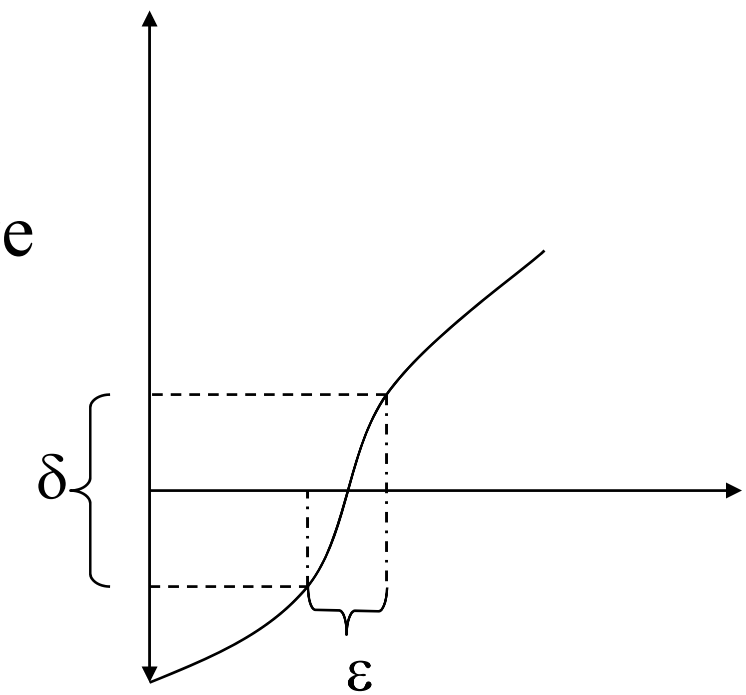
$c_i = (a_i + b_i)/2$

if $f(c_i) > 0$; $a_{i+1} = a$ and $b_{i+1} = c$

if $f(c_i) < 0$; $a_{i+1} = c$ and $b_{i+1} = b$

Bisection Method

- Bisection is an iterative process, where the initial interval is halved until the size of the interval decreases until it is below some predefined tolerance ε: |a-b| ≥ ε or f(x) falls below a tolerance δ: |f(c) – f(c-1)| ≤ δ.



Bisection Method

- Advantages

1. Is guaranteed to work if f(x) is continuous and the root is bracketed.
2. The number of iterations to get the root to specified tolerance is known in advance

