One-Dimensional, Steady-State Conduction without Thermal Energy Generation

Methodology of a Conduction Analysis

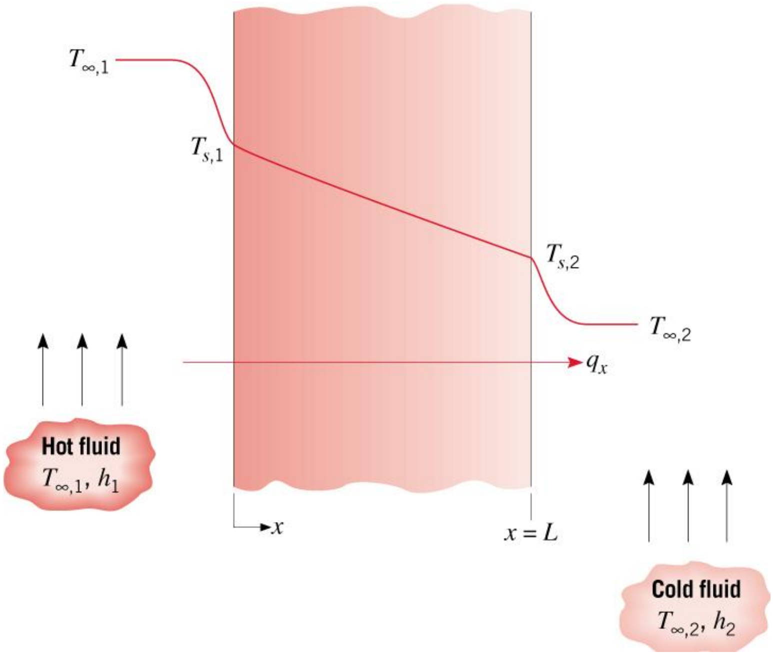
- Specify appropriate form of the heat equation.
- Solve for the temperature distribution.
- Apply Fourier's Law to determine the heat flux.

Simplest Case: One-Dimensional, Steady-State Conduction with No Thermal Energy Generation.

- Common Geometries:
  - The Plane Wall: Described in rectangular (x) coordinate. Area perpendicular to direction of heat transfer is constant (independent of x).
  - The Tube Wall: Radial conduction through tube wall.
  - The Spherical Shell: Radial conduction through shell wall.

The Plane Wall

- Consider a plane wall between two fluids of different temperature:



- Heat Equation:

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0 \qquad (3.1)$$

- Implications:

  Heat flux $\left(q''_x\right)$ is independent of *x*.

  Heat rate $\left(q_x\right)$ is independent of *x*.

- Boundary Conditions: $T(0)=T_{s,1}, \quad T(L)=T_{s,2}$

- Temperature Distribution for Constant k:

$$T(x)=T_{s,1}+\left(T_{s,2}-T_{s,1}\right)\frac{x}{L} \qquad (3.3)$$

- Heat Flux and Heat Rate:

$$q''_x = -k\frac{dT}{dx} = \frac{k}{L}\left(T_{s,1} - T_{s,2}\right) \quad (3.5)$$

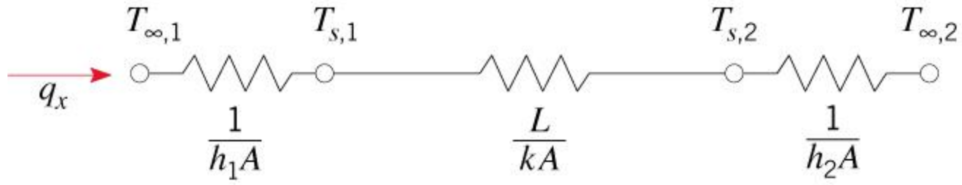
$$q_x = -kA\frac{dT}{dx} = \frac{kA}{L}\left(T_{s,1} - T_{s,2}\right) \quad (3.4)$$

- Thermal Resistances $\left(R_t = \frac{\Delta T}{q}\right)$ and Thermal Circuits:

