First order system responses

Equation describing behavior of a system consists of two parts.

- Complementary function- It corresponds to the short time or transient response.
- Particular function – It refers to the long time steady state response.

The transfer operator form of the first-order system is given by :

$$\frac{I_o}{I_i} = \frac{S}{1 + \tau D}$$

When S (static sensitivity or steady state gain) equals *unity*,

$$(1 + \tau D)\, I_o = I_i$$

Transient response (Complementary function):

Put input equal to zero. $(1 + \tau D) I_{o,t} = 0$

Let the solution be of the form :

$$I_{o,t} = A\, e^{mt}$$

where, m is an algebraic variable)

or, $$(1 + \tau D)\, A\, e^{mt} = 0$$

or, $$A\, e^{mt} + \tau \cdot \frac{d}{dt}(A\, e^{mt}) = 0$$

or, $$A\, e^{mt} + \tau \cdot A\, m\, e^{mt} = 0$$

$$A\, e^{mt}\,(1 + \tau \cdot m) = 0$$

$\therefore$ $$m = -\frac{1}{\tau}$$

Then, $$I_{o,t} = A\, e^{mt} = Ae^{-t/\tau}$$

Steady state response (Particular integral) :

The steady state response is given by :

$$(1 + \tau D)\, I_{o,s} = I_i$$

(Subscript s refers to the steady state value)

or, $$I_{o,s} = (1 + \tau D)^{-1}\, I_i$$

$$= (1 - \tau D + \text{terms in } D^2 \text{ and higher})\, I_i$$

Step Input

Since the input I_i is a step of constant magnitude; its differential equals zero,

$$I_{o,s} = (1 - \tau D)\, I_i = I_i$$

Total response = Transisent response + steady state response

or

$$I_o = Ae^{-t/\tau} + SI_i$$

The constant A is evaluated from the initial conditions as follow :

At $t = 0, \quad I_o = 0$

$\therefore \quad 0 = A + SI_i \quad$ or, $A = -SI_i$

$$\therefore \quad I_o = \underbrace{-I_i\, e^{-t/\tau}}_{\text{Transient}} + \underbrace{I_i}_{\text{steady state}}$$