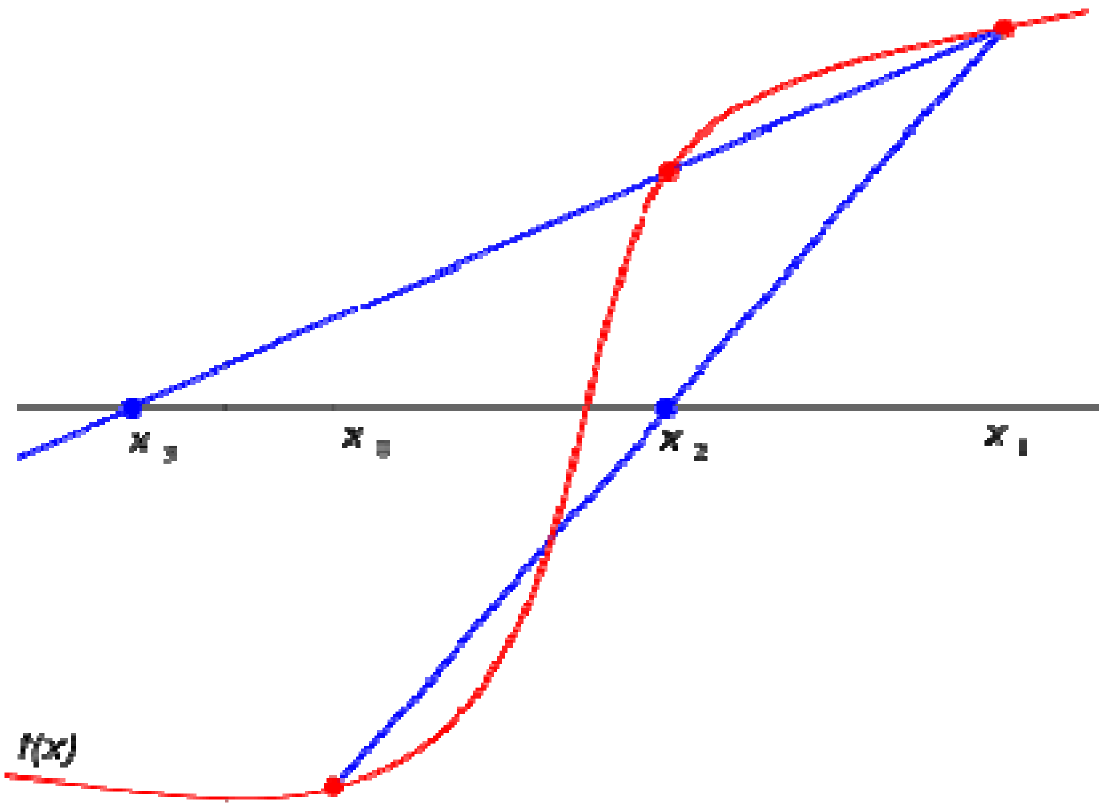
Bisection Method

- Disadvantages

1. Slow convergence.
2. Not applicable when here are multiple roots. Will only find one root at a time.

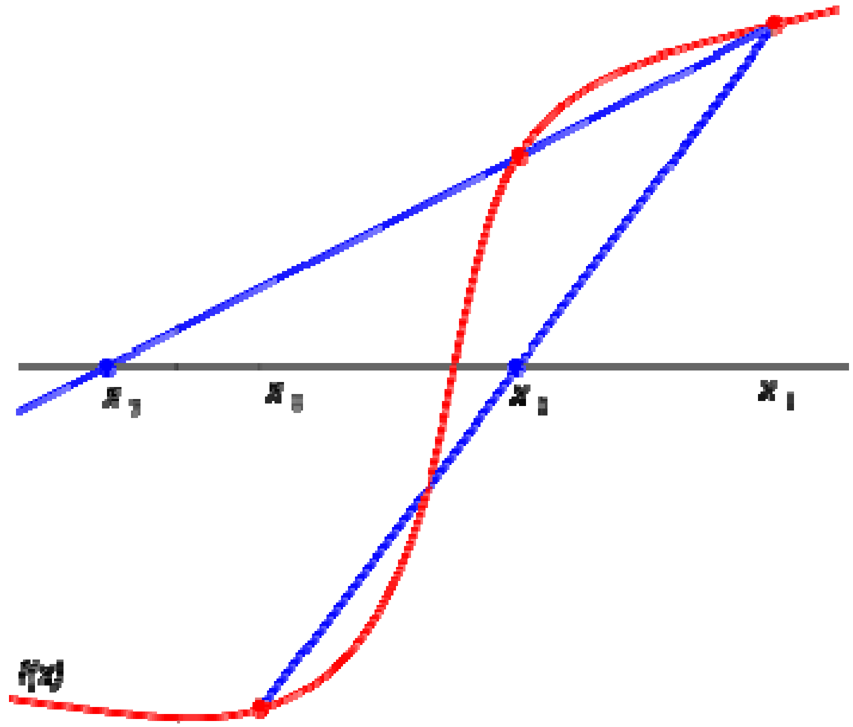
Secant Method

Secant Method



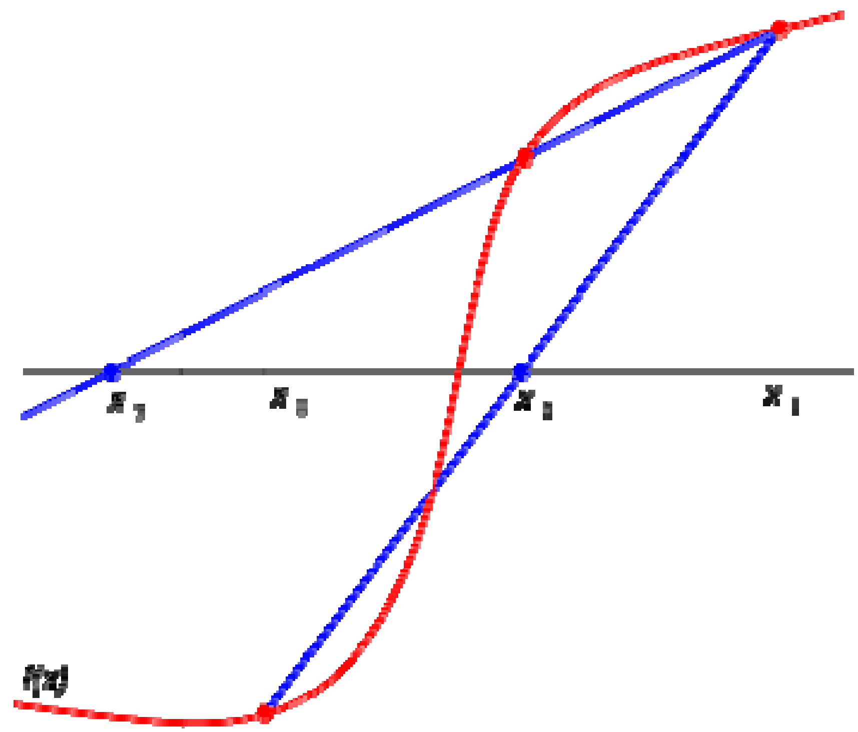
Overview of Secant Method

- Again to initial guesses are chosen.
- **However there is not requirement that the root is bracketed!**
- The method proceeds by drawing a line through the points to get a new point closer to the root.
- This is repeated until the root is found.



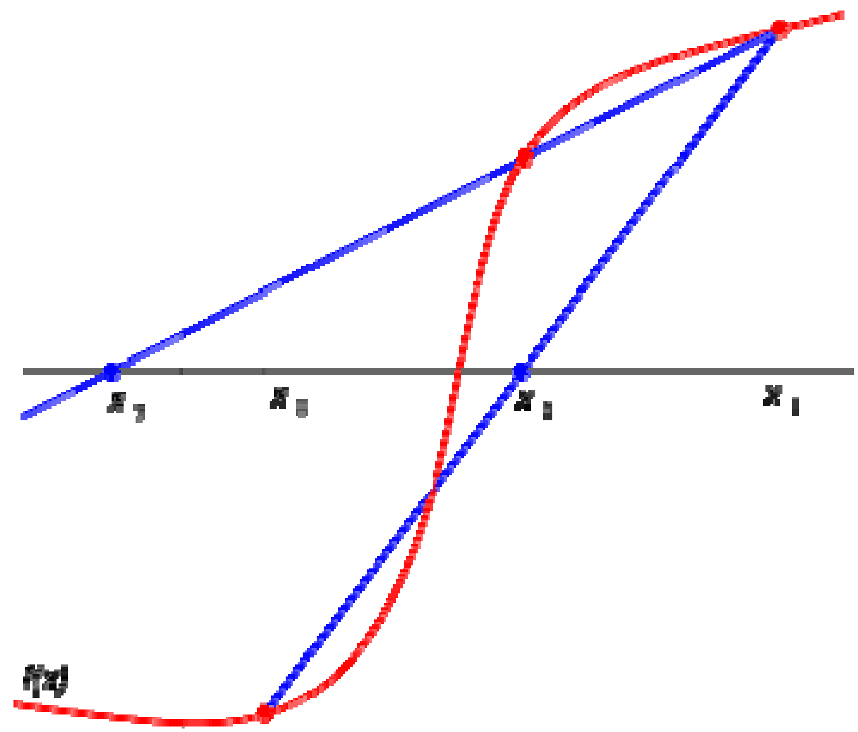
Secant Method

- First we find two points(x_0,x_1), which are hopefully near the root (we may use the bisection method).
- A line is then drawn through the two points and we find where the line intercepts the x-axis, x_2.



