Test for Controllability

Theorem 17.2: Consider the state space model

$$\delta x[k] = \mathbf{A}_\delta x[k] + \mathbf{B}_\delta u[k]$$
$$y[k] = \mathbf{C}_\delta x[k] + \mathbf{D}_\delta u[k]$$

(*i*) The set of all controllable states is the range space of the controllability matrix $\Gamma_c[\mathbf{A}, \mathbf{B}]$, where

$$\mathbf{\Gamma}_c[\mathbf{A}, \mathbf{B}] \stackrel{\triangle}{=} \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A^2B} & \dots & \mathbf{A^{n-1}B} \end{bmatrix}$$

(*ii*) The model is completely controllable if and only if where $\Gamma_c[\mathbf{A}, \mathbf{B}]$ has full row rank.

Proof: Uses Cayley-Hamilton Theorem - see book.

Example 17.5

Consider the state space model

$$\mathbf{A} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The controllability matrix is given by

$$\mathbf{\Gamma}_c[\mathbf{A}, \mathbf{B}] = [\mathbf{B}, \mathbf{AB}] = \begin{bmatrix} 1 & -4 \\ -1 & -2 \end{bmatrix}$$

Clearly, rank $\Gamma_c[\mathbf{A}, \mathbf{B}] = 2$; thus, the system is completely controllable.

Example 17.6

For

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The controllability matrix is given by:

$$\mathbf{\Gamma}_c[\mathbf{A}, \mathbf{B}] = [\mathbf{B}, \mathbf{AB}] = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

Rank $\Gamma_c[\mathbf{A}, \mathbf{B}] = 1 < 2$; thus, the system is not completely controllable.

Although we have derived the above result by using the delta model, it holds equally for shift and/or continuous-time models.

We see that controllability is a black and white issue: a model either is completely controllable or it is not. Clearly, to know that something is uncontrollable is a valuable piece of information. However, to know that something is controllable really tells us nothing about the *degree* of controllability, i.e., about the difficulty that might be involved in achieving a certain objective. The latter issue lies at the heart of the fundamental design trade-offs in control that were the subject of Chapters 8 and 9.

If a system is not completely controllable, it can be decomposed into a controllable and a completely uncontrollable subsystem, as explained below.

Controllable Decompositon

Lemma 17.1: Consider a system having rank$\{\Gamma_c[\mathbf{A}, \mathbf{B}]\}$ = k < n; then there exists a similarity transformation T such that $\bar{x} = \mathbf{T}^{-1}x$,

$$\overline{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}; \qquad \overline{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B}$$

and $\overline{\mathbf{A}}, \overline{\mathbf{B}}$ have the form

$$\overline{\mathbf{A}} = \begin{bmatrix} \overline{\mathbf{A}}_c & \overline{\mathbf{A}}_{12} \\ 0 & \overline{\mathbf{A}}_{nc} \end{bmatrix}; \qquad \overline{\mathbf{B}} = \begin{bmatrix} \overline{\mathbf{B}}_c \\ 0 \end{bmatrix}$$

where $\overline{\mathbf{A}}_c$ has dimension k and $(\overline{\mathbf{A}}_c, \overline{\mathbf{B}}_c)$ is completely controllable.

Proof: See the book.

The above result has important consequences regarding control. To appreciate this, express the (*transformed*) state and output equations in partitioned form as

$$\delta \begin{bmatrix} \overline{x}_c[k] \\ \overline{x}_{nc}[k] \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{A}}_c & \overline{\mathbf{A}}_{12} \\ \mathbf{0} & \overline{\mathbf{A}}_{nc} \end{bmatrix} \begin{bmatrix} \overline{x}_c[k] \\ \overline{x}_{nc}[k] \end{bmatrix} + \begin{bmatrix} \overline{\mathbf{B}}_c \\ \mathbf{0} \end{bmatrix} u[k]$$

$$y[k] = \begin{bmatrix} \overline{\mathbf{C}}_c & \overline{\mathbf{C}}_{nc} \end{bmatrix} \begin{bmatrix} \overline{x}_c[k] \\ \overline{x}_{nc}[k] \end{bmatrix} + \mathbf{D}u[k]$$

A pictorial representation of these equations is shown in Figure 17.1.

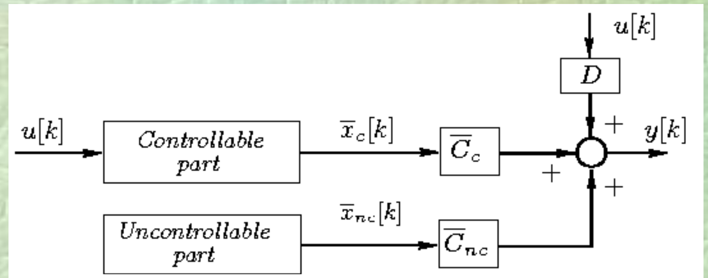

Figure 17.1: *Controllable-uncontrollable decomposition*

We see that caution must be exercised when controlling a system (*or designing a controller with a model that is not completely controllable*), because the output has a component that does not depend on the manipulated input $u[k]$.

$$\mathbf{C}_{nc}\,\bar{x}_{nc}[k]$$

The controllable subspace of a state space model is composed of all states generated through every possible linear combination of the states in

The stability of this subspace is determined by the location of the eigenvalues of $\mathbf{A}_{nc}$.