Conduction in a plane wall: $$R_{t,cond} = \frac{L}{kA} \quad (3.6)$$

Convection: $$R_{t,conv} = \frac{1}{hA} \quad (3.9)$$

Thermal circuit for plane wall with adjoining fluids:



$$R_{tot} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A} \quad (3.12)$$

$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{R_{tot}} \quad (3.11)$$

- Thermal Resistance for Unit Surface Area:

$$R''_{t,cond} = \frac{L}{k} \qquad R''_{t,conv} = \frac{1}{h}$$

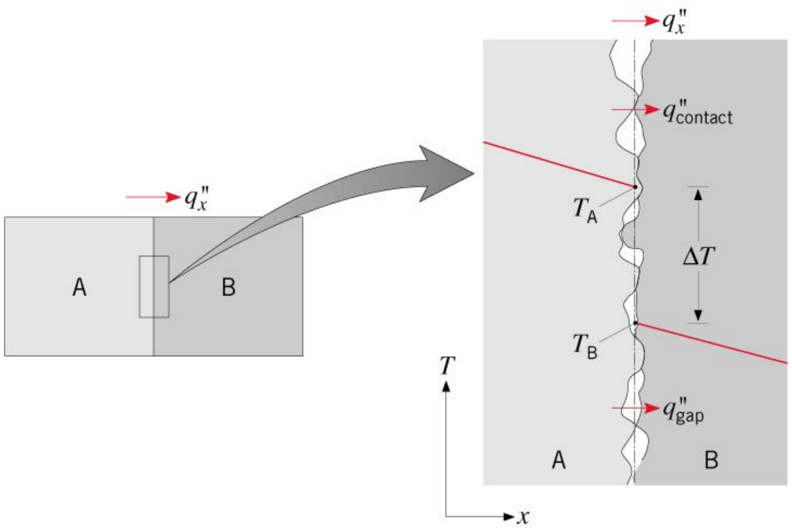
Units: $R_t \leftrightarrow \mathrm{W/K} \qquad R''_t \leftrightarrow \mathrm{m^2 \cdot K/W}$

- Radiation Resistance:

$$R_{t,rad} = \frac{1}{h_r A} \qquad R''_{t,rad} = \frac{1}{h_r}$$

$$h_r = \varepsilon\sigma\left(T_s + T_{sur}\right)\left(T_s^2 + T_{sur}^2\right) \quad (1.9)$$

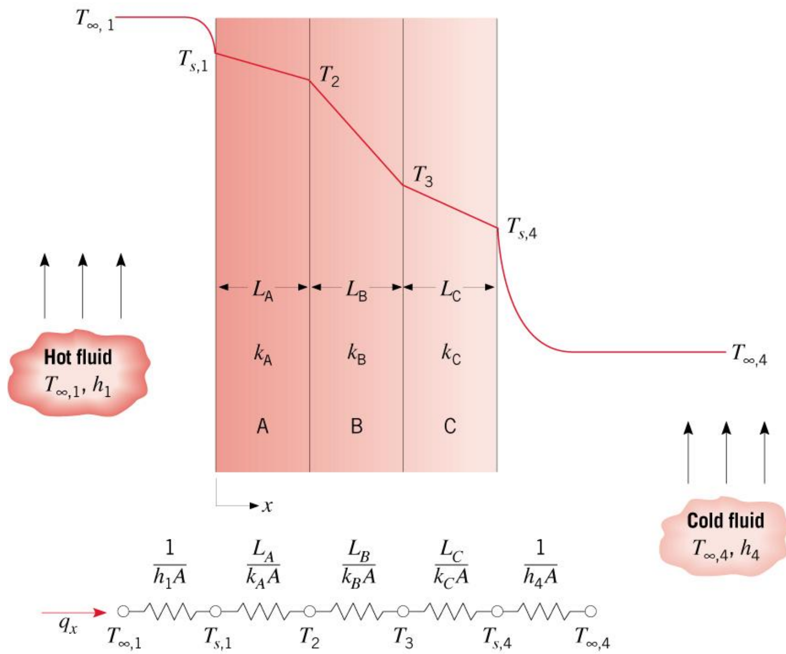
- Contact Resistance:



$$R''_{t,c} = \frac{T_A - T_B}{q''_x} \qquad \boldsymbol{R_{t,c} = \frac{R''_{t,c}}{A_c}}$$

Values depend on: Materials A and B, surface finishes, interstitial conditions, and contact pressure (Tables 3.1 and 3.2)