Secant Method

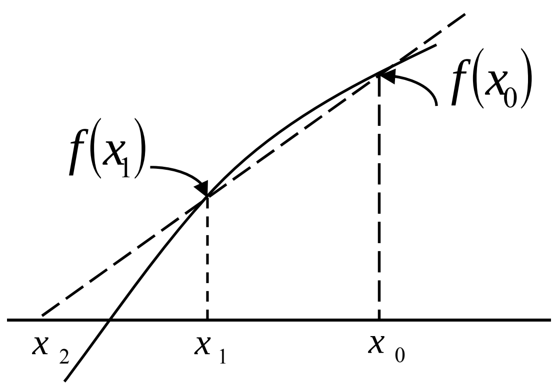
- If f(x) were truly linear, the straight line would intercept the x-axis at the root.
- However since it is not linear, the intercept is not at the root but it should be close to it.

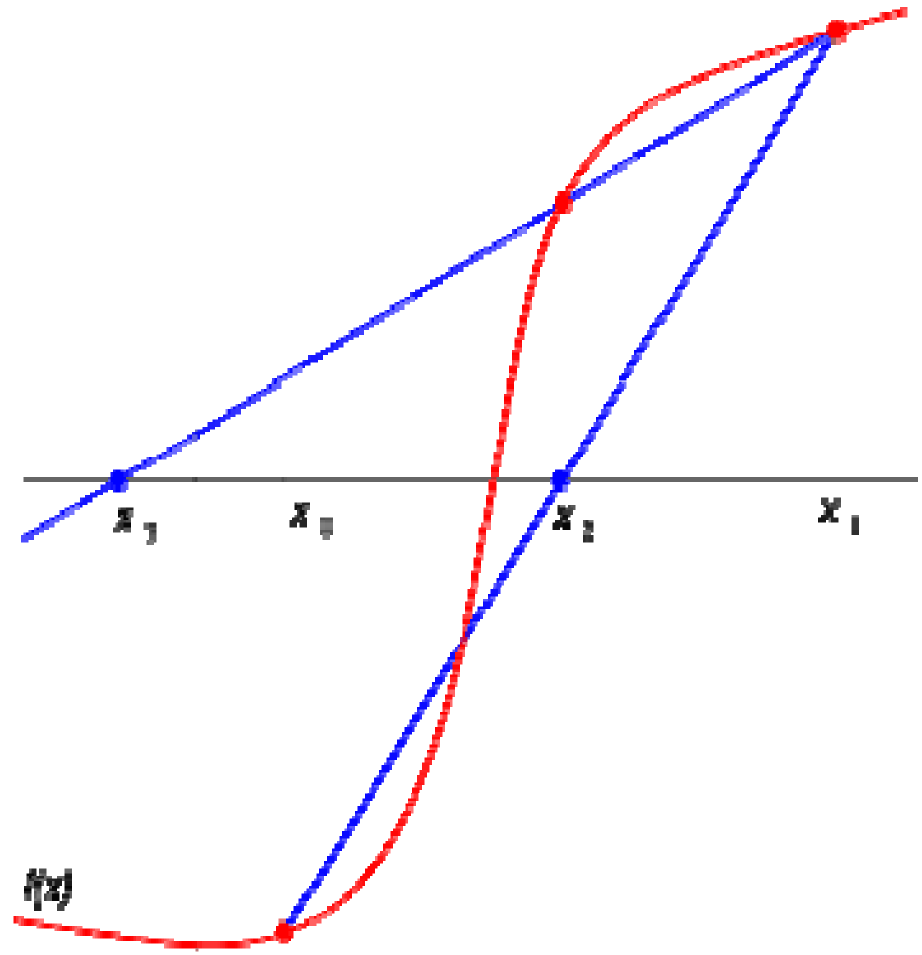


Secant Method

- From similar triangles we can write that,

$$\frac{(x_1 - x_2)}{f(x_1)} = \frac{(x_0 - x_1)}{f(x_0) - f(x_1)}$$