$\bar{x}_c$.

The uncontrollable subspace of a state space model is composed of all states generated through every possible linear combination of the states in The stability of this subspace is determined by the location of the eigenvalues of $\mathbf{A}_{nc}$.

$\bar{x}_{nc}$.

Stabilizability

A state space model is said to be *stabilizable* if its uncontrollable subspace is stable.

A fact that we will find useful in what follows is that, if the system is completely controllable, there exist similarity transformations that convert it into special forms, known as *canonical forms*. This is established in the following two lemmas.

Controllability Canonical Form

Lemma 17.2: Consider a completely controllable state space model for a SISO system. Then there exists a similarity transformation that converts the state space model into the following controllability-canonical form:

$$\mathbf{A}' = \begin{bmatrix} 0 & 0 & \dots & 0 & -\alpha_0 \\ 1 & 0 & \dots & 0 & -\alpha_1 \\ 0 & 1 & \dots & 0 & -\alpha_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -\alpha_{n-1} \end{bmatrix} \qquad \mathbf{B}' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $\lambda^n + \alpha_{n-1}\lambda_{n-1} + \ldots + \alpha_1\lambda + \alpha_0 = \det(\lambda\mathbf{I} - \mathbf{A})$ is the characteristic polynomial of **A**.

Proof: See the book.

Controller - Canonical Form

Lemma 17.3: Consider a completely controllable state space model for a SISO system. Then there exists a similarity transformation that converts the state space model into the following controller-canonical form:

$$\mathbf{A}'' = \begin{bmatrix} -\alpha_{n-1} & -\alpha_{n-2} & \dots & -\alpha_1 & -\alpha_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \qquad \mathbf{B}'' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $\lambda^n + \alpha_{n-1}\lambda_{n-1} + \dots + \alpha_1\lambda + \alpha_0 = \det(\lambda \mathbf{I} - \mathbf{A})$ is the characteristic polynomial of **A**.

Proof: See the book.

Finally, we remark that, as we have seen in Chapter 10, it is very common indeed to employ uncontrollable models in control-system design. This is because they are a convenient way of describing various commonly occurring disturbances. For example, a constant disturbance can be modeled by the following state space model:

$$\dot{x}_d = 0$$

which is readily seen to be uncontrollable and, indeed, nonstabilizable.

Observability and Detectability

Consider again the state space model

$$\delta x[k] = \mathbf{A}_\delta x[k] + \mathbf{B}_\delta u[k]$$
$$y[k] = \mathbf{C}_\delta x[k] + \mathbf{D}_\delta u[k]$$

In general, the dimension of the observed output, *y*, can be less than the dimension of the state, *x*. However, one might conjecture that, if one observed the output over some nonvanishing time interval, then this might tell us something about the state. The associated properties are called observability (*or reconstructability*). A related issue is that of detectability. We begin with observability.

Observability

Observability is concerned with the issue of what can be said about the state when one is given measurements of the plant output.

A formal definition is as follows:

Definition 17.6: The state $x_0 \neq 0$ is said to be unobservable if, given $x(0) = x_0$, and $u[k] = 0$ for $k \geq 0$, then $y[k] = 0$ for $k \geq 0$. The system is said to be completely observable if there exists no nonzero initial state that it is unobservable.

Reconstructability

A concept related to observability is that of reconstructability. This concept is sometimes used in discrete-time systems. Reconstructability is concerned with what can be said about $x(T)$, on the basis of the past values of the output, i.e., $y[k]$ for $0 \leq k \leq T$. For linear time-invariant continuous-time systems, the distinction between observability and reconstructability is unnecessary. However, the following example illustrates that, in discrete time, the two concepts are different.

Consider

$$x[k+1] = 0 \qquad x[0] = x_o$$
$$y[k] = 0$$

this system i… for all $T \geq 1$, because we … $= 0$ for $T \geq 1$. However, it is completely unobservable, because $y[k] = 0 \ \forall k$, irrespective of the value of x_0.

In view of the subtle difference between observability and reconstructability, we will use the term observability in the sequel to cover the stronger of the two concepts.

Test for Observability

A test for observability of a system is established in the following theorem.

Theorem 17.3: Consider the state model

$$\delta x[k] = \mathbf{A}_\delta x[k] + \mathbf{B}_\delta u[k]$$
$$y[k] = \mathbf{C}_\delta x[k] + \mathbf{D}_\delta u[k]$$

(*i*) The set of all unobservable states is equal to the null space of the observability matrix $\Gamma_0[\mathbf{A}, \mathbf{C}]$, where

$$\mathbf{\Gamma}_o[\mathbf{A}, \mathbf{C}] \triangleq \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

(*ii*) The system is completely observable if and only if $\Gamma_0[\mathbf{A}, \mathbf{C}]$, has full column rank n.

Proof: See the book.

As for controllability, the above result also applies to continuous-time and discrete (*shift*) operator models.

Example 17.1

Consider the following state space model:

$$\mathbf{A} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \qquad \mathbf{C} = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Then

$$\mathbf{\Gamma}_o[\mathbf{A}, \mathbf{C}] = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -4 & -2 \end{bmatrix}$$

Hence, rank $\Gamma_0[\mathbf{A}, \mathbf{C}] = 2$, and the system is completely observable.