Example 17.1

Consider the following state space model:

$$\mathbf{A} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Then

$$\mathbf{\Gamma}_o[\mathbf{A}, \mathbf{C}] = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -4 & -2 \end{bmatrix}$$

Hence, rank $\Gamma_0[\mathbf{A}, \mathbf{C}] = 2$, and the system is completely observable.

Example 17.8

Consider

$$\mathbf{A} = \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Here

$$\mathbf{\Gamma}_o[\mathbf{A}, \mathbf{C}] = \begin{bmatrix} 1 & -1 \\ -2 & -2 \end{bmatrix}$$

Hence, rank $\Gamma_0[\mathbf{A}, \mathbf{C}] = 1 < 2$, and the system is not completely observable.

Duality

We see a remarkable similarity between the results in Theorem 17.2 and in Theorem 17.3. We can formalize this as follows:

Theorem 17.4 (*Duality*). Consider a state space model described by the 4-tuple $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$. Then the system is completely controllable if and only if the dual system $(\mathbf{A}^T, \mathbf{C}^T, \mathbf{B}^T, \mathbf{D}^T)$ is completely observable.

Observable Decomposition

The above theorem can often be used to go from a result on controllability to one on observability, and vice versa. For example, the dual of Lemma 17.1 is the following:

Lemma 17.4: If rank$\{\Gamma_0[\mathbf{A}, \mathbf{C}]\} = k < n$, there exists a similarity transformation T such that with $\bar{x} = \mathbf{T}^{-1} x$, $\overline{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \overline{\mathbf{C}} = \mathbf{C}\mathbf{T}$, then $\overline{\mathbf{C}}$ and $\overline{\mathbf{A}}$ take the form

$$\overline{\mathbf{A}} = \begin{bmatrix} \overline{\mathbf{A}}_o & 0 \\ \overline{\mathbf{A}}_{21} & \overline{\mathbf{A}}_{no} \end{bmatrix} \qquad \overline{\mathbf{C}} = \begin{bmatrix} \overline{\mathbf{C}}_o & 0 \end{bmatrix}$$

where $\overline{\mathbf{A}}_0$ has dimension k and the pair $(\overline{\mathbf{C}}_0 \overline{\mathbf{A}}_0)$ is completely observable.

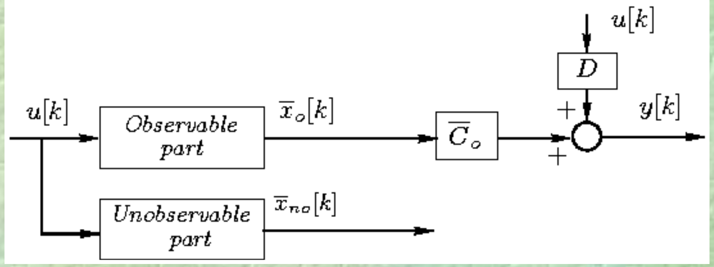
The above result has a relevance similar to that of the controllability property and the associated decomposition. To appreciate this, we apply the dual of Lemma 17.1 to express the (*transformed*) state and output equations in partitioned form as

$$\delta \begin{bmatrix} \overline{x}_o[k] \\ \overline{x}_{no}[k] \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{A}}_o & 0 \\ \overline{\mathbf{A}}_{21} & \overline{\mathbf{A}}_{n0} \end{bmatrix} \begin{bmatrix} \overline{x}_o[k] \\ \overline{x}_{no}[k] \end{bmatrix} + \begin{bmatrix} \overline{\mathbf{B}}_o \\ \overline{\mathbf{B}}_{no} \end{bmatrix} u[k]$$

$$y[k] = \begin{bmatrix} \overline{\mathbf{C}}_o & 0 \end{bmatrix} \begin{bmatrix} \overline{x}_o[k] \\ \overline{x}_{no}[k] \end{bmatrix} + \mathbf{D}u[k]$$

A pictorial description of these equations is shown on the next slide.

Figure 17.2: *Observable-unobservable decomposition*



The observable subspace of a plant is composed of all states generated through every possible linear combination of the states in $\bar{x}_0$. The stability of this subspace is determined by the location of the eigenvalues of $\bar{\mathbf{A}}_0$.

The unobservable subspace of a plant is composed of all states generated through every possible linear combination of the states in $\bar{x}_{n0}$. The stability of this subspace is determined by the location of the eigenvalues of $\overline{\mathbf{A}}_{n0}$.

Detectability

A plant is said to be *detectable* if its unobservable subspace is stable.

We remarked earlier that noncontrollable (*indeed nonstabilizable*) models are frequently used in control-system design. This is not true for nondetectable models. Essentially all models used in the sequel can be taken to be detectable, without loss of generality.

