Ordinary Differential Equations

- Equations which are composed of an unknown function and its derivatives are called *differential equations*.

- Differential equations play a fundamental role in engineering because many physical phenomena are best formulated mathematically in terms of their rate of change.

$$\frac{dv}{dt} = g - \frac{c}{m}v$$

v- dependent variable

t- independent variable

- When a function involves one dependent variable, the equation is called an *ordinary differential equation (or ODE).* A *partial differential equation (or PDE)* involves two or more independent variables.

- Differential equations are also classified as to their order.
  - *A first order equation* includes a first derivative as its highest derivative.
  - *A second order equation* includes a second derivative.

- Higher order equations can be reduced to a system of first order equations, by redefining a variable.

ODEs and Engineering Practice

Figure 7.1

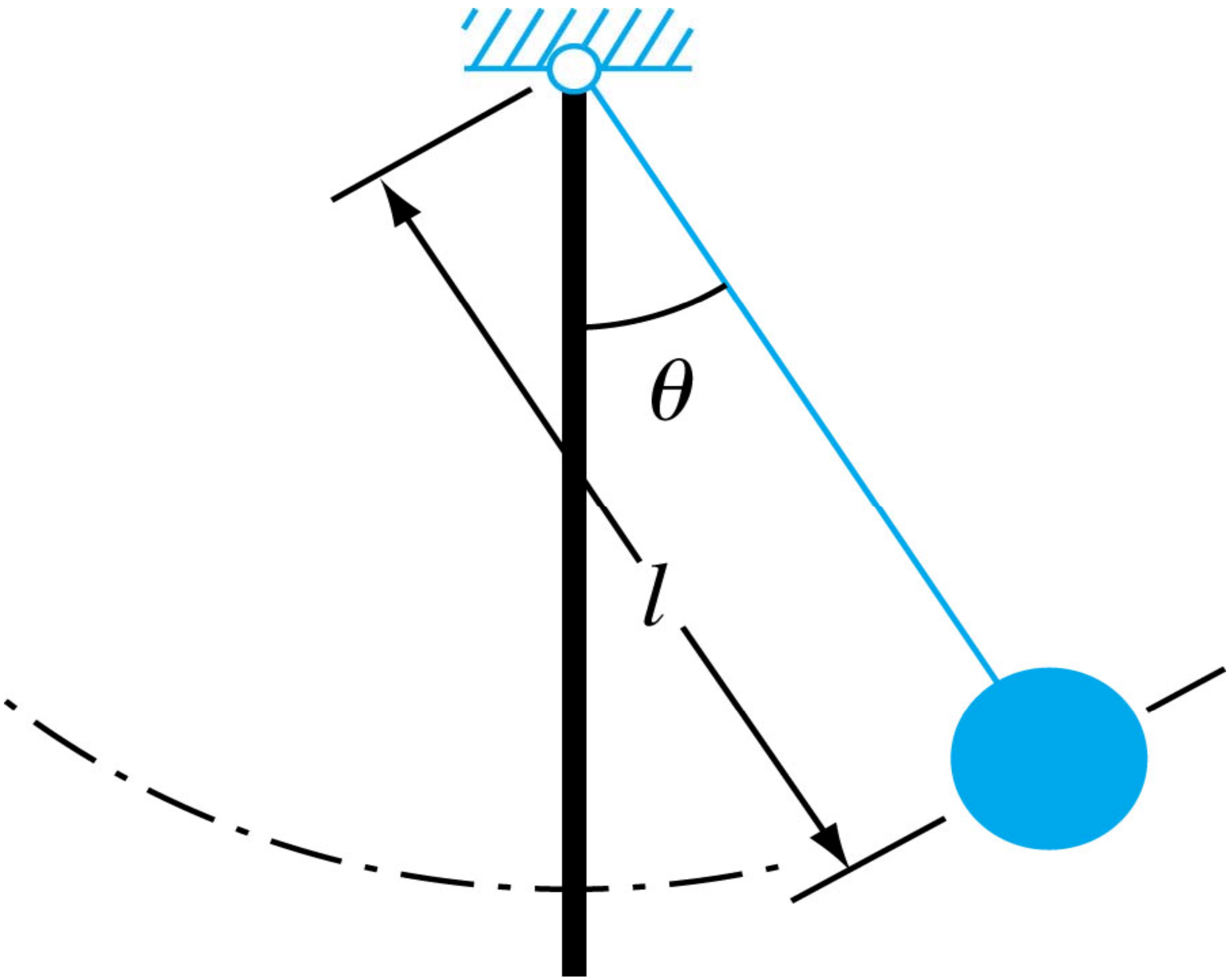
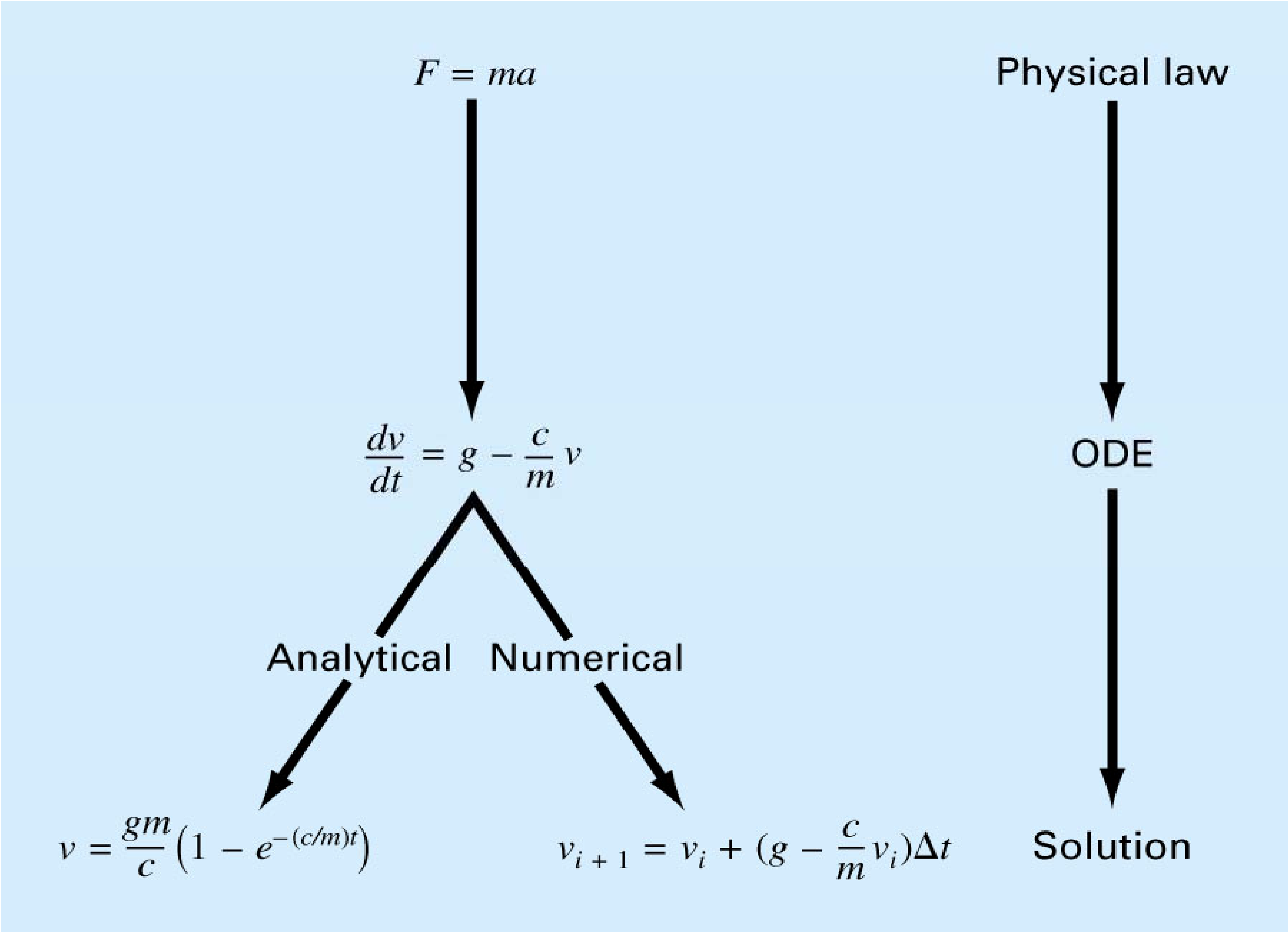