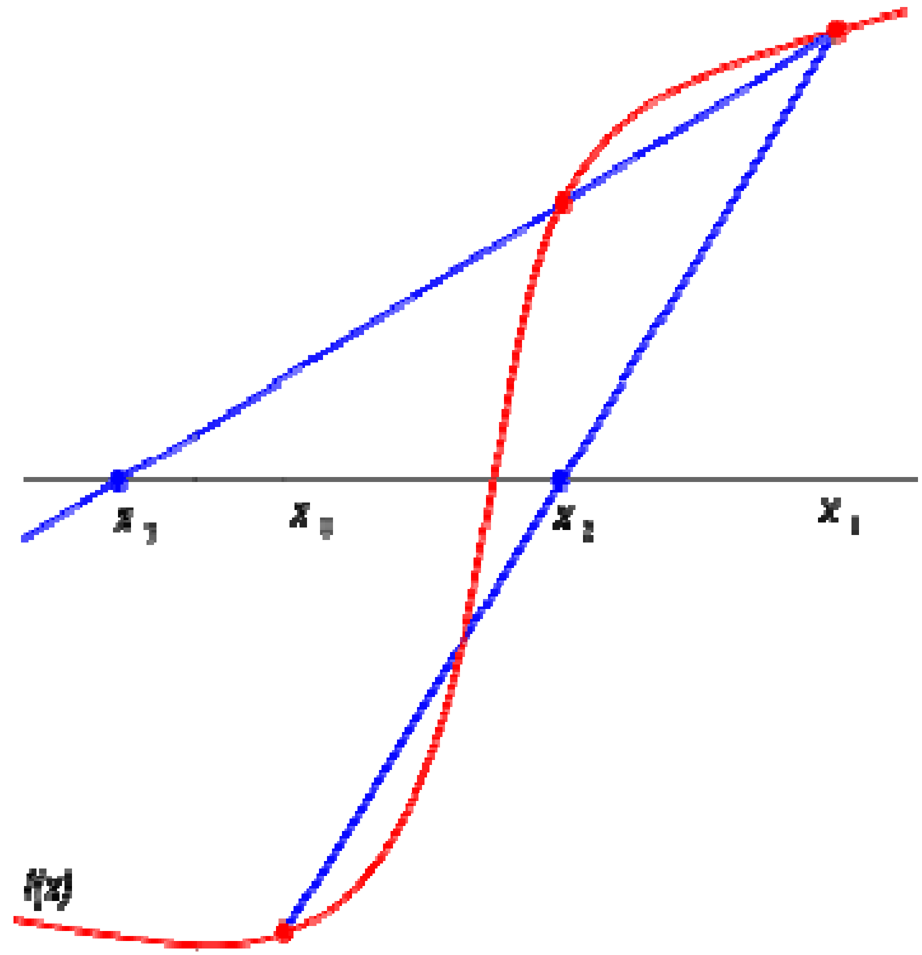
Secant Method

- From similar triangles we can write that,

$$\frac{(x_1 - x_2)}{f(x_1)} = \frac{(x_0 - x_1)}{f(x_0) - f(x_1)}$$

- Solving for x_2 we get:

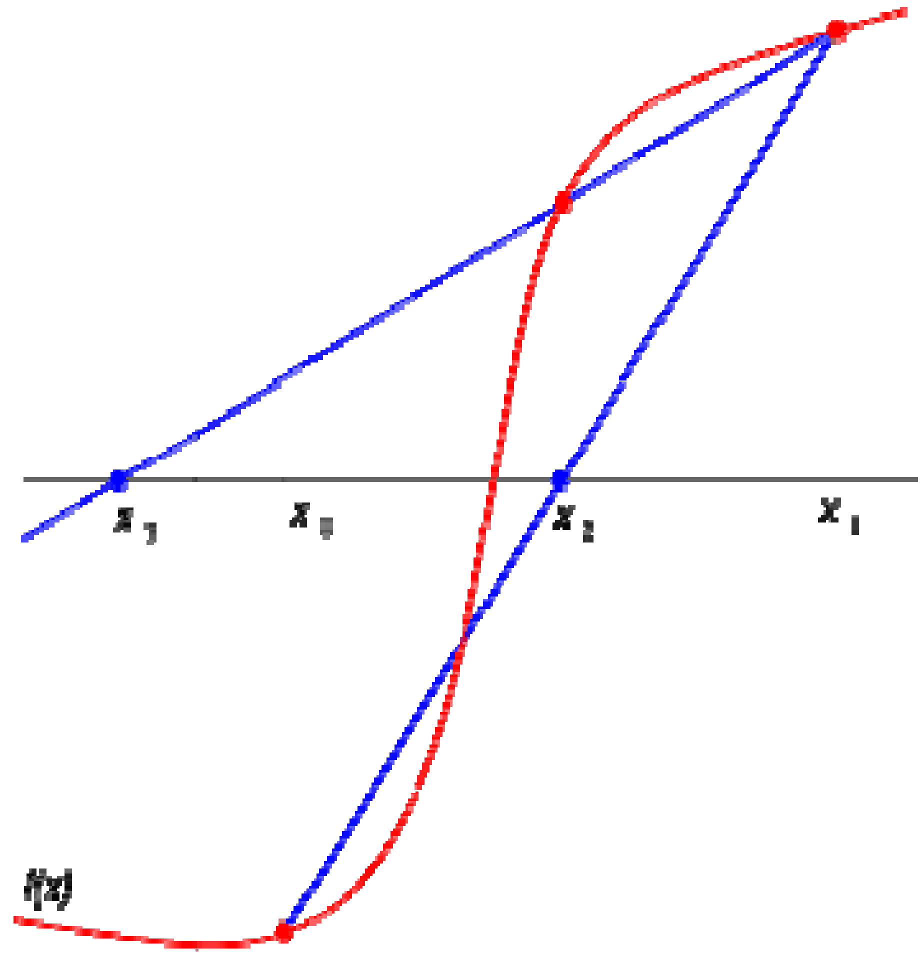
$$x_2 = x_1 - f(x_1)\frac{(x_0 - x_1)}{f(x_0) - f(x_1)}$$



Secant Method

- Iteratively this is written as:

$$x_{n+1} = x_n - f(x_n)\frac{(x_{n-1} - x_n)}{f(x_{n-1}) - f(x_n)}$$



Algorithm

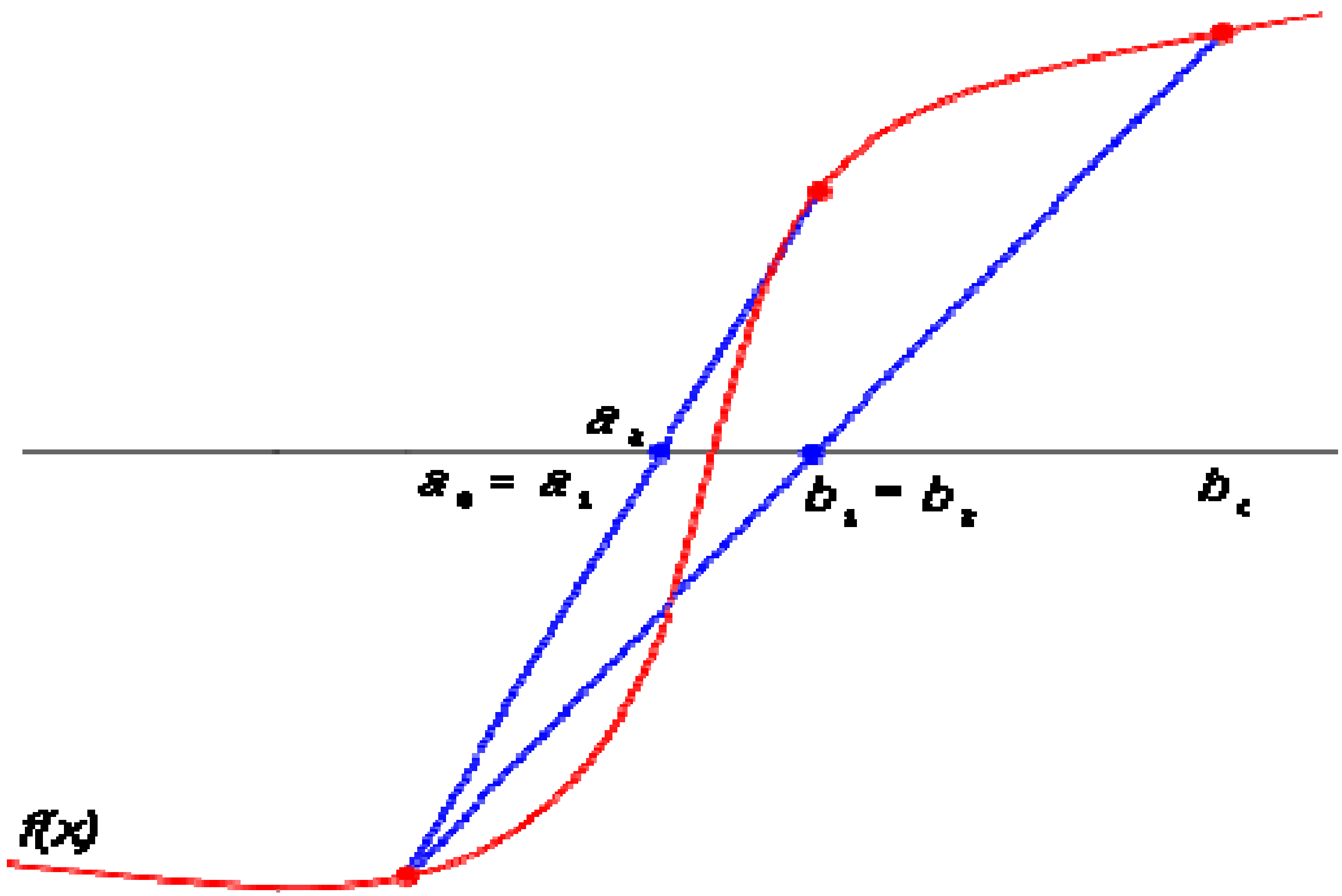
- Given two guesses x_0, x_1 near the root,
- If $|f(x_0)| < |f(x_1)|$ then
- Swap x_0 and x_1.
- Repeat
- Set $x_2 = x_1 - f(x_1) * \dfrac{x_0 - x_1}{f(x_0) - f(x_1)}$
- Set $x_0 = x_1$
- Set $x_1 = x_2$
- Until $|f(x_2)| <$ tolerance value.

Cont...

- Because the new point should be closer the root after the 2nd iteration we usually choose the last two points.
- After the first iteration there is only one new point. However x_1 is chosen so that it is closer to root that x_0.
- **This is not a "hard and fast rule"!**

False Position

False Position



False Position

- The method of false position is seen as an improvement on the secant method.
- The method of false position avoids the problems of the secant method by ensuring that the root is bracketed between the two starting points and remains bracketing between successive pairs.

False Position

- This technique is similar to the bisection method except that the next iterate is taken as the line of interception between the pair of x-values and the x-axis rather than at the midpoint.