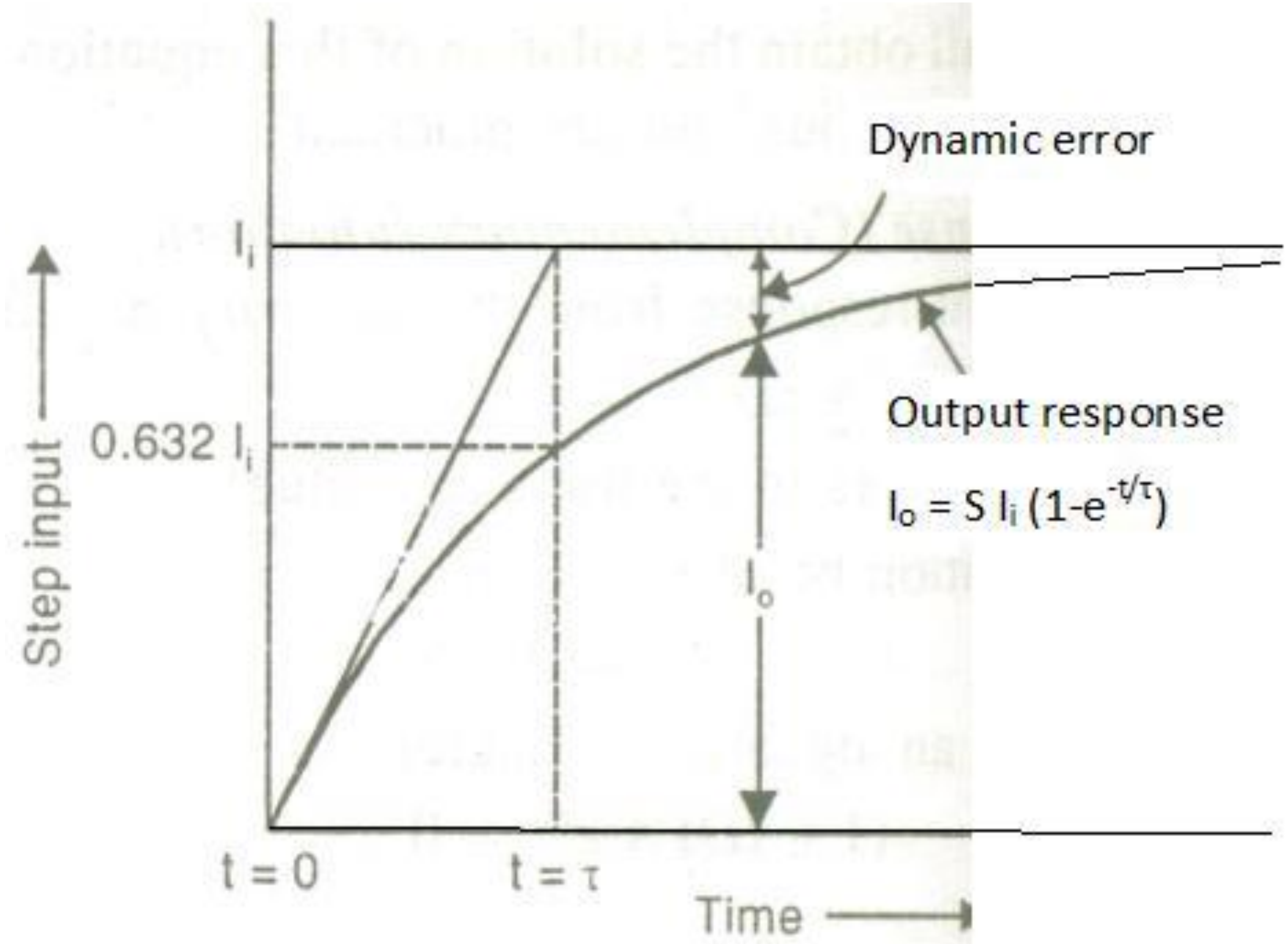
or,

$$I_o = I_i\,(1 - e^{-t/\tau})$$

or,

$$\frac{I_o}{I_i} = (1 - e^{-t/\tau})$$

Time response of first order system with step input



Salient features of first order system with step input

- The transient response is time dependent.
- The speed of response relates to time constant τ. For good fidelity, minimize value of τ.
- The time constant (τ), for a rising exponential function is defined as the time to reach 63.2% of its steady state value. The time constant (τ), for a decaying function would correspond to time taken to fall to 36.8% of its initial value.
- Dynamic Error
- In case the measurand has an initial value of $I_{initial}$ at t=0, then output I_o at any instant is given by

$$I_o = I_i [1 - e^{-t/\tau}] + I_{\text{initial}}\, e^{-t/\tau}$$

$$I_o = I_i + (I_{\text{initial}} - I_i)\, e^{-t/\tau}$$

- The speed response of a system is defined in terms of settling time (it is time taken by the system to reach and remain within a certain percentage tolerance band of the final steady state value.

Ramp Input

Input signal varies linearly with time so ramp input $I_i = \psi t$, where ψ is constant.