- Composite Wall with Negligible Contact Resistance:



$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{R_{tot}} \qquad (3.14)$$

$$R_{tot} = \frac{1}{A}\left[\frac{1}{h_1} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{h_4}\right] = \frac{R''_{tot}}{A}$$

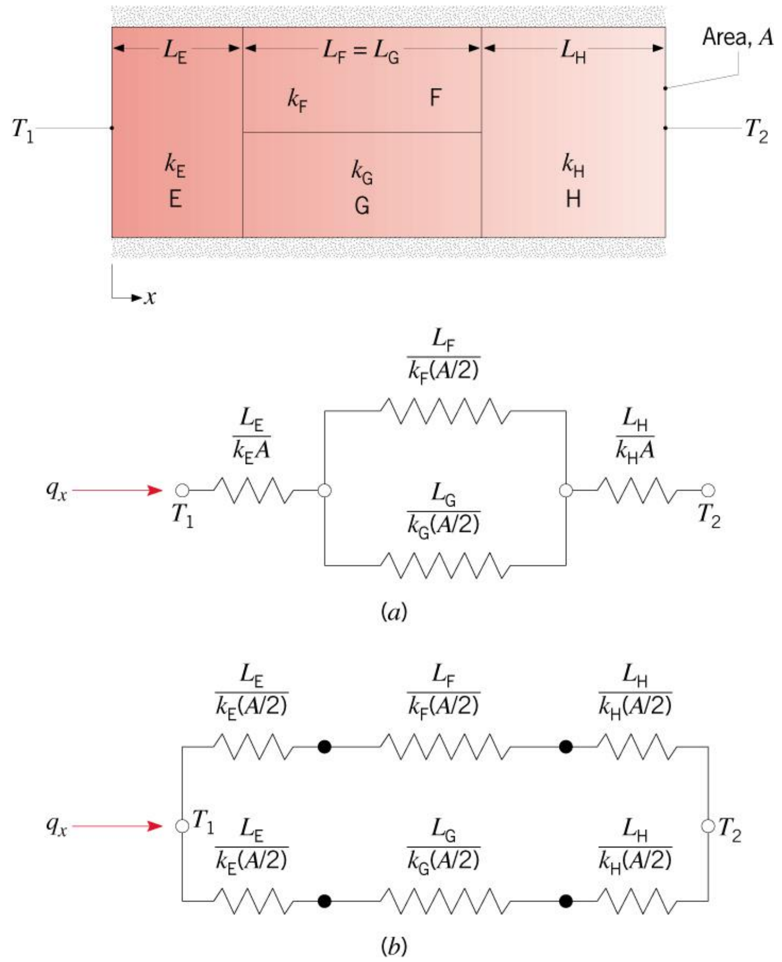
- Overall Heat Transfer Coefficient (U) :

A modified form of Newton's Law of Cooling to encompass multiple resistances to heat transfer.

$$q_x = UA\Delta T_{overall} \qquad (3.17)$$

$$R_{tot} = \frac{1}{UA} \qquad (3.19)$$

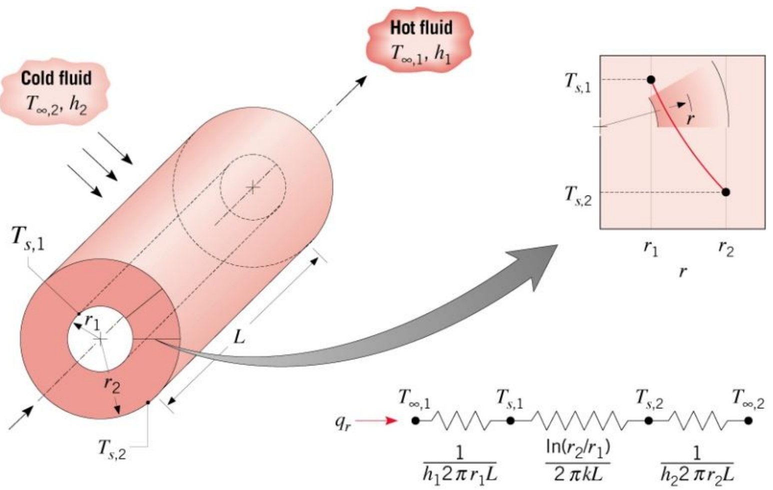
- Series – Parallel Composite Wall:



- Note departure from one-dimensional conditions for $k_F \neq k_G$.

