Extended Surfaces/Fins

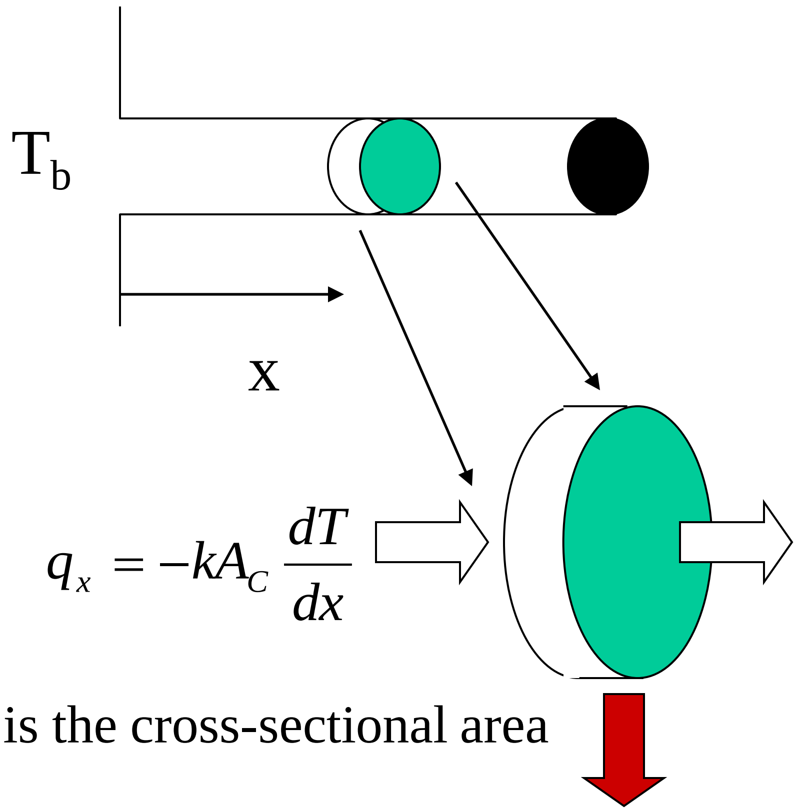
Convection: Heat transfer between a solid surface and a moving fluid is governed by the Newton's cooling law: $q = hA(T_s - T_\infty)$. Therefore, to increase the convective heat transfer, one can

- Increase the temperature difference $(T_s - T_\infty)$ between the surface and the fluid.

- Increase the convection coefficient h. This can be accomplished by increasing the fluid flow over the surface since h is a function of the flow velocity and the higher the velocity, the higher the h. Example: a cooling fan.

- Increase the contact surface area A. Example: a heat sink with fins.

Extended Surface Analysis



P: the fin perimeter

A_c: the fin cross-sectional area

$$q_x = -kA_C \frac{dT}{dx}$$

$$q_{x+dx} = q_x + \frac{dq_x}{dx} dx$$

A_C is the cross-sectional area

$dq_{conv} = h(dA_S)(T - T_\infty)$, where dA_S is the surface area of the element

Energy Balance: $q_x = q_{x+dx} + dq_{conv} = q_x + \frac{dq_x}{dx} dx + hdA_S(T - T_\infty)$

$-kA_C \frac{d^2T}{dx^2} dx + hP(T - T_\infty)dx = 0$, if k, A_C are all constants.

Extended Surface Analysis (cont.)

$$\frac{d^2T}{dx^2} - \frac{hP}{kA_C}(T - T_\infty) = 0,$$ A second-order, ordinary differential equation

Define a new variable $\theta(x) = T(x) - T_\infty$, so that

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0, \text{ where } m^2 = \frac{hP}{kA_C}, \ (D^2 - m^2)\theta = 0$$

Characteristics equation with two real roots: $+m$ & $-m$

The general solution is of the form

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$

To evaluate the two constants C_1 and C_2, we need to specify two boundary conditions:

The first one is obvious: the base temperature is known as $T(0) = T_b$

The second condition will depend on the end condition of the tip

Extended Surface Analysis (cont.)

For example: assume the tip is insulated and no heat transfer $d\theta/dx(x=L)=0$

The temperature distribution is given by

$$\frac{T(x)-T_\infty}{T_b-T_\infty}=\frac{\theta}{\theta_b}=\frac{\cosh m(L-x)}{\cosh mL}$$

The fin heat transfer rate is

$$q_f=-kA_C\frac{dT}{dx}(x=0)=\sqrt{hPkA_C}\tanh mL=M\tanh mL$$

These results and other solutions using different end conditions are tabulated in the following fins table