$$(1 + \tau D)\, I_o = \psi t$$

Now, *Transisent response* (complimentary function) is given as;

$$I_{o,t} = A\, e^{-t/\tau} \qquad \text{... as before}$$

and, *steady state response* (particular integral) will be;

$$I_{o,s} = (1 - \tau D + \text{terms in } D^2 \text{ and higher})\, \psi\, t = \psi t - \tau \frac{d}{dt}(\psi t)$$

or, $$I_{o,s} = \psi t - \psi \tau$$

∴ Complete response = Transisent response + steady state response

$$I_o = A\, e^{-t/\tau} + (\psi t - \psi \tau)$$

The value of constant A can be evaluated by applying the initial condition,

At, $$t = 0 \qquad I_0 = 0$$

or, $$0 = A - \psi \tau \qquad \therefore A = \psi \tau$$

∴ $$I_o = \psi t - \psi \tau + \psi \tau \times e^{-t/\tau} = \psi(t - \tau) + \psi \tau\, e^{-t/\tau}$$

or, $$I_o = \psi[t - \tau(1 - e^{-t/\tau})]$$

Time response of first order system with step input

The dynamic error,

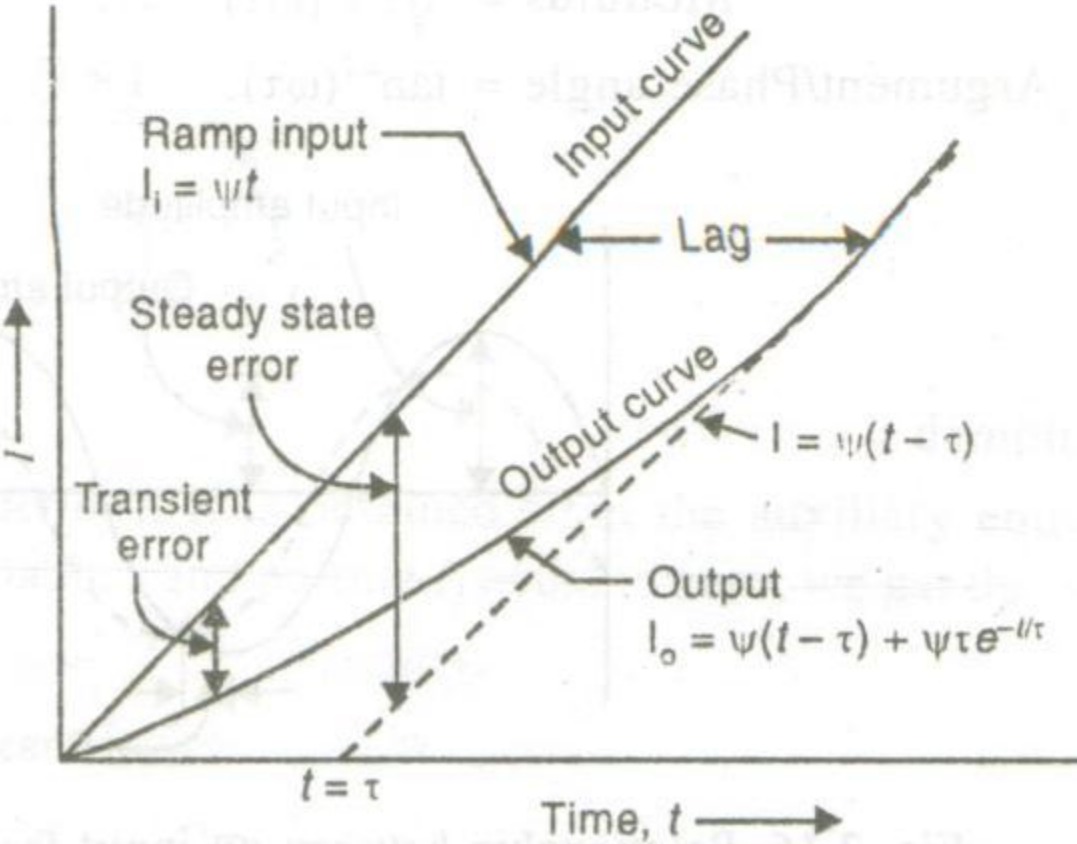
$$E_{dy.} = I_i - I_o$$

$$= \psi t - [\psi t - \psi\tau + \psi\tau\, e^{-t/\tau}]$$

$$= \underbrace{\psi\tau}_{\text{steady}} - \underbrace{\psi\tau\, e^{-t/\tau}}_{\text{Transisent}}$$

or, $$\frac{E_{dy.}}{\psi\tau} = 1 - e^{-t/\tau}$$

... (in dimensionless for



Salient features of first order system with Ramp Input

- The term ψt being independent of time continues to exist and so it is called the steady state error.
- The term $\psi \tau\, e^{-t/\tau}$ gradually decreases with time and hence is called transient error.
- The output response curve always lags behind the input curve by a constant amount known as lag.