- Circuits based on assumption of isothermal surfaces normal to *x* direction or adiabatic surfaces parallel to *x* direction provide approximations for q_x .

The Tube Wall



- Heat Equation:

$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right)=0 \qquad (3.23)$$

What does the form of the heat equation tell us about the variation of q_r with r in the wall?

Is the foregoing conclusion consistent with the energy conservation requirement?

How does q_r'' vary with r?

- Temperature Distribution for Constant k:

$$T(r)=\frac{T_{s,1}-T_{s,2}}{\ln(r_1/r_2)}\ln\left(\frac{r}{r_2}\right)+T_{s,2} \qquad (3.26)$$

- Heat Flux and Heat Rate:

$$q_r'' = -k\frac{dT}{dr} = \frac{k}{r\ln\left(r_2/r_1\right)}\left(T_{s,1} - T_{s,2}\right)$$

$$q_r' = 2\pi r q_r'' = \frac{2\pi k}{\ln\left(r_2/r_1\right)}\left(T_{s,1} - T_{s,2}\right)$$

$$q_{r=} 2\pi r L q_r'' = \frac{2\pi L k}{\ln\left(r_2/r_1\right)}\left(T_{s,1} - T_{s,2}\right) \qquad (3.27)$$

- Conduction Resistance:

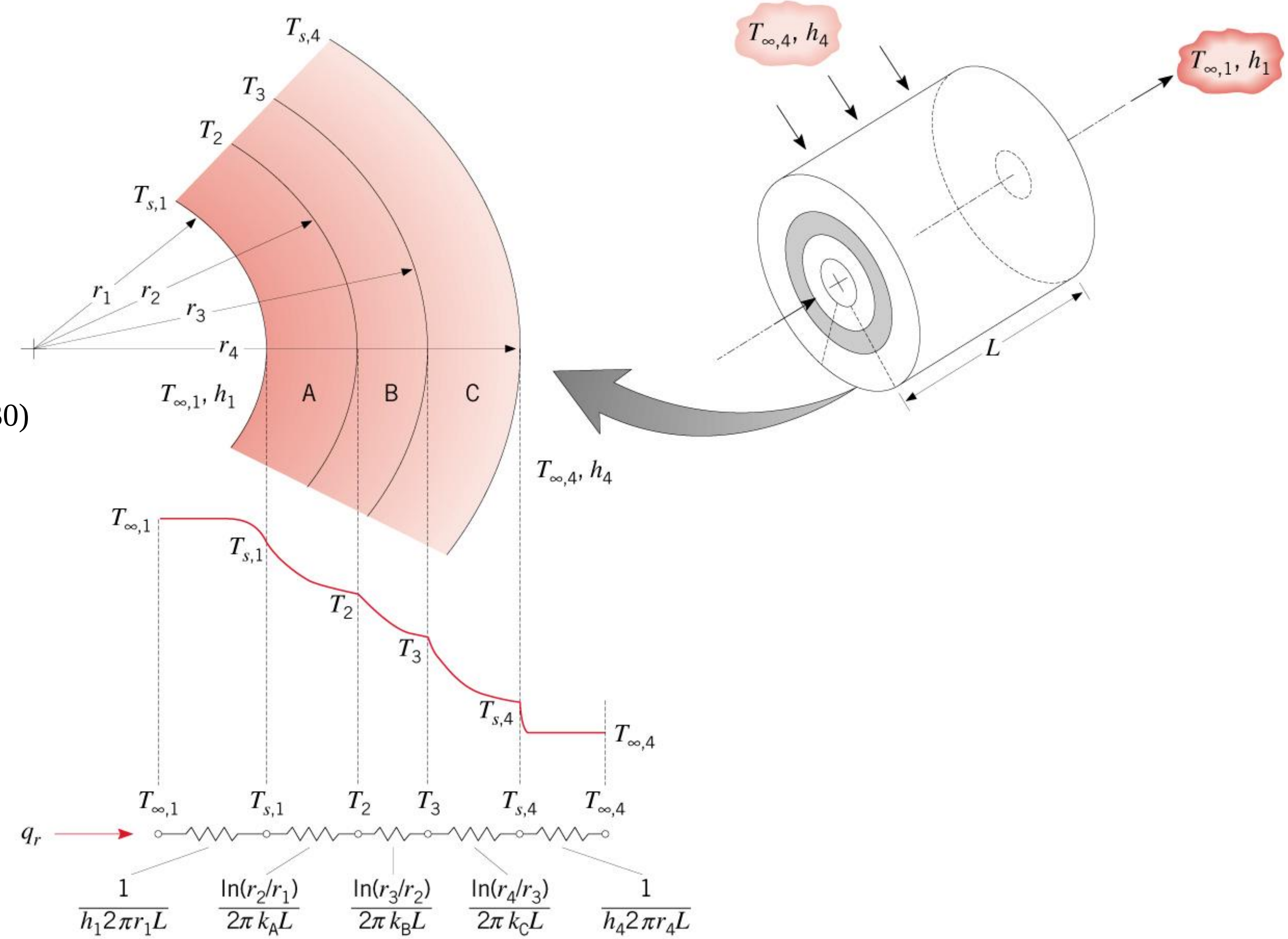
$$R_{t,cond} = \frac{\ln\left(r_2/r_1\right)}{2\pi L k} \qquad \text{Units} \leftrightarrow \text{K/W} \qquad (3.28)$$