False Position

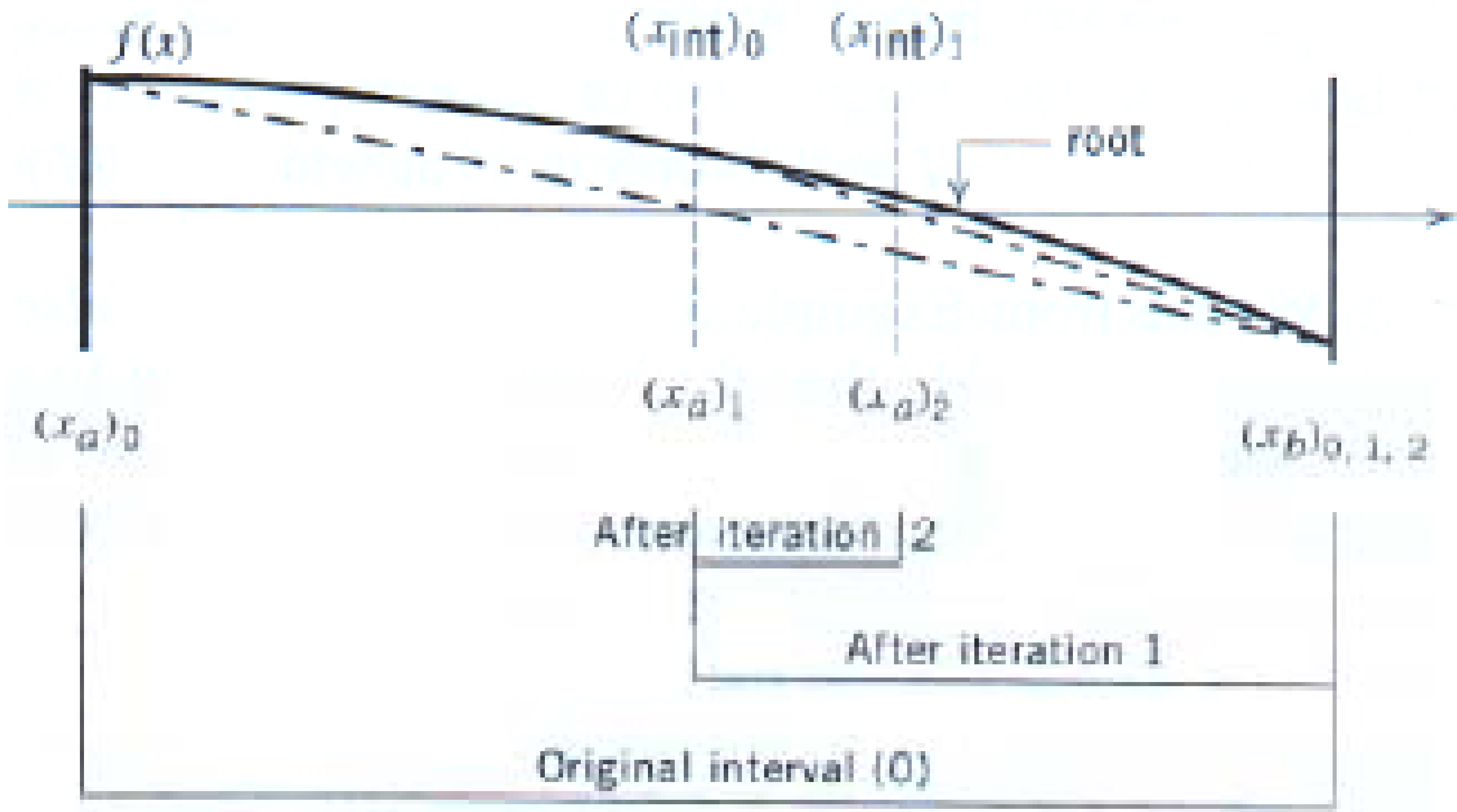


Illustration of the false-position method.

Algorithm

- **Given two guesses x_0, x_1 that bracket the root,**
- Repeat
- Set $x_2 = x_1 - f(x_1) * \dfrac{x_0 - x_1}{f(x_0) - f(x_1)}$
- If $f(x_2)$ is of opposite sign to $f(x_0)$ then
- Set $x_1 = x_2$
- Else Set $x_0 = x_1$
- End If
- Until $|f(x_2)| <$ tolerance value.

Discussion of False Position Method

- This method achieves better convergence but a more complicated algorithm.
- May fail if the function is not continuous.

Newton's Method

Newton’s Method

- The bisection method is useful up to a point.
- In order to get a good accuracy a large number of iterations must be carried out.
- A second inadequacy occurs when there are multiple roots to be found.

Newton’s Method

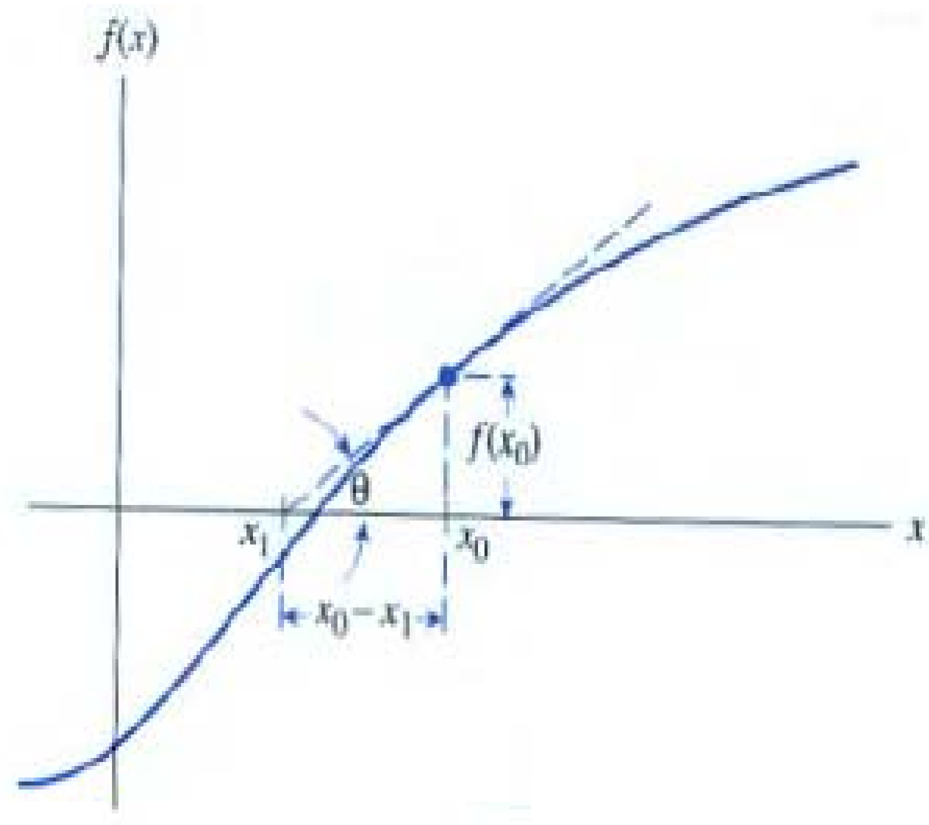
- The bisection method is useful up to a point.
- In order to get a good accuracy a large number of iterations must be carried out.
- A second inadequacy occurs when there are multiple roots to be found.
- Newton’s method is a much better algorithm.

Newton’s Method

- Newton’s method relies on calculus and uses linear approximation to the function by finding the tangent to the curve.