Sinusoidal (Harmonic) Input

In order to determine the frequency response of sinusoidal input to a first order system, replace the transfer operator D by a factor *jw* in input/output relationship.

$$\frac{I_o}{I_i} = \frac{1}{1 + D\tau} = \frac{1}{1 + j\omega\tau}$$

ω = Input frequency, rad/s, and

$j = \sqrt{(-1)}$

i) **Amplitude ratio or modulus (I_o/I_i) :**

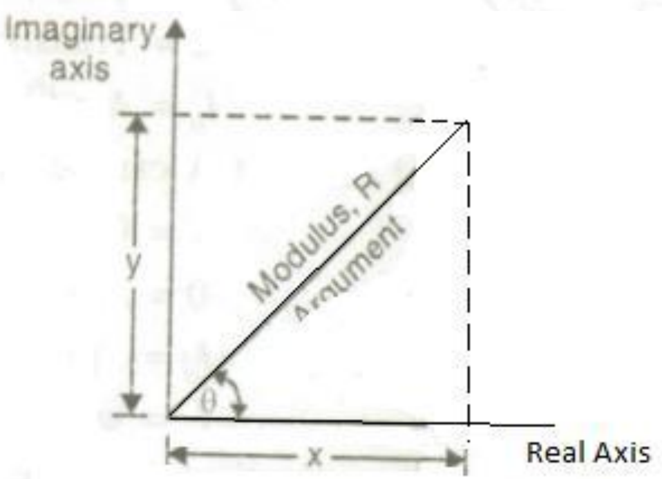
prescribes size of output amplitude

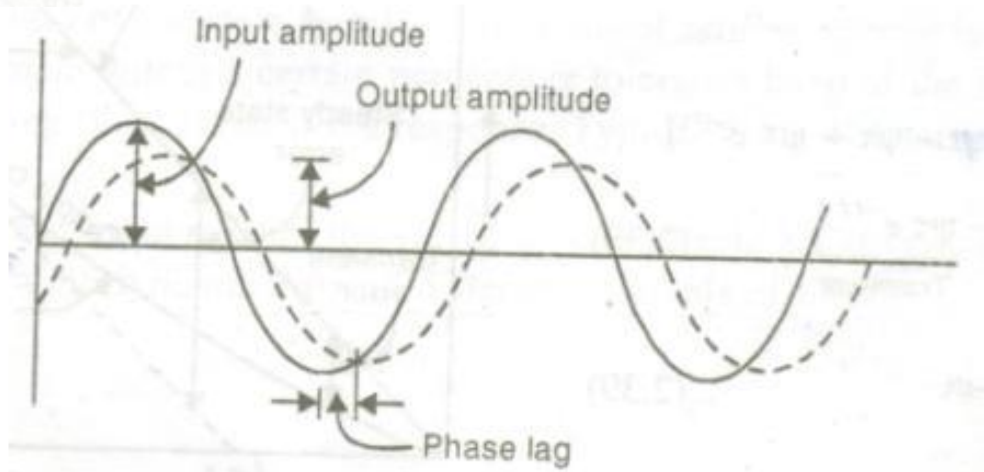
relative to input amplitude.

ii) **Phase Shift :**

$$\text{Modulus} = \sqrt{1 + (\omega\tau)^2}$$

$$\text{Argument/Phase angle} = \tan^{-1}(\omega\tau)$$





Salient features of first order system with Sinusoidal Input

- Output will be sinusoidal but magnitude of output amplitude may not be same.