$$R'_{t,cond} = \frac{\ln\left(r_2/r_1\right)}{2\pi k} \qquad \text{Units} \leftrightarrow \text{m}\cdot\text{K/W}$$

Why is it inappropriate to base the thermal resistance on a unit surface area?

- Composite Wall with Negligible Contact Resistance

$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{R_{tot}} = UA\left(T_{\infty,1} - T_{\infty,4}\right) \qquad (3.30)$$



Note that

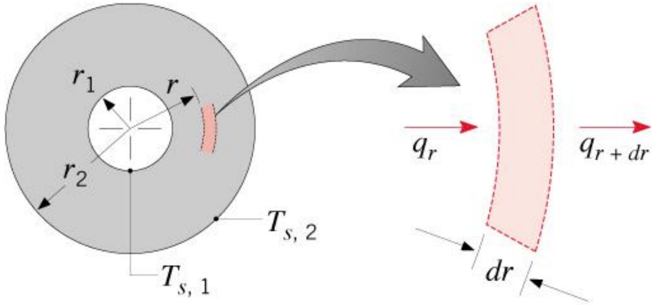
$$UA = R_{tot}^{-1}$$

is a constant independent of radius.

But, U itself is tied to specification of an interface.

$$U_i = \left(A_i R_{tot}\right)^{-1} \qquad (3.32)$$

Spherical Shell



- Heal Equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)=0$$

What does the form of the heat equation tell us about the variation of q_r with r? Is this result consistent with conservation of energy?

How does q_r'' vary with r ?

- Temperature Distribution for Constant k:

$$T(r)=T_{s,1}-\left(T_{s,1}-T_{s,2}\right)\frac{1-\left(r_1/r\right)}{1-\left(r_1/r_2\right)}$$

- Heat flux, Heat Rate and Thermal Resistance:

$$q''_r = -k\frac{dT}{dr} = \frac{k}{r^2\left[(1/r_1)-(1/r_2)\right]}\left(T_{s,1}-T_{s,2}\right)$$

$$q_r = 4\pi r^2 q''_r = \frac{4\pi k}{(1/r_1)-(1/r_2)}\left(T_{s,1}-T_{s,2}\right) \qquad (3.35)$$

$$R_{t,cond} = \frac{(1/r_1)-(1/r_2)}{4\pi k} \qquad (3.36)$$

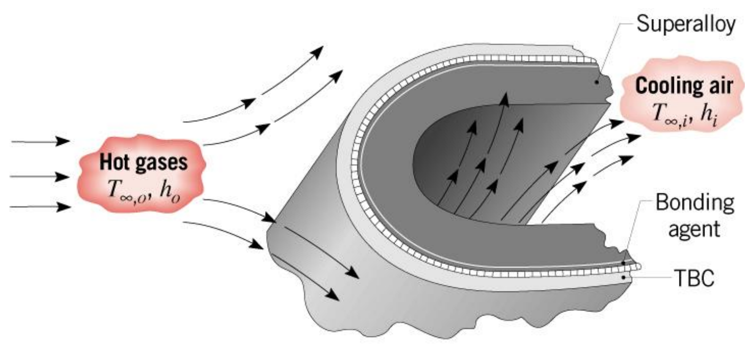
- Composite Shell:

$$q_r = \frac{\Delta T_{overall}}{R_{tot}} = UA\Delta T_{overall}$$

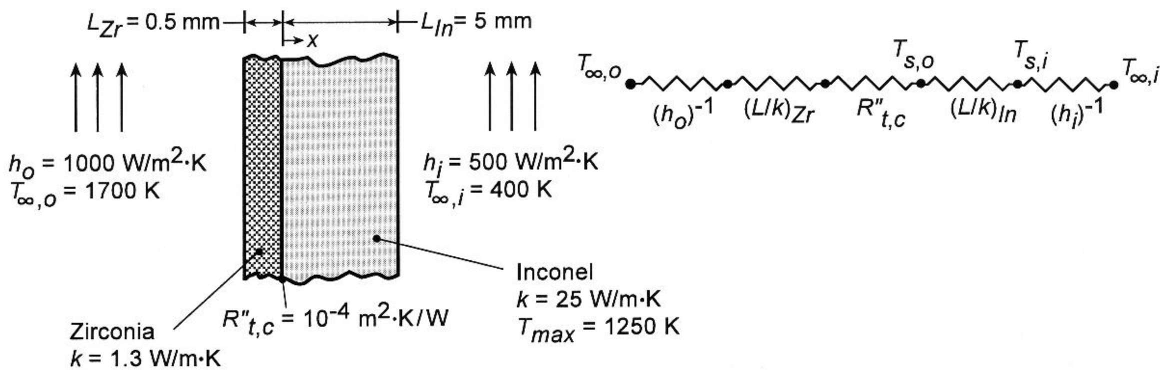
$$UA = R_{tot}^{-1} \leftrightarrow \text{Constant}$$

$$U_i = \left(A_i R_{tot}\right)^{-1} \leftrightarrow \text{ Depends on } A_i$$

Problem 3.23: Assessment of thermal barrier coating (TBC) for protection of turbine blades. Determine maximum blade temperature with and without TBC.



Schematic:



ASSUMPTIONS: (1) One-dimensional, steady-state conduction in a composite plane wall, (2) Constant properties, (3) Negligible radiation.

ANALYSIS: For a unit area, the total thermal resistance with the TBC is

$$R''_{tot,w} = h_o^{-1} + (L/k)_{Zr} + R''_{t,c} + (L/k)_{In} + h_i^{-1}$$

$$R''_{tot,w} = \left(10^{-3} + 3.85\times10^{-4} + 10^{-4} + 2\times10^{-4} + 2\times10^{-3}\right) m^2 \cdot \mathrm{K/W} = 3.69\times10^{-3}\,\mathrm{m^2\times K/W}$$

With a heat flux of

$$q''_w = \frac{T_{\infty,o} - T_{\infty,i}}{R''_{tot,w}} = \frac{1300\,\mathrm{K}}{3.69\times10^{-3}\,\mathrm{m^2\times K/W}} = 3.52\times10^{5}\,\mathrm{W/m^2}$$

the inner and outer surface temperatures of the Inconel are

$$T_{s,i(w)} = T_{\infty,i} + (q''_w/h_i) = 400\,\mathrm{K} + \left(3.52\times10^{5}\,\mathrm{W/m^2} / 500\,\mathrm{W/m^2\times K}\right) = 1104\,\mathrm{K}$$

$$T_{s,o(w)} = T_{\infty,i} + \left[(1/h_i) + (L/k)_{In}\right] q''_w = 400\,\mathrm{K} + \left(2\times10^{-3} + 2\times10^{-4}\right)\mathrm{m^2\times K/W}\left(3.52\times10^{5}\,\mathrm{W/m^2}\right) = 1174\,\mathrm{K}$$