Figure 7.2



$F = ma$
Physical law
$\frac{dv}{dt} = g - \frac{c}{m}v$
ODE
Analytical
Numerical
$v = \frac{gm}{c}\left(1 - e^{-(c/m)t}\right)$
$v_{i+1} = v_i + (g - \frac{c}{m}v_i)\Delta t$
Solution

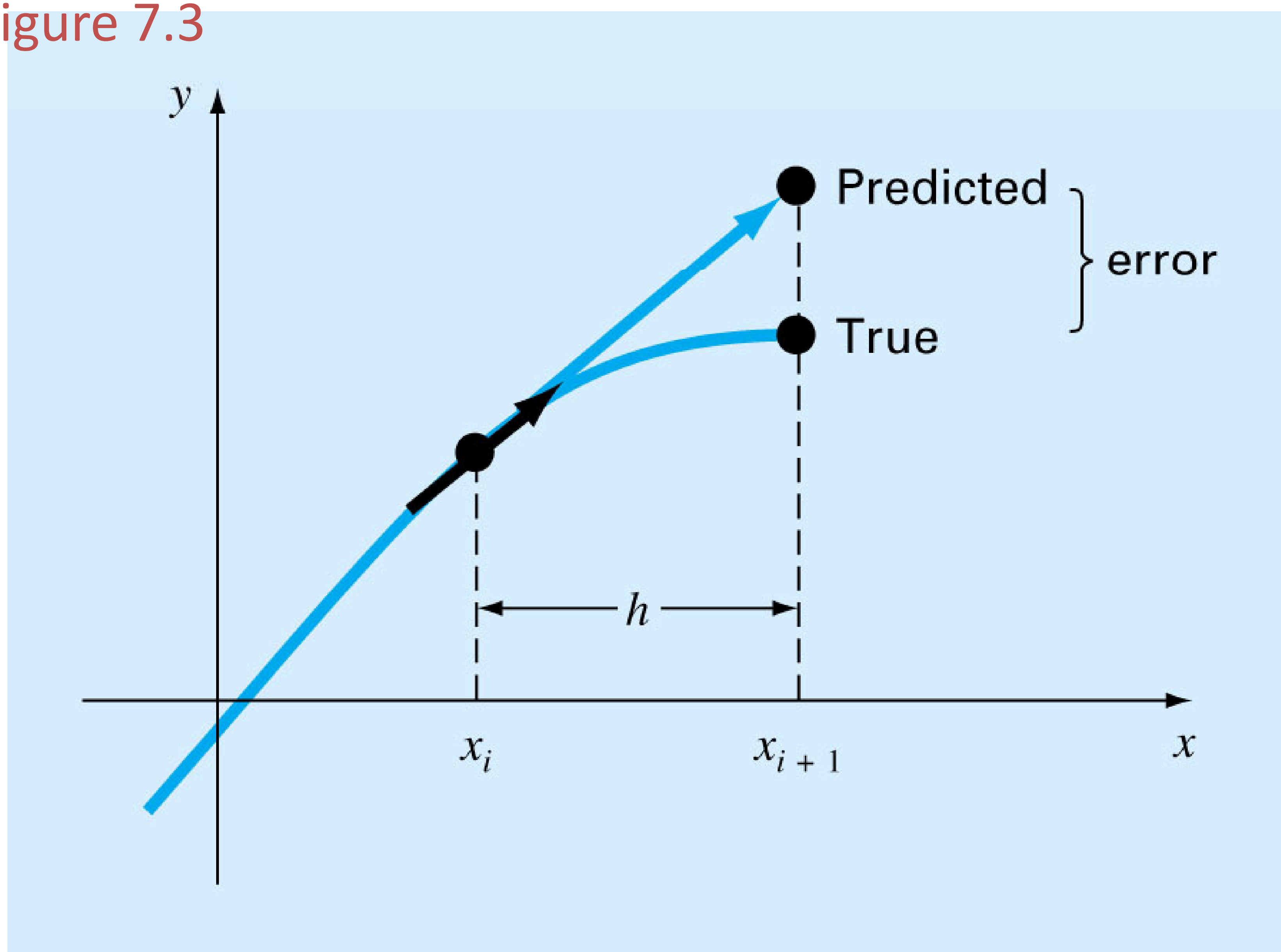
Runga-Kutta Methods

- This chapter is devoted to solving ordinary differential equations of the form

$$\frac{dy}{dx} = f(x, y)$$

Euler's Method

Figure 7.3



- The first derivative provides a direct estimate of the slope at x_i

$$\phi = f(x_i, y_i)$$

where $f(x_i,y_i)$ is the differential equation evaluated at x_i and y_i. This estimate can be substituted into the equation:

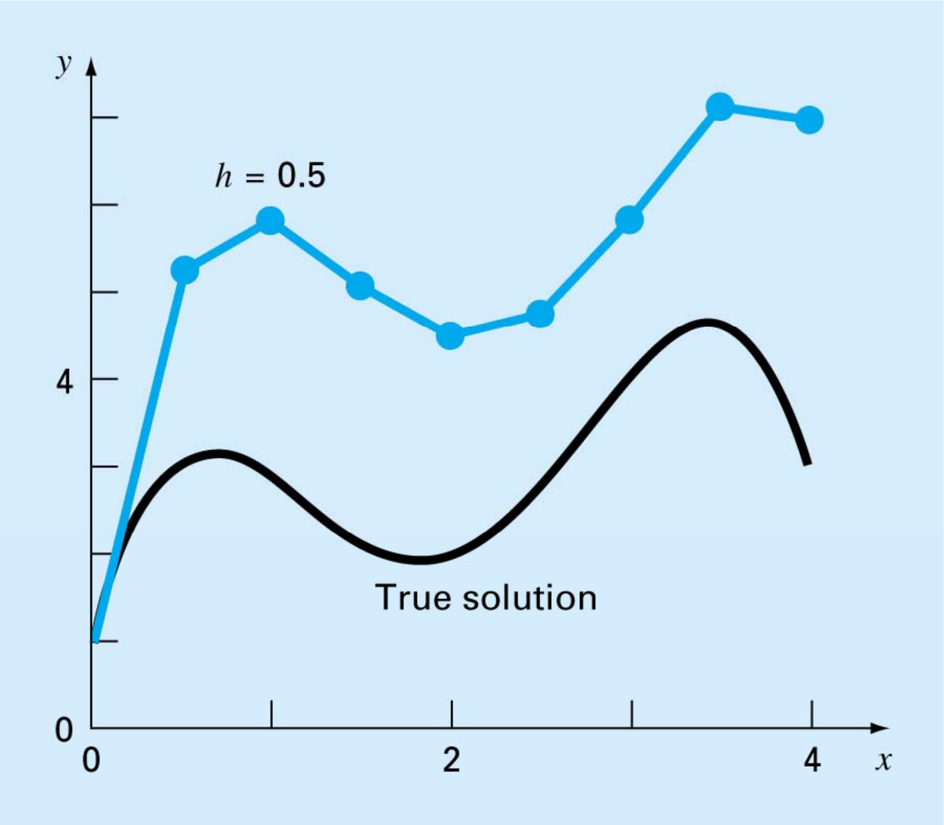
$$y_{i+1} = y_i + f(x_i, y_i)h$$

- A new value of y is predicted using the slope to extrapolate linearly over the step size h.

$$\frac{dy}{dx} = f(x, y) = -2x^3 + 12x^2 - 20x + 8.5$$

$$Starting \quad point \quad x_0 = 0, \quad y_0 = 1$$

$$y_{i+1} = y_i + f(x_i, y_i)h = 1 + 8.5 * 0.5 = 5.25$$



Not good

Error Analysis for Euler's Method/

- Numerical solutions of ODEs involves two types of error:
  - *Truncation* error
    - *Local truncation error*