The ratio of amplitude [often called attenuation]

Is given as

$$\frac{I_o}{I_i} = \frac{1}{\sqrt{1 + (\omega\tau)^2}}$$

Thus with Increase in input frequency amplitude ratio decreases.

- The phase difference is given by ϕ (phase angle) $= -\tan^{-1}(\omega\tau)$

  -Ve sign indicates that output lags behind input.

- When input & output signals are given as:

$I_i = A \sin \omega t$, and $I_o = B \sin(\omega t + \phi) = zA \sin(\omega t + \phi)$,

$$K = \left|\frac{I_o}{I_i}\right| = \frac{z}{\sqrt{1 + (\omega\tau)^2}}$$

Second order system with Sinusoidal Input

$$\frac{I_o}{I_i} = \frac{1}{\left(\frac{1}{\omega_n^2}\right)D^2 + \left(\frac{2\gamma}{\omega_n}\right)D + 1}$$

$$\left[\frac{1}{(\omega_n)^2}D^2 + \left(\frac{2\gamma}{\omega_n}\right)D + 1\right]I_o = I_i$$

a. Transient response- Replace D by an algebraic variable s & put $I_i = 0$

New roots are

$$s_1, s_2 = \frac{\frac{-2\gamma}{\omega_n} \pm \sqrt{\left(\frac{2\gamma}{\omega_n}\right)^2 - \frac{4}{\omega_n^2}}}{\frac{2}{\omega_n^2}}$$

$$= -\gamma\omega_n \pm \sqrt{\gamma^2\omega_n^2 - \omega_n^2}$$

$$s_1, s_2 = -\gamma\omega_n \pm \omega_n\sqrt{\gamma^2 - 1}$$

Thus transient solution is in the form

$$I_{o,t} = A e^{S_1 t} + B e^{S_2 t}$$

Where, A & B = arbitrary constants to be determined from initial conditions.

S1 & s2 = roots of auxiliary equations

The response of the system is of the following types depending upon the roots of the characteristics equations:

1. Over damped system
2. Critically damped system
3. Under damped system

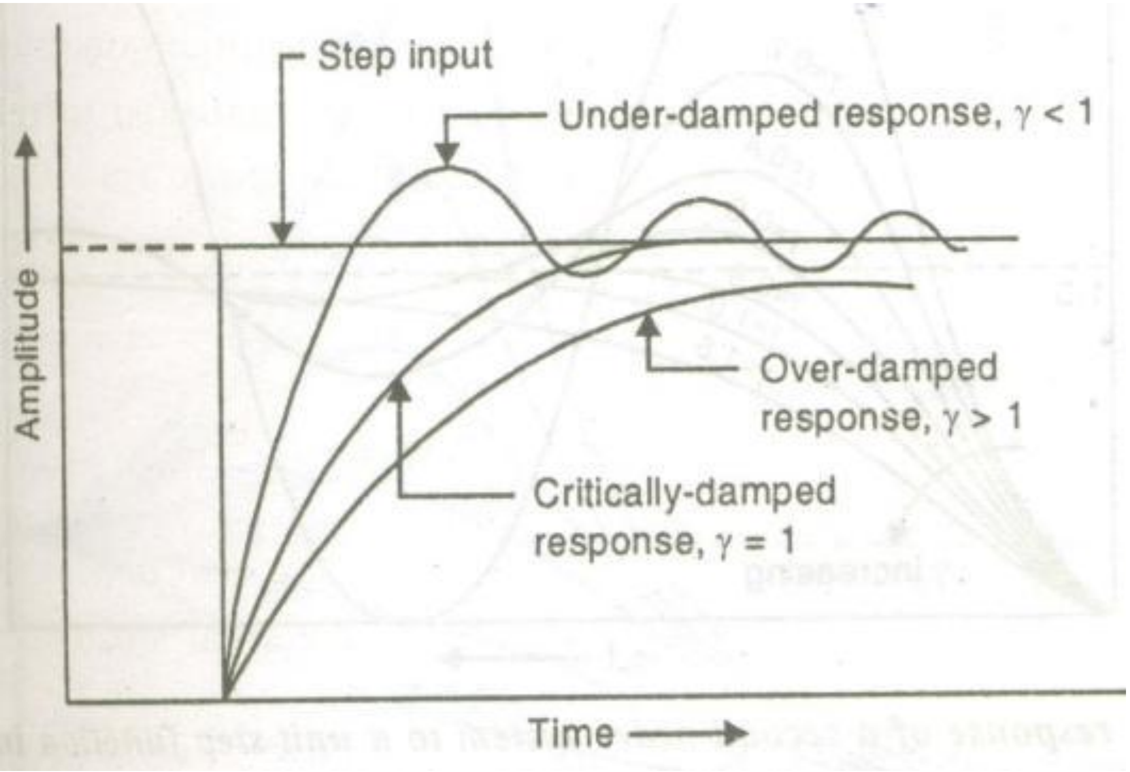
b) Steady state response-

$$\left(\frac{1}{\omega_n^2} D^2 + \frac{2\gamma}{\omega_n} D + 1\right) I_{o,s} = I_i$$

$$I_{o,s} = \left(1 + \frac{2\gamma}{\omega_n} D + \frac{1}{\omega_n^2} D^2\right)^{-1} I_i$$

$$= \left(1 - \frac{2\gamma}{\omega_n} D + \text{terms in } D^2 \text{ and higher}\right) I_i$$

Time response of second order system with step input



Since the input I_i is a step of constant magnitude, its differential equals zero and we have,

$$I_{o,s} = \left(1 - \frac{2\gamma}{\omega_n} D\right) I_i = I_i$$

∴ The complete response

$$I_o = I_{o,s} + I_{o,t}$$

or

$$I_o = I_i + A\,e^{s_1 t} + Be^{s_2 t}$$

For the under-damped system, the complex conjugate pair of roots are given by

$$s_1, s_2 = -\gamma\omega_n \pm j\omega_d$$

∴

$$I_o = I_i + Ae^{-(\gamma\omega_n + j\omega_d)\,t} + Be^{-(\gamma\omega_n - j\omega_d)\,t}$$

