Linear State Space Models

There are many alternative model formats that can be used for linear dynamic systems. In simple SISO problems, any representation is probably as good as any other. However, as we move to more complex problems (*especially multivariable problems*), it is desirable to use special model formats. One of the most flexible and useful structures is the state space model.

We will examine linear state space models in a little more depth for the SISO case. Many of the ideas will carry over to the MIMO case which we will study later. In particular we will study

- similarity transformations and equivalent state representations,
- state space model properties:
  - controllability, reachability, and stabilizability,
  - observability, reconstructability, and detectability,
- special (*canonical*) model formats.

Linear Continuous-Time State Space Models

A continuous-time linear time-invariant state space model takes the form

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t) \qquad x(t_o) = x_o$$
$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control signal, $y \in \mathbb{R}^p$ is the output, $x_0 \in \mathbb{R}^n$ is the state vector at time $t = t_0$ and **A**, **B**, **C**, and **D** are matrices of appropriate dimensions.

Similarity Transformations

It is readily seen that the definition of the state of a system is nonunique. Consider, for example, a linear transformation of $x(t)$ to $\overline{x}(t)$ defined as

$$\overline{x}(t) = \mathbf{T}^{-1} x(t) \qquad\qquad x(t) = \mathbf{T}\,\overline{x}(t)$$

where $\mathbf{T}$ is any nonsingular matrix, called a similarity transformation.

The following alternative state description is obtained

where

$$\dot{\overline{x}}(t) = \overline{\mathbf{A}}\,\overline{x}(t) + \overline{\mathbf{B}}\,u(t) \qquad \overline{x}(t_o) = \mathbf{T}^{-1}x_o$$

$$y(t) = \overline{\mathbf{C}}\,\overline{x}(t) + \overline{\mathbf{D}}\,u(t)$$

The above model is an equally valid description of the system

$$\overline{\mathbf{A}} \triangleq \mathbf{T}^{-1}\mathbf{A}\mathbf{T} \qquad \overline{\mathbf{B}} \triangleq \mathbf{T}^{-1}\mathbf{B} \qquad \overline{\mathbf{C}} \triangleq \mathbf{C}\mathbf{T} \qquad \overline{\mathbf{D}} \triangleq \mathbf{D}$$

An illustration, say that the matrix **A** can be diagonalized by a similarity transformation **T**; then

where if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of **A**, then

$$\overline{\mathbf{A}} = \mathbf{\Lambda} \triangleq \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$$

$$\mathbf{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \ldots \lambda_n)$$

Because Λ is diagonal, we have

$$\overline{x}_i(t) = e^{\lambda_i(t-t_o)}\overline{x}_o + \int_{t_o}^{t} e^{\lambda_i(t-\tau)}\overline{b}_i u(\tau)d\tau$$

where the onent of the state vector.

Example

$$\mathbf{A} = \begin{bmatrix} -4 & -1 & 1 \\ 0 & -3 & 1 \\ 1 & 1 & -3 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} -1 & -1 & 0 \end{bmatrix} \quad \mathbf{D} = 0$$

The matrix **T** can also be obtained by using the MATLAB command **eig**, which yields

$$\mathbf{T} = \begin{bmatrix} 0.8018 & 0.7071 & 0.0000 \\ 0.2673 & -0.7071 & 0.7071 \\ -0.5345 & -0.0000 & 0.7071 \end{bmatrix}$$

We obtain the similar state space description given by

$$\overline{\mathbf{A}} = \mathbf{\Lambda} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}; \qquad \overline{\mathbf{B}} = \begin{bmatrix} 0.0 \\ -1.414 \\ 0.0 \end{bmatrix};$$

$$\overline{\mathbf{C}} = \begin{bmatrix} -0.5345 & -1.4142 & 0.7071 \end{bmatrix} \qquad \overline{\mathbf{D}} = 0$$

Transfer Functions Revisited

The solution to the state equation model can be obtained via

$$
\begin{aligned}
Y(s) &= [\overline{\mathbf{C}}(s\mathbf{I}-\overline{\mathbf{A}})^{-1}\overline{\mathbf{B}}+\overline{\mathbf{D}}]U(s)+\overline{\mathbf{C}}(s\mathbf{I}-\overline{\mathbf{A}})^{-1}\overline{x}(0) \\
&= [\mathbf{C}\mathbf{T}(s\mathbf{I}-\mathbf{T}^{-1}\mathbf{A}\mathbf{T})^{-1}\mathbf{T}^{-1}\mathbf{B}+\mathbf{D}]U(s)+\mathbf{C}\mathbf{T}(s\mathbf{I}-\mathbf{T}^{-1}\mathbf{A}\mathbf{T})^{-1}\mathbf{T}^{-1}x(0) \\
&= [\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}]U(s)+\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}x(0)
\end{aligned}
$$

We thus see that different choices of state variables lead to different internal descriptions of the model, but to the same input-output model, because the system transfer function can be expressed in either of the two equivalent fashions.

for any nonsingular **T**.

$$\overline{\mathbf{C}}(s\mathbf{I} - \overline{\mathbf{A}})^{-1}\overline{\mathbf{B}} + \overline{\mathbf{D}} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

From Transfer Function to State Space Representation

We have seen above how to go from a state space description to the corresponding transfer function. The converse operation leads to the following question:

> Given a transfer function $G(s)$, how can a state representation for this system be obtained?