Temperature distribution for fins of different configurations

Case	Tip Condition	Temp. Distribution	Fin heat transfer
A	Convection heat transfer: $h\theta(L)=-k(d\theta/dx)_{x=L}$	$\dfrac{\cosh m(L-x)+(h/mk)\sinh m(L-x)}{\cosh mL+(h/mk)\sinh mL}$	$M\dfrac{\sinh mL+(h/mk)\cosh mL}{\cosh mL+(h/mk)\sinh mL}$
B	Adiabatic $(d\theta/dx)_{x=L}=0$	$\dfrac{\cosh m(L-x)}{\cosh mL}$	$M\ \tanh mL$
C	Given temperature: $\theta(L)=\theta_L$	$\dfrac{(\theta_L/\theta_b)\sinh m(L-x)+\sinh m(L-x)}{\sinh mL}$	$M\ \dfrac{(\cosh mL-\theta_L/\theta_b)}{\sinh mL}$
D	Infinitely long fin $\theta(L)=0$	e^{-mx}	M

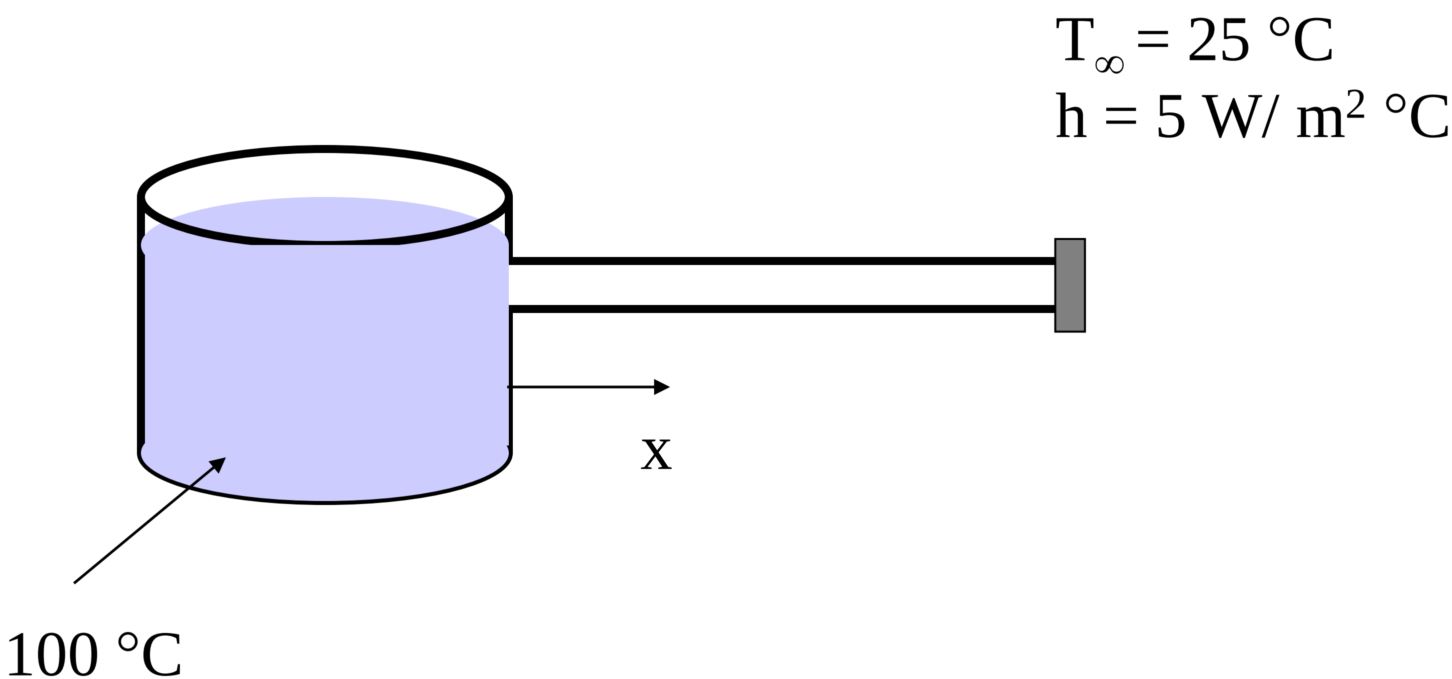
$$\theta \equiv T - T_\infty, \quad m^2 \equiv \frac{hP}{kA_C}$$

$$\theta_b = \theta(0) = T_b - T_\infty, \quad M = \sqrt{hPkA_C}\,\theta_b$$

Note: This table is adopted from *Introduction to Heat Transfer* by Frank Incropera and David DeWitt

Example

An Aluminum pot is used to boil water as shown below. The handle of the pot is 20-cm long, 3-cm wide, and 0.5-cm thick. The pot is exposed to room air at 25°C, and the convection coefficient is 5 W/m^2 °C. Question: can you touch the handle when the water is boiling? (k for aluminum is 237 W/m °C)



Example (cont.)