Replacing the complex exponentials by sines and cosines, we get

$$I_o = I_i + e^{-\gamma\omega_n t}\,(A \cos w_d t + B \sin w_d t)$$

By applying the initial conditions,

At $t = 0$, $\quad I_o = 0$ and $\dfrac{dI_o}{dt} = 0$

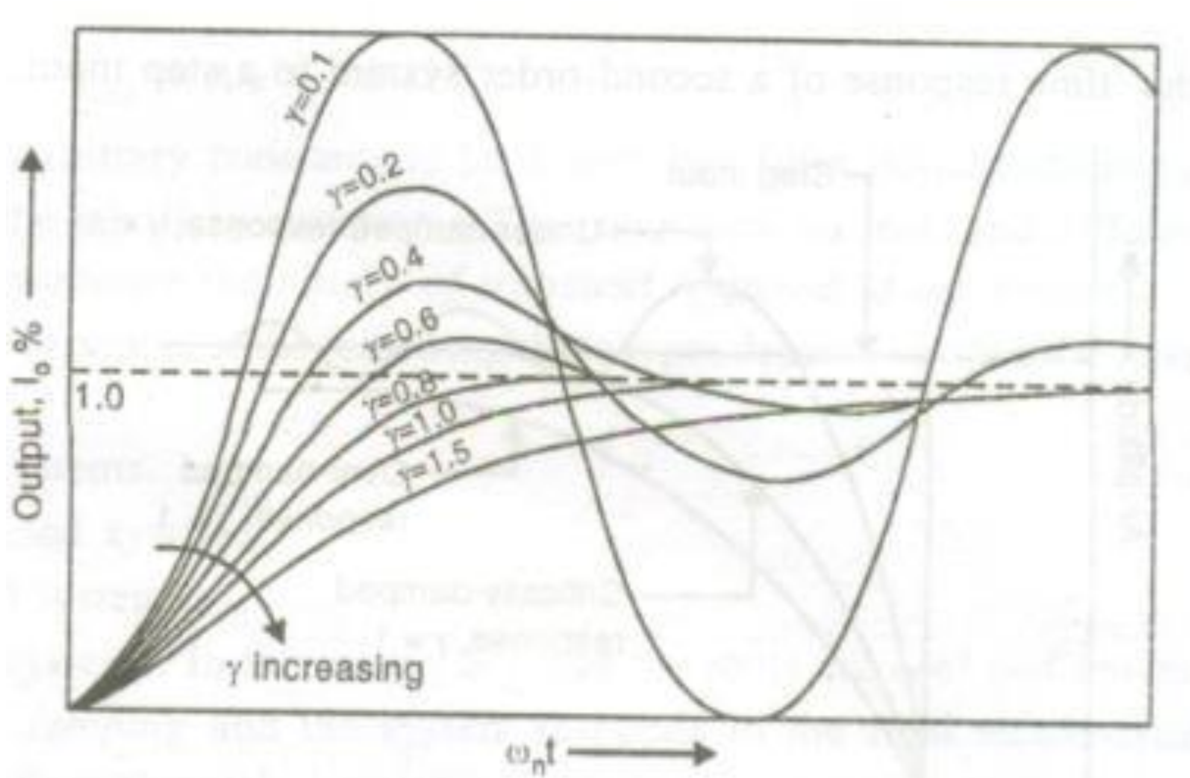
we get the values of the constant A and B as :

$$A = -I_i, \text{ and}$$

$$B = \frac{-I_i\,\gamma}{\sqrt{1-\gamma^2}}$$

$$I_o = I_i\left[1 - e^{-\gamma\omega_n t}\left\{\cos\omega_d t + \frac{\gamma}{\sqrt{1-\gamma^2}}\sin\omega_d t\right\}\right]$$

Transient response of a second order system to a unit step function input for different values of damping factor ϒ



The curve indicates overshoot & oscillation increase with reduced damping in system.

Sinusoidal (Harmonic) input

Steady state response is determined by replacing operator D by *jw* in input/output relationship

$$\frac{I_o}{I_i} = \frac{1}{\frac{1}{\omega_n^2}D^2 + \frac{2\gamma}{\omega_n}D + 1} = \frac{\omega_n^2}{D^2 + 2\gamma\omega_n D + \omega_n^2}$$

$$= \frac{\omega_n^2}{(j\omega)^2 + 2\gamma\omega_n(j\omega) + \omega_n^2} = \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + j(2\gamma\omega_n\omega)}$$

where, ω = Input frequency in rad/s, and

$j = \sqrt{-1}$

The denominator is a complex number having:

$$\text{Modulus} = \sqrt{\left[(\omega_n^2 - \omega^2)^2 + (2\gamma\omega_n\omega)^2\right]}$$

$$\text{Argument} = \tan^{-1}\left\{\frac{2\gamma\omega_n\omega}{\omega_n^2 - \omega^2}\right\}$$