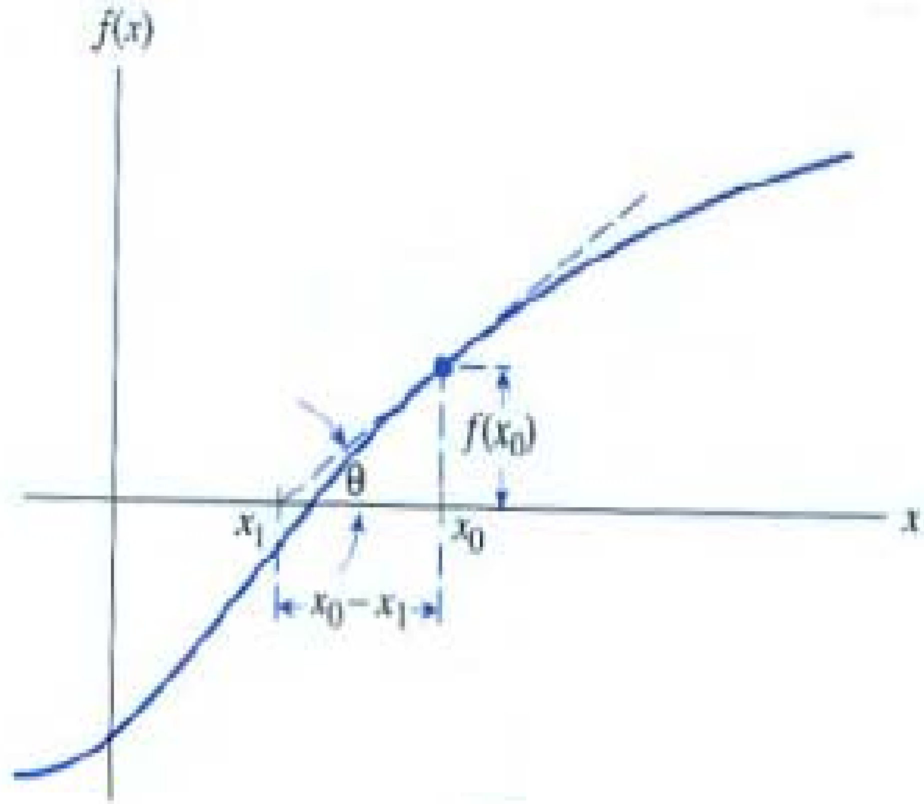
Newton's Method

- Algorithm requires an initial guess, x_0, which is close to the root.
- The point where the tangent line to the function (f (x)) meets the x-axis is the next approximation, x_1.
- This procedure is repeated until the value of 'x' is sufficiently close to zero.



Newton’s Method

- The equation for Newton’s Method can be determined graphically!



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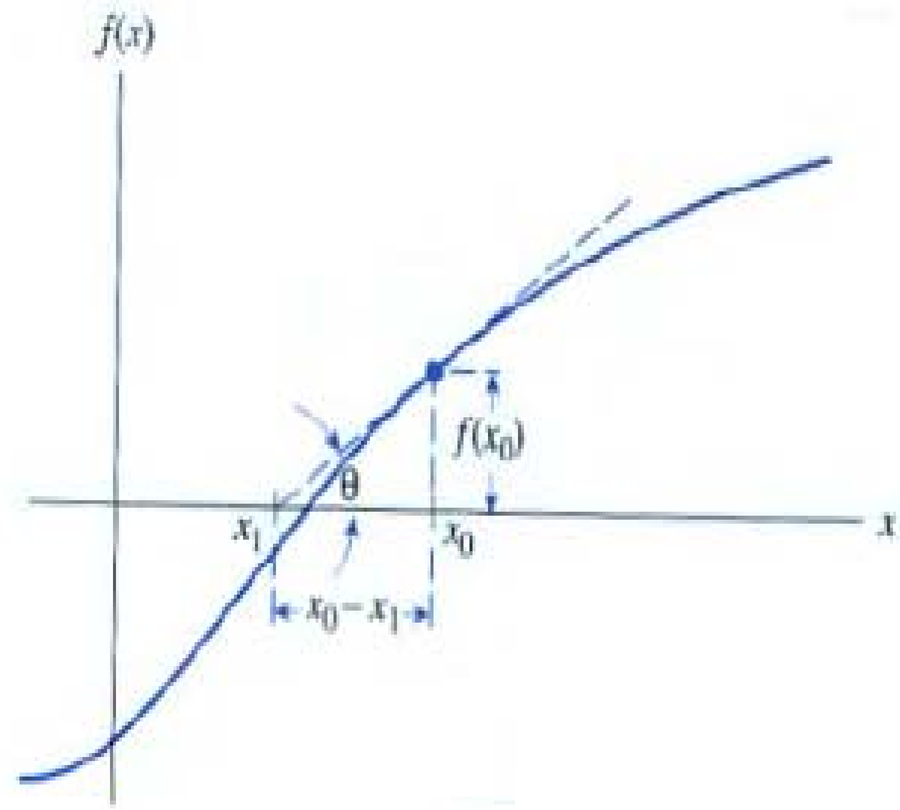
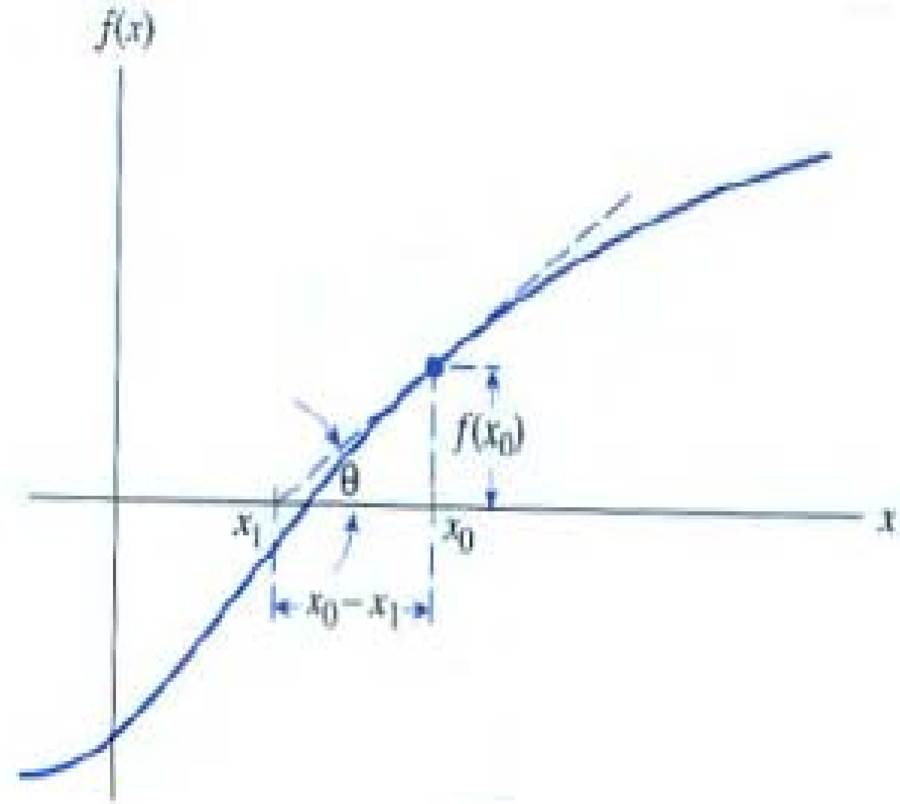
- From the diagram tan Θ $= f'(x_0) = f(x_0)/(x_0 - x_1)$