Observer Canonical Form

There are also duals of the canonical forms given in Lemmas 17.2 and 17.3. For example the dual of Lemma 17.3 is

Lemma 17.5: Consider a completely observable SISO system given by

$$\delta x[k] = \mathbf{A}_\delta x[k] + \mathbf{B}_\delta u[k]$$
$$y[k] = \mathbf{C}_\delta x[k] + \mathbf{D}_\delta u[k]$$

Then there exists a similarity transformation that converts the model to the observer-canonical form

$$\delta x'[k] = \begin{bmatrix} -\alpha_{n-1} & 1 & & \\ \vdots & & \ddots & \\ \vdots & & & 1 \\ -\alpha_0 & 0 & & 0 \end{bmatrix} x'[k] + \begin{bmatrix} b_{n-1} \\ \vdots \\ \vdots \\ b_0 \end{bmatrix} u[k]$$

$$y[k] = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} x'[k] + \mathbf{D}u[k]$$

Canonical Decomposition

Further insight into the structure of linear dynamical systems is obtained by considering those systems that are only partially observable or controllable. These systems can be separated into completely observable and completely controllable systems.

The two results of Lemmas 17.1 and 17.4 can be combined as on the next slide.

Canonical Decomposition Theorem

Theorem 17.5: (*Canonical Decomposition Theorem*). Consider a system described in state space form. Then, there always exists a similarity transformation *T* such that the transformed model for $\bar{x} = \mathbf{T}^{-1} x$ takes the form

$$\overline{\mathbf{A}} = \begin{bmatrix} \overline{\mathbf{A}}_{co} & \mathbf{0} & \overline{\mathbf{A}}_{13} & \mathbf{0} \\ \overline{\mathbf{A}}_{21} & \overline{\mathbf{A}}_{22} & \overline{\mathbf{A}}_{23} & \overline{\mathbf{A}}_{24} \\ \mathbf{0} & \mathbf{0} & \overline{\mathbf{A}}_{33} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \overline{\mathbf{A}}_{34} & \overline{\mathbf{A}}_{44} \end{bmatrix}; \qquad \overline{\mathbf{B}} = \begin{bmatrix} \overline{\mathbf{B}}_1 \\ \overline{\mathbf{B}}_2 \\ 0 \\ 0 \end{bmatrix}; \qquad \overline{\mathbf{C}} = \begin{bmatrix} \overline{\mathbf{C}}_1 & \mathbf{0} & \overline{\mathbf{C}}_2 & \mathbf{0} \end{bmatrix}$$

Where

(i) The subsystem $[\bar{\mathbf{A}}_{c0}, \bar{\mathbf{B}}_1, \bar{\mathbf{C}}_1]$ is both completely controllable and completely observable and has the same transfer function as the original system.

(*ii*) The subsystem

$$\left[\begin{array}{cc} \overline{\mathbf{A}}_{co} & \mathbf{0} \\ \overline{\mathbf{A}}_{21} & \overline{\mathbf{A}}_{22} \end{array}\right], \left[\begin{array}{c} \overline{\mathbf{B}}_1 \\ \overline{\mathbf{B}}_2 \end{array}\right], \left[\begin{array}{cc} \overline{\mathbf{C}}_1 & \mathbf{0} \end{array}\right]$$

is completely controllable.

(*iii*) The subsystem

$$\left[\begin{matrix}\overline{\mathbf{A}}_{co} & \overline{\mathbf{A}}_{13} \\ \mathbf{0} & \overline{\mathbf{A}}_{33}\end{matrix}\right], \left[\begin{matrix}\overline{\mathbf{B}}_1 \\ 0\end{matrix}\right], \left[\begin{matrix}\overline{\mathbf{C}}_1 & \overline{\mathbf{C}}_2\end{matrix}\right]$$

is completely observable.

The canonical decomposition described above leads to

Lemma 17.6: Consider the transfer-function matrix **H**(**s**) satisfying

$$Y(s) = \mathbf{H}(s)U(s)$$

Then

$$\mathbf{H} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \overline{\mathbf{C}}_1(s\mathbf{I} - \overline{\mathbf{A}}_{co})^{-1}\overline{\mathbf{B}}_1 + \mathbf{D}$$

where $\overline{\mathbf{C}}_1$, $\overline{\mathbf{A}}_{c0}$, and $\overline{\mathbf{B}}_1$ correspond to the observable and controllable part of the model.

Lemma 17.6 shows that the uncontrollable and the unobservable parts of a linear system do not appear in the transfer function. Conversely, given a transfer function, it is possible to generate a state space description that is both completely controllable and observable. We then say that this state description is a minimal realization of the transfer function. As mentioned earlier, nonminimal models are frequently used in control-system design to include disturbances.

Controllability depends on the structure of the input ports: where, in the system, the manipulable inputs are applied. Thus the states of a given subsystem might be uncontrollable for one given input but completely controllable for another. This distinction is of fundamental importance in control-system design, because not all plant inputs can be manipulated (*consider*, *for example*, *disturbances*) to steer the plant to reach certain states.