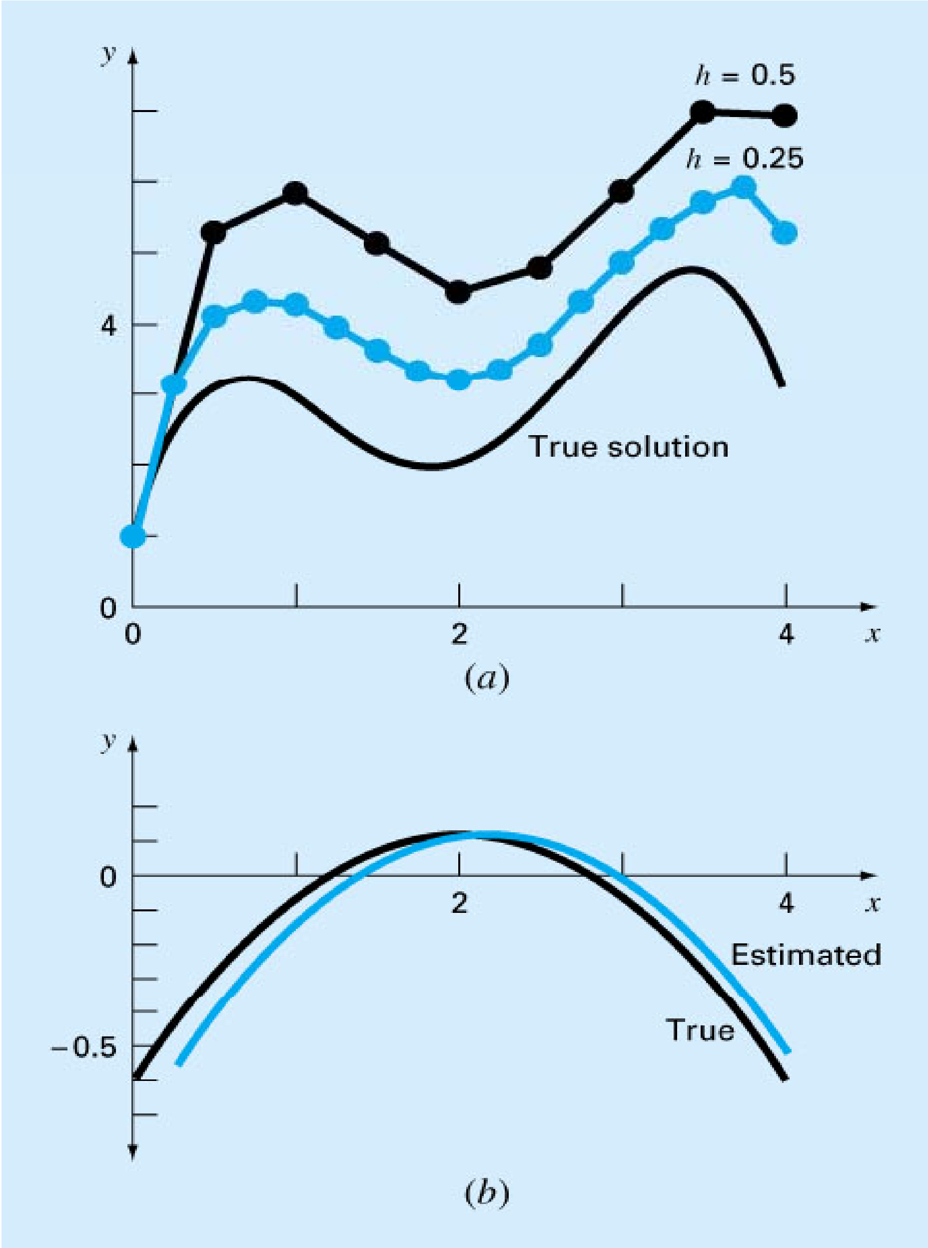
$$E_a = \frac{f'(x_i, y_i)}{2!} h^2$$

$$E_a = O(h^2)$$

    - *Propagated truncation error*
  - The sum of the two is the *total or global truncation error*
  - *Round-off* errors

- The Taylor series provides a means of quantifying the error in Euler's method. However;
  - The Taylor series provides only an estimate of the local truncation error-that is, the error created during a single step of the method.
  - In actual problems, the functions are more complicated than simple polynomials. Consequently, the derivatives needed to evaluate the Taylor series expansion would not always be easy to obtain.
- In conclusion,
  - the error can be reduced by reducing the step size
  - If the solution to the differential equation is linear, the method will provide error free predictions as for a straight line the 2^{nd} derivative would be zero.