Without the TBC,

$$R''_{tot,wo} = h_o^{-1} + (L/k)_{In} + h_i^{-1} = 3.20\times10^{-3}\,\mathrm{m^2\times K/W}$$

$$q''_{wo} = (T_{\infty,o} - T_{\infty,i})/R''_{tot,wo} = 4.06\times10^{5}\ \mathrm{W/m^2}.$$

the inner and outer surface temperatures of the Inconel are

$$\mathrm{T_{s,i(wo)}} = \mathrm{T_{\infty,i}} + (\mathrm{q''_{wo}/h_i}) = 1212\,\mathrm{K}$$

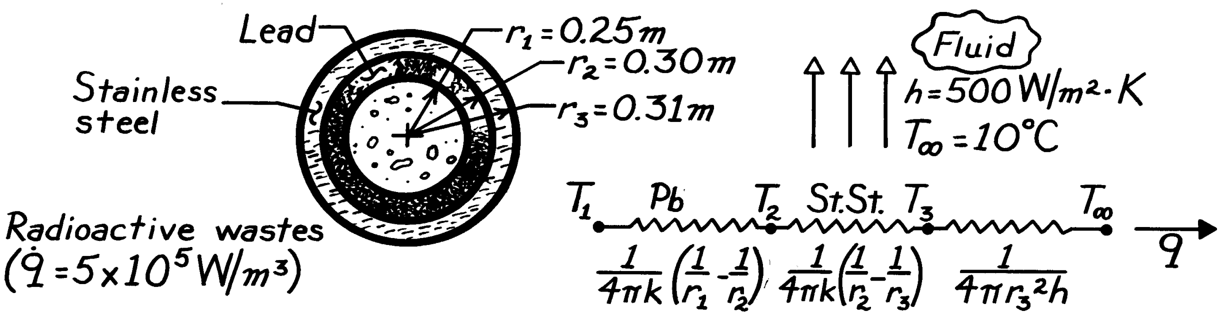
$$T_{s,o(wo)} = T_{\infty,i} + [(1/h_i) + (L/k)_{In}]q''_{wo} = 1293\,K$$

Use of the TBC facilitates operation of the Inconel below T_{max} = 1250 K.

COMMENTS: Since the durability of the TBC decreases with increasing temperature, which increases with increasing thickness, limits to its thickness are associated with reliability considerations.

Problem 3.62: Suitability of a composite spherical shell for storing radioactive wastes in oceanic waters.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Steady-state conditions, (3) Constant properties at 300K, (4) Negligible contact resistance.

PROPERTIES: *Table A-1*, Lead: k = 35.3 W/m·K, MP = 601K; St.St.: 15.1 W/m·K.

ANALYSIS: From the thermal circuit, it follows that

$$q=\frac{T_1-T_\infty}{R_{tot}}=\dot{q}\left[\frac{4}{3}\pi\, r_1^3\right]$$

The thermal resistances are:

$$R_{Pb} = \left[1/\left(4\pi \times 35.3\ \text{W/m}\cdot\text{K}\right)\right] \left[\frac{1}{0.25\text{m}} - \frac{1}{0.30\text{m}}\right] = 0.00150\ \text{K/W}$$

$$R_{St.St.} = \left[1/\left(4\pi \times 15.1\ \text{W/m}\cdot\text{K}\right)\right] \left[\frac{1}{0.30\text{m}} - \frac{1}{0.31\text{m}}\right] = 0.000567\ \text{K/W}$$

$$R_{conv} = \left[1/\left(4\pi \times 0.31^2\,\text{m}^2 \times 500\ \text{W/m}^2\cdot\text{K}\right)\right] = 0.00166\ \text{K/W}$$

$$R_{tot} = 0.00372\ \text{K/W.}$$

The heat rate is then

$$q = 5 \times 10^5\ \text{W/m}^3\,(4\pi/3)(0.25\text{m})^3 = 32{,}725\ \text{W}$$

and the inner surface temperature is

$$T_1 = T_\infty + R_{tot}\ q = 283\text{K} + 0.00372\text{K/W}\left(32{,}725\ \text{W}\right) = 405\ \text{K} < \text{MP} = 601\text{K.}$$

Hence, from the thermal standpoint, the proposal is adequate.

COMMENTS: In fabrication, attention should be given to maintaining a good thermal contact. A protective outer coating should be applied to prevent long term corrosion of the stainless steel.