We can model the pot handle as an extended surface. Assume that there is no heat transfer at the free end of the handle. The condition matches that specified in the fins Table, case B.

h=5 W/ m^2 °C, P=2W+2t=2(0.03+0.005)=0.07(m), k=237 W/m °C, A_C=Wt=0.00015(m^2), L=0.2(m)

Therefore, $m=(hP/kA_C)^{1/2}=3.138$,

$M=\sqrt{(hPkA_C)}(T_b-T_\infty)=0.111\theta_b=0.111(100-25)=8.325(W)$

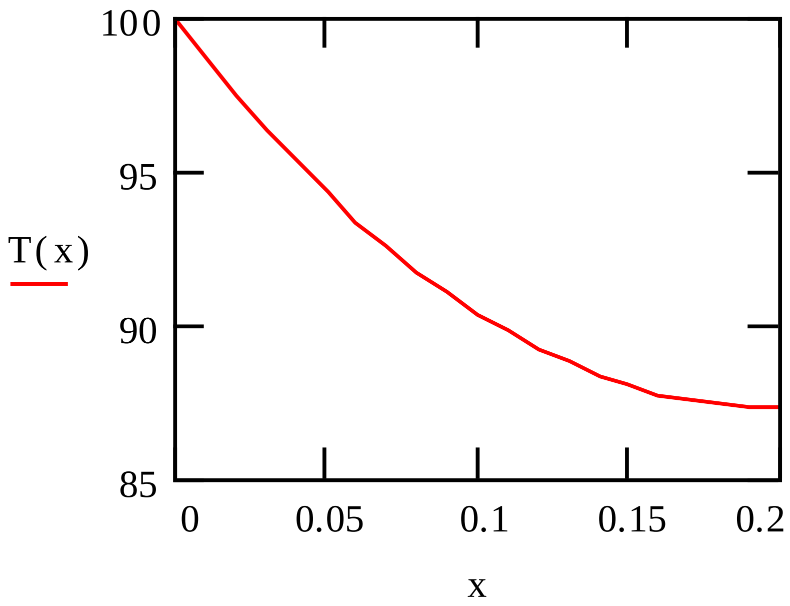
$$\frac{T(x)-T_\infty}{T_b-T_\infty}=\frac{\theta}{\theta_b}=\frac{\cosh m(L-x)}{\cosh mL}$$

$$\frac{T-25}{100-25}=\frac{\cosh[3.138(0.2-x)]}{\cosh(3.138*0.2)},$$

$$T(x)=25+62.32*\cosh[3.138(0.2-x)]$$

Example (cont.)

Plot the temperature distribution along the pot handle



As shown, temperature drops off very quickly. At the midpoint T(0.1)=90.4°C. At the end T(0.2)=87.3°C.

Therefore, it should not be safe to touch the end of the handle

Example (cont.)

The total heat transfer through the handle can be calculated also. q_f=Mtanh(mL)=8.325*tanh(3.138*0.2)=4.632(W) Very small amount: latent heat of evaporation for water: 2257 kJ/kg. Therefore, the amount of heat loss is just enough to vaporize 0.007 kg of water in one hour.

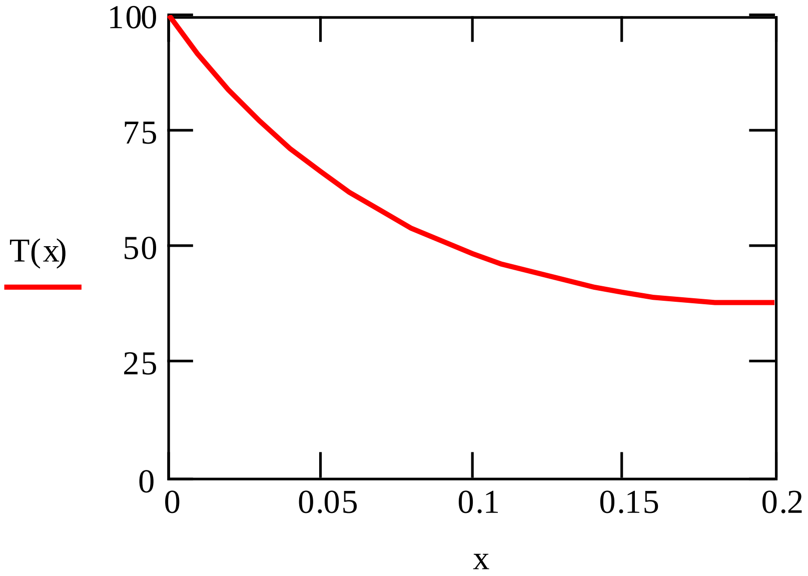
If a stainless steel handle is used instead, what will happen: For a stainless steel, the thermal conductivity k=15 W/m°C. Use the same parameter as before:

$$m = \left(\frac{hP}{kA_C} \right)^{1/2} = 12.47, \quad M = \sqrt{hPkA_C} = 0.0281$$

Example (cont.)

$$\frac{T(x)-T_\infty}{T_b-T_\infty}=\frac{\cosh m(L-x)}{\cosh mL}$$

$$T(x)=25+12.3\cosh[12.47(L-x)]$$



Temperature at the handle (x=0.2 m) is only 37.3 °C, not hot at all. This example illustrates the important role played by the thermal conductivity of the material in terms of conductive heat transfer.