Newton's Method

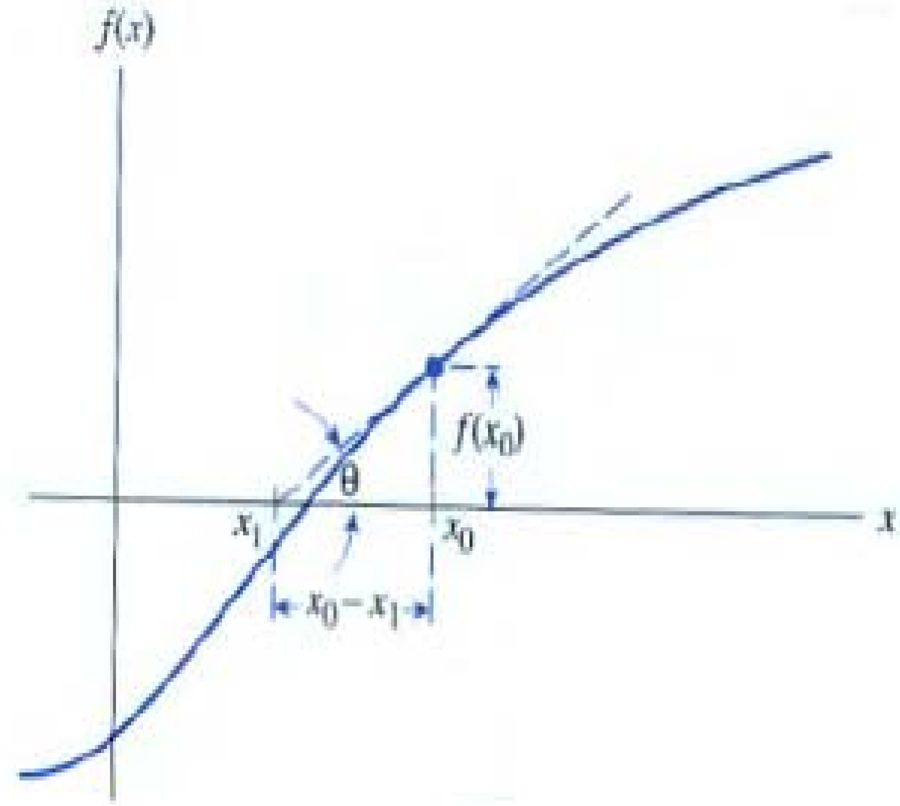
- The equation for Newton's Method can be determined graphically!
- From the diagram tan Θ $= f'(x_0) = f(x_0)/(x_0 - x_1)$
- Thus, $x_1 = x_0 - f(x_0)/f'(x_0)$.



Newton's Method

- The general form of Newton's Method is:

$$x_{n+1} = x_n - f(x_n)/f'(x_n)$$



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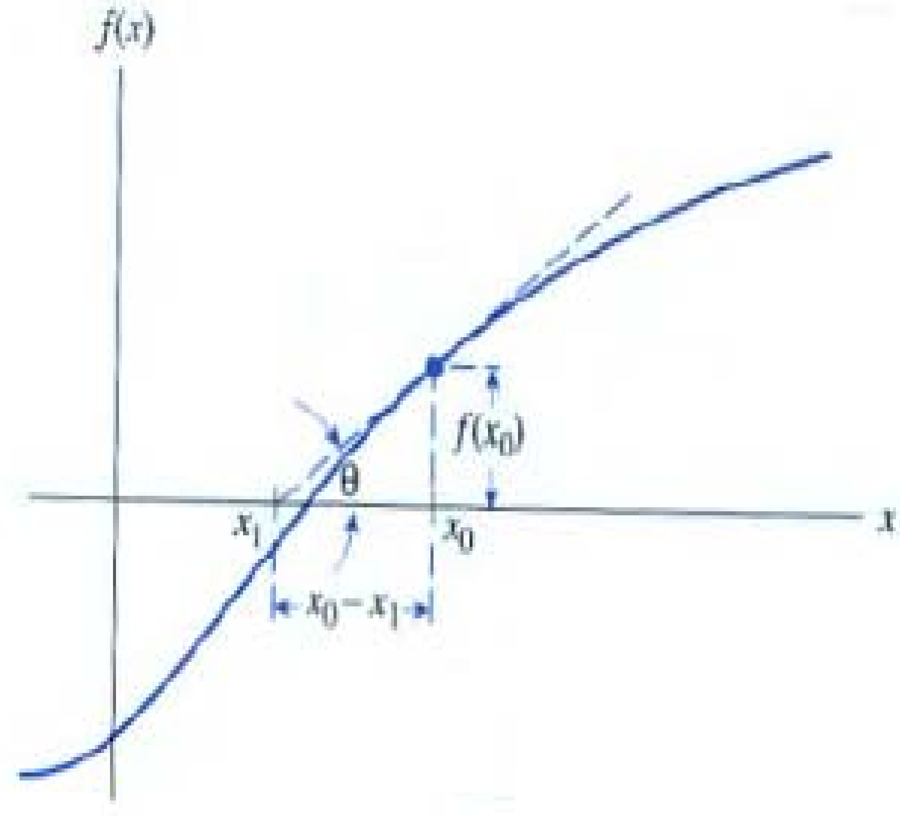

- **Algorithm**
- Pick a starting value for x
- Repeat
- $x := x - f(x)/f'(x)$
- Return x