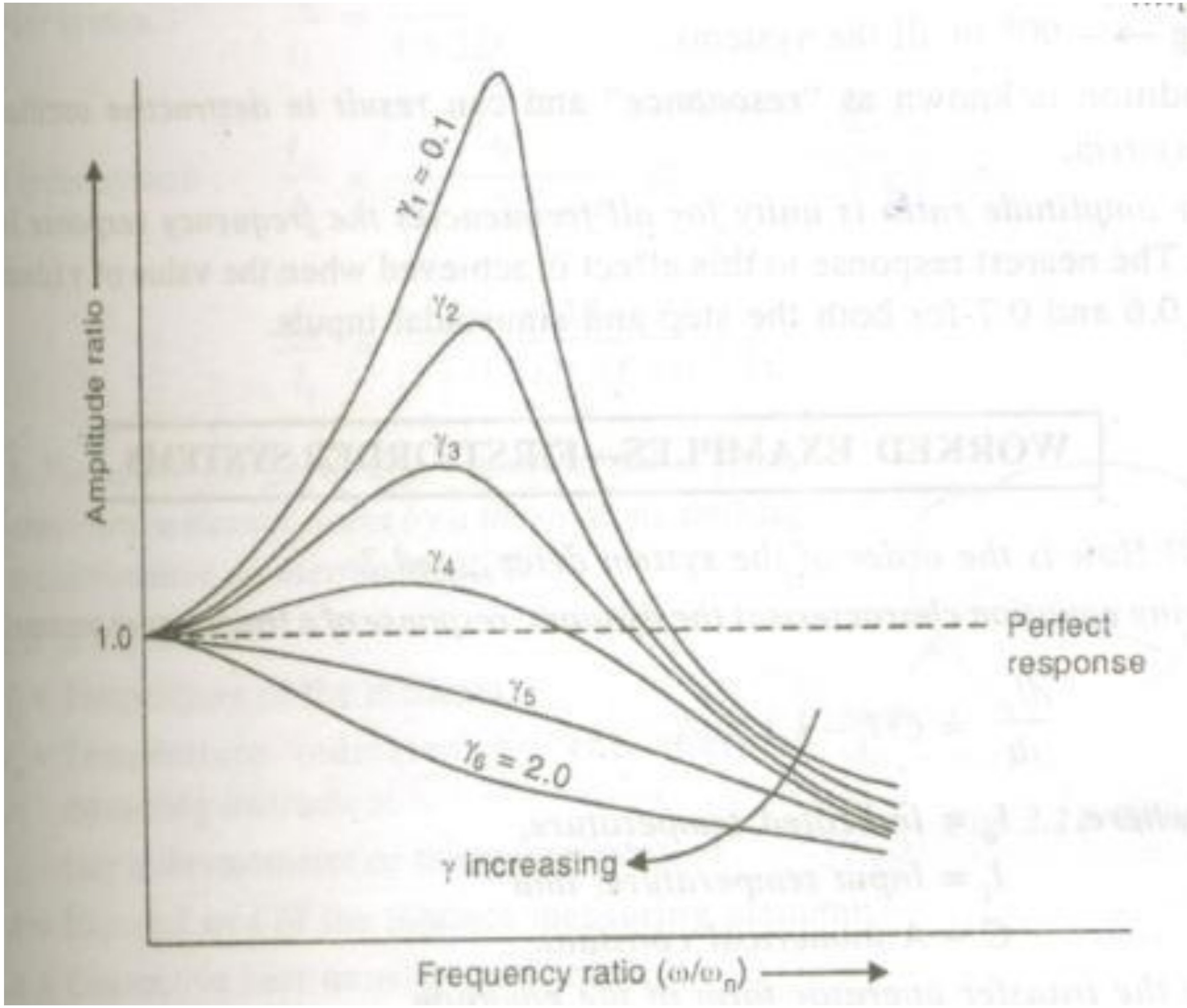
∴ The amplitude ratio,

$$\left|\frac{I_o}{I_i}\right| = \frac{\omega_n^2}{\sqrt{\left[(\omega_n^2 - \omega^2)^2 + (2\gamma\omega_n\omega)^2\right]}}$$

and, the system phase lag,

$$\phi = \tan^{-1}\left(\frac{2\gamma\omega_n\omega}{\omega_n^2 - \omega^2}\right)$$

- Step Response of second order system



- Plot between phase lag & Frequency ratio for second order system