Development

Consider a transfer function $G(s) = {}^{B(s)}/_{A(s)}$. We can then write

$$Y(s) = \sum_{i=1}^{n} b_{i-1} V_i(s) \qquad \text{where} \qquad V_i(s) = \frac{s^{i-1}}{A(s)} U(s)$$

We note from the above definitions that

$$v_i(t) = \mathcal{L}^{-1}\left[V(s)\right] = \frac{dv_{i-1}(t)}{dt} \qquad \text{for} \qquad i = 1, 2, \ldots, n$$

We can then choose, as state variables, $x_i(t) = v_i(t)$, which lead to the following state space model for the system.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \end{bmatrix} \qquad \mathbf{D} = 0$$

The above model has a special form. We will see later that any *completely controllable* system can be expressed in this way. Before we do this, we need to introduce the idea of controllability.

Controllability and Stabilizability

An important question that lies at the heart of control using state space models is whether we can steer the state via the control input to certain locations in the state space. Technically, this property is called controllability or reachability. A closely related issue is that of stabilizability. We will begin with controllability.

Controllability

The issue of controllability concerns whether a given initial state x_0 can be steered to the origin in finite time using the input $u(t)$.

Formally, we have the following:

Definition 17.1: A state x_0 is said to be controllable if there exists a finite interval $[0, T]$ and an input $\{u(t), t \in [0, T]\}$ such that $x(T) = 0$. If all states are controllable, then the system is said to be completely controllable.

Reachability

A related concept is that of reachability. This concept is sometimes used in discrete-time systems. It is formally defined as follows:

Definition 17.2: A state $\bar{x} \neq 0$ is said to be reachable (*from the origin*) if, given $x(0) = 0$, there exist a finite time interval $[0, T]$ and an input $\{u(t), t \in [0, T]\}$ such that $x(T) = \bar{x}$. If all states are reachable, the system is said to be completely reachable.

For continuous, time-invariant, linear systems, there is no distinction between complete controllability and complete reachability. However, the following example illustrates that there is a subtle difference in discrete time.

Consider the following shift-operator state space model:

This system is obviously completely controllable: the state immediately goes to the origin. However, no nonzero state is reachable.

$$x[k+1] = 0$$

In view of the subtle distinction between controllability and reachability in discrete time, we will use the term *controllability* in the sequel to cover the stronger of the two concepts. The discrete-time proofs for the results presented below are a little easier. We will thus prove the results on the following discrete-time (*delta-domain*) model:

$$\delta x[k] = \mathbf{A}_\delta x[k] + \mathbf{B}_\delta u[k]$$
$$y[k] = \mathbf{C}_\delta x[k] + \mathbf{D}_\delta u[k]$$

Our next step will be to derive a simple algebraic test for controllability that can easily be applied to a given state space model. In deriving this result, we will use a result from linear algebra known as the Cayley-Hamilton Theorem.

Theorem 17.1: (*Cayley-Hamilton theorem*). Every matrix satisfies its own characteristic equation - i.e., if

then

$$\det(s\mathbf{I} - \mathbf{A}) = s^n + a_{n-1}s^{n-1} + \ldots + a_0$$

Proof: See the book.

$$\mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + \ldots + a_0\mathbf{I} = 0$$

Test for Controllability

Theorem 17.2: Consider the state space model

$$\delta x[k] = \mathbf{A}_\delta x[k] + \mathbf{B}_\delta u[k]$$
$$y[k] = \mathbf{C}_\delta x[k] + \mathbf{D}_\delta u[k]$$

(i) The set of all controllable states is the range space of the controllability matrix $\Gamma_c[\mathbf{A}, \mathbf{B}]$, where

$$\mathbf{\Gamma}_c[\mathbf{A}, \mathbf{B}] \triangleq \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A^2B} & \dots & \mathbf{A^{n-1}B} \end{bmatrix}$$

(ii) The model is completely controllable if and only if where $\Gamma_c[\mathbf{A}, \mathbf{B}]$ has full row rank.

Proof: Uses Cayley-Hamilton Theorem - see book.