Figure 7.4



Improvements of Euler’s method

- A fundamental source of error in Euler’s method is that the derivative at the beginning of the interval is assumed to apply across the entire interval.
- Two simple modifications are available to circumvent this shortcoming:
  - Heun’s Method
  - The Midpoint (or Improved Polygon) Method

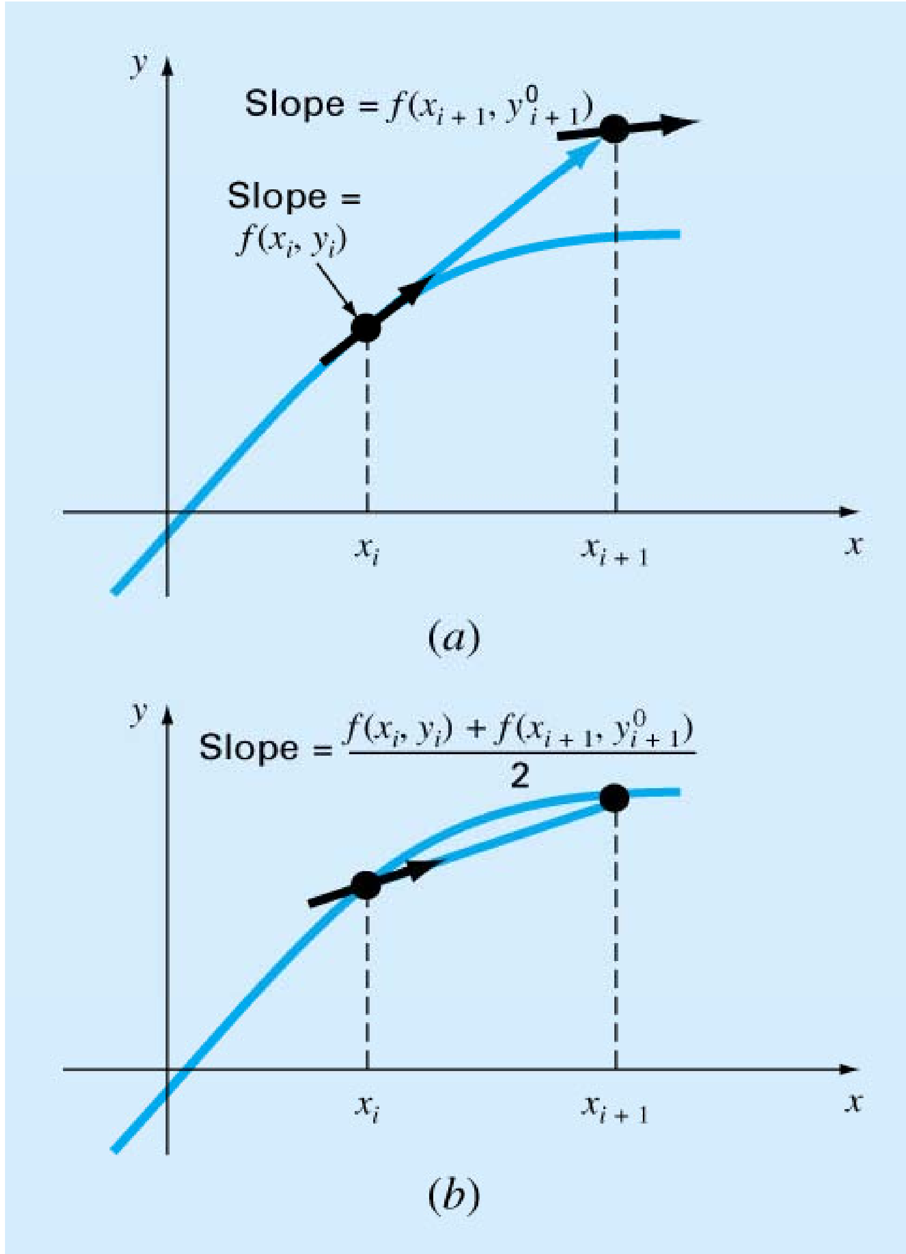
Heun's Method/

- One method to improve the estimate of the slope involves the determination of two derivatives for the interval:
  - At the initial point
  - At the end point
- The two derivatives are then averaged to obtain an improved estimate of the slope for the entire interval.