Similarly, observability depends on which outputs are being considered. Certain states may be unobservable from a given output, but they may be completely observable from some other output. This also has a significant impact on output-feedback control systems, because some states might not appear in the plant output being measured and fed back. However, they could appear in crucial internal variables and thus be important to the control problem.

Pole-Zero Cancellation and System Properties

The system properties described above are also intimately related to issues of pole-zero cancellations. To facilitate the subsequent development, we introduce the following test, which is useful for studying issues of controllability and observability.

PBH Test

Lemma 17.7: (*PBH Test*). Consider a state space model (**A**, **B**, **C**).

(*i*) The system is not completely observable if and only if there exist a nonzero vector $x \in \mathbb{C}^n$ and a scalar $\lambda \in \mathbb{C}$ such that

$$\mathbf{A}x = \lambda x$$
$$\mathbf{C}x = 0$$

(*ii*) The system is not completely controllable if and only if there exist a nonzero vector $x \in \mathbb{C}^n$ and a scalar $\lambda \in \mathbb{C}$ such that

$$x^T\mathbf{A} = \lambda x^T$$
$$x^T\mathbf{B} = 0$$

Proof: See the book.

We will use the preceding result to study the system properties of cascaded systems.

Consider the cascaded system shown below.

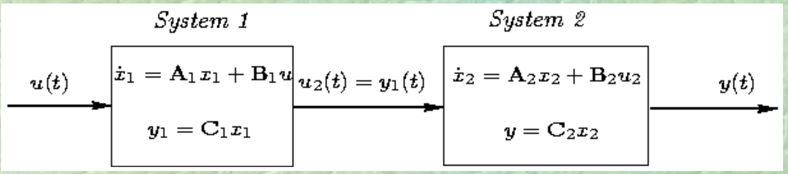


Figure 17.3: *Pole-zero cancellation*

We assume that $u(t)$, $u_2(t)$, $y_1(t)$, $y(t) \in \mathbb{R}$, that both subsystems are minimal, and that

System 1 has a zero at α and pole at β,

System 2 has a pole at α and zero at β.

Then the combined model has the property that

(a) The system pole at β is not observable from Y, and

(b) The system pole at α is not controllable from u.

The above results are readily established using the PBH test - see the book.

Summary

- State variables are system internal variables, upon which a full model for the system behavior can be built. The state variables can be ordered in a state vector.
- Given a linear system, the choice of state variables is not unique - however,
  - the minimal dimension of the state vector is a system invariant,
  - there exists a nonsingular matrix that defines a similarity transformation between any two state vectors, and
  - any designed system output can be expressed as a linear combination of the state variables and the inputs.

- For linear, time-invariant systems, the state space model is expressed in the following equations:

continuous-time systems

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

discrete-time systems, shift form

$$x[k+1] = \mathbf{A}_q x[k] + \mathbf{B}_q u[k]$$
$$y[k] = \mathbf{C}_q x[k] + \mathbf{D}_q u[k]$$

discrete-time systems, delta form

$$\delta x[k] = \mathbf{A}_\delta x[k] + \mathbf{B}_\delta u[k]$$
$$y[k] = \mathbf{C}_\delta x[k] + \mathbf{D}_\delta u[k]$$

- Stability and natural response characteristics of the system can be studied from the eigenvalues of the matrix $\mathbf{A}$ or ($\mathbf{A}_q$, $\mathbf{A}_\delta$).
- State space models faciclitate the study of certain system properties that are paramount in the solution to the control-design problem. These properties relate to the following questions:
  - By proper choice of the input u, can we steer the system state to a desired state (*point value*)? (*controllability*)
  - If some states are uncontrollable, will these states generate a time-decaying component? (*stabilizability*)
  - If one knows the input, $u(t)$, for $t \geq t_0$, can we infer the state at time $t = t_0$ by measuring the system output, $y(t)$, for $t \geq t_0$? (*observability*)
  - If some of the states are unobservable, do these states generate a time-decaying signal? (*detectability*)

- Controllability tells us about the feasibility of attempting to control a plant.
- Observability tells us about whether it is possible to know what is happening inside a given system by observing its outputs.
- The above system properties are system invariants. However, changes in the number of inputs, in their injection points, in the number of measurements, and in the choice of variables to be measured can yield different properties.

- A transfer function can always be derived from a state space model.
- A state space model can be built from a transfer-function model. However, only the completely controllable and observable part of the system is described in that state space model. Thus *the transfer-function model might be only a partial description of the system.*

- The properties of individual systems do not necessarily translate unmodified to composed systems. In particular, given two systems completely observable and controllable, their cascaded connection
  - is not completely observable if a pole of the first system coincides with a zero of the second system (pole-zero cancellation),
  - is not detectable if the pole-zero cancellation affects an unstable pole,
  - is not completely controllable if a zero of the first system coincides with a pole of the second system (zero-pole cancellation), and
  - is not stabilizable if the zero-pole cancellation affects a NMP zero.

- This chapter provides a foundation for the design requirement that one should never attempt to cancel unstable poles and zeros.