$$\text{Predictor}: \quad y_{i+1}^{0} = y_i + f(x_i, y_i)h$$

$$\text{Corrector}: \quad y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{0})}{2}h$$

Figure 7.5



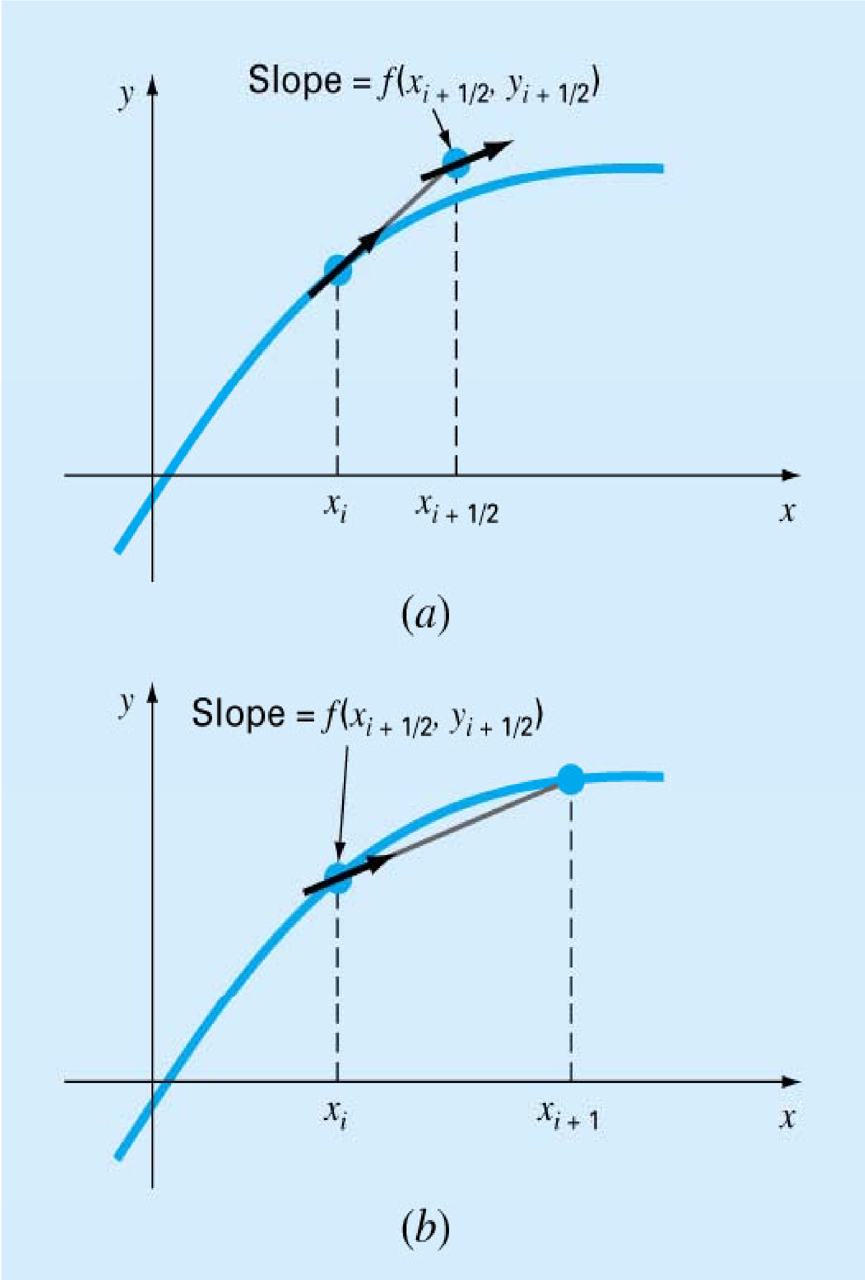
y
Slope = $f(x_{i+1}, y^0_{i+1})$
Slope =
$f(x_i, y_i)$
x_i
x_{i+1}
x
(a)
y
Slope = $\frac{f(x_i, y_i) + f(x_{i+1}, y^0_{i+1})}{2}$
x_i
x_{i+1}
x
(b)

The Midpoint (or Improved Polygon) Method/

- Uses Euler's method t predict a value of *y* at the midpoint of the interval:

$$y_{i+1} = y_i + f(x_{i+1/2}, y_{i+1/2})h$$

Figure 7.6



Runge-Kutta Methods (RK)

- Runge-Kutta methods achieve the accuracy of a Taylor series approach without requiring the calculation of higher derivatives.

$$y_{i+1} = y_i + \phi(x_i, y_i, h)h$$

$$\phi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n$$ ***Increment function***

$$a's = \text{constants}$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$ ***p's and q's are constants***

$$k_3 = f(x_i + p_3 h, y_i + q_{21} k_1 h + q_{22} k_2 h)$$

$$\vdots$$

$$k_n = f(x_i + p_{n-1} h, y_i + q_{n-1} k_1 h + q_{n-1,2} k_2 h + \cdots + q_{n-1,n-1} k_{n-1} h)$$

- *k*'s are recurrence functions. Because each *k* is a functional evaluation, this recurrence makes RK methods efficient for computer calculations.
- Various types of RK methods can be devised by employing different number of terms in the increment function as specified by *n*.
- First order RK method with $n=1$ is in fact Euler's method.
- Once *n* is chosen, values of *a*'s, *p*'s, and *q*'s are evaluated by setting general equation equal to terms in a Taylor series expansion.

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

- Values of a_1, a_2, p_1, and q_{11} are evaluated by setting the second order equation to Taylor series expansion to the second order term. Three equations to evaluate four unknowns constants are derived.

$$We\ have: y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

$$However\ y_{i+1} = y_i + f(x_i, y_i)h + \frac{f'(x_i, y_i)}{2!}h^2$$

$$But \quad f'(x_i, y_i) = \frac{\partial f(x_i, y_i)}{\partial x} + \frac{\partial f(x_i, y_i)}{\partial y}\frac{dy}{dx}$$

$$Then \quad y_{i+1} = y_i + f(x_i, y_i)h + \left[\frac{\partial f(x_i, y_i)}{\partial x} + \frac{\partial f(x_i, y_i)}{\partial y}\frac{dy}{dx}\right]\frac{h^2}{2!}$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

$$We\ now\ expand\ k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

$$k_2 = f(x_i, y_i) + \frac{\partial f(x_i, y_i)}{\partial x} p_1 h + \frac{\partial f(x_i, y_i)}{\partial y} q_{11} k_1 h$$

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2) h$$

$$y_{i+1} = y_i + \left\{ a_1 f(x_i, y_i) + a_2 \left(f(x_i, y_i) + \frac{\partial f(x_i, y_i)}{\partial x} p_1 h + \frac{\partial f(x_i, y_i)}{\partial y} q_{11} k_1 h \right) \right\} h$$

$$y_{i+1} = y_i + \underbrace{a_1 h\, f(x_i, y_i) + a_2 h\, f(x_i, y_i)} + a_2\, p_1 h^2 \frac{\partial f(x_i, y_i)}{\partial x} + a_2\, q_{11}\, f(x_i, y_i) h^2 \frac{\partial f(x_i, y_i)}{\partial y}$$

$$y_{i+1} = y_i + f(x_i, y_i) h + \left[\frac{\partial f(x_i, y_i)}{\partial x} + \frac{\partial f(x_i, y_i)}{\partial y} f(x_i, y_i) \right] \frac{h^2}{2!}$$

$$a_1 + a_2 = 1$$

$$a_2 p_1 = \frac{1}{2}$$

$$a_2 q_{11} = \frac{1}{2}$$

- Because we can choose an infinite number of values for a_2, there are an infinite number of second-order RK methods.
- Every version would yield exactly the same results if the solution to ODE were quadratic, linear, or a constant.
- However, they yield different results if the solution is more complicated (typically the case).
- Three of the most commonly used methods are:
  - Huen Method with a Single Corrector ($a_2=1/2$)
  - The Midpoint Method ($a_2=1$)
  - Raltson's Method ($a_2=2/3$)

Figure 7.7