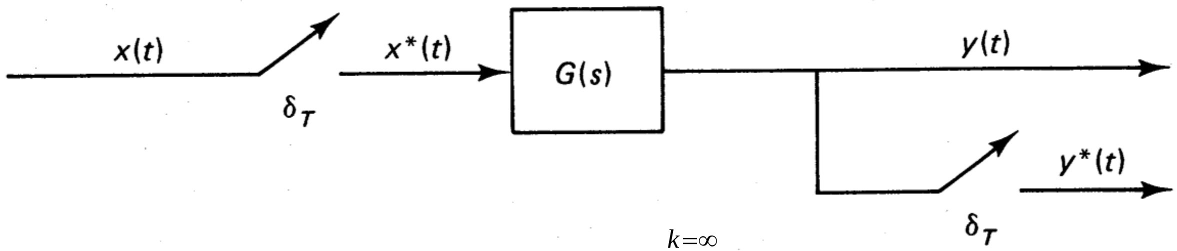
The Pulse Transfer Function

- Convolution Summation



$$Z[y(t)] = Y(z) = \sum_{k=0}^{k=\infty} y(kT) z^{-k}$$

  – For the continuous time-system

$$y(t) = \int_0^t g(t-\tau) x(\tau) d\tau = \int_0^t x(t-\tau) g(\tau) d\tau$$

  – For the discrete-time system

$$x^*(t) = \sum_{k=0}^{\infty} x(t)\delta(t-kT) = \sum_{k=0}^{\infty} x(kT)\delta(t-kT)$$

    - For a physical system a response cannot precede the input

$$y(t) = \sum_{h=0}^{\infty} g(t-hT) x(ht) \qquad 0 \le t \le kT$$

The Pulse Transfer Function

- Convolution Summation (cont.)
  - The value of the output $y(t)$ at the sampling instants $t=kT$ are given by

$$y(kT)=\sum_{h=0}^{k} g(kT-hT)x(hT)=\sum_{h=0}^{k} x(kT-hT)g(hT)$$ Convolution summation

$$\boxed{y(kT)=x(kT)*g(kT)}$$

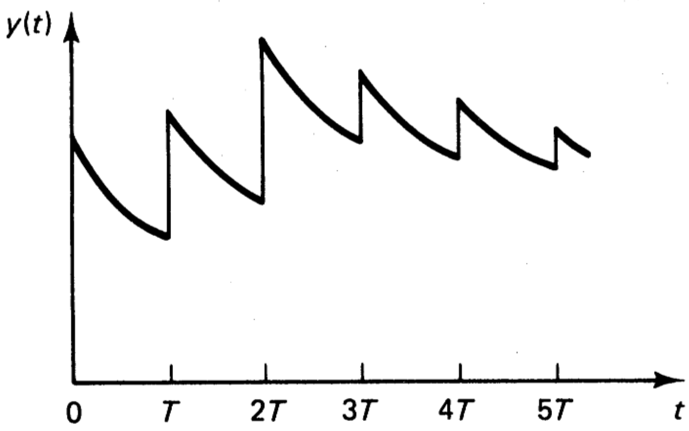
  - Since we assume that $x(t)=0$ for $t<0$

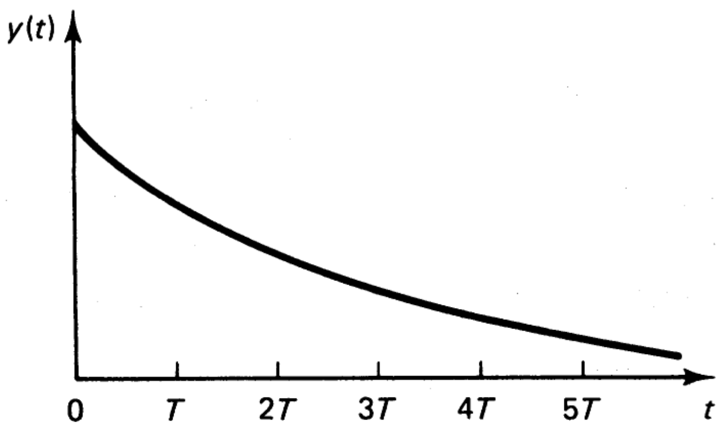
$$y(kT)=\sum_{h=0}^{\infty} g(kT-hT)x(hT)=\sum_{h=0}^{\infty} x(kT-hT)g(hT)$$

  - It is noted that if $G(s)$ is a ratio of polynimials in s and if the degree of the denominator polynomial exceeds that of the numerator polynomial only by 1 the output $y(t)$ is discontinuous.

The Pulse Transfer Function

- Convolution Summation (cont.)
  - In analyzing discrete-time control systems it is important to remember that the system response to the impulse-sampled signal may not portray the correct time-response behavior of the actual system unless the transfer function $G(s)$ of the continuous-time part of the system has at least two more poles than zeros, so that $\lim_{s \to \infty} sG(s) = 0$





The Pulse Transfer Function

- Pulse Transfer Function

$$y(kT) = \sum_{h=0}^{\infty} g(kT - hT)x(hT) \qquad k = 0,1,2,\cdots$$

  – The z transform of $y(kT)$

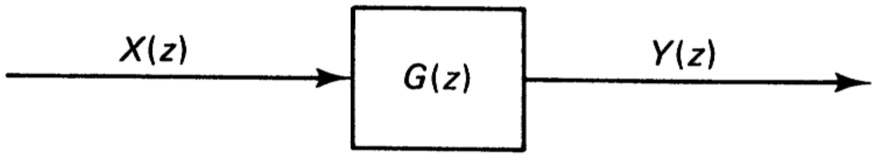
$$Y(z) = \sum_{k=0}^{\infty} y(kT)z^{-k} = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} g(kT - hT)x(hT)z^{-k}$$

$$= \sum_{m=0}^{\infty}\sum_{h=0}^{\infty} g(mT)x(hT)z^{-(m+h)} = \sum_{m=0}^{\infty} g(mT)z^{-m}\sum_{h=0}^{\infty} x(hT)z^{-h}$$

$$= G(z)X(z)$$

$$G(z) = \frac{Y(z)}{X(z)}$$

Pulse transfer function

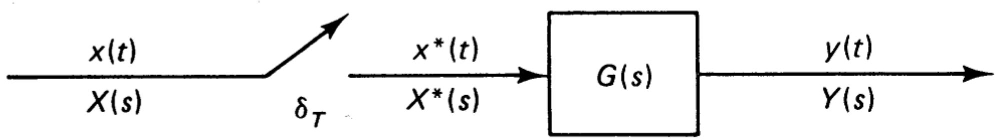


to the Kronecker delta input

$Y(z) = G(z)$

The Pulse Transfer Function

- Starred Laplace Transform of the Signal involving both Ordinary and Starred Laplace Transform



$$Y(s)=G(s)X^*(s) \qquad X^*(s)=X^*(s\pm j\omega_s k), \qquad k=0,1,2\cdots$$

$$Y^*(s)=\left[G(s)X^*(s)\right]^*=\left[G(s)\right]^*X^*(s)=G^*(s)X^*(s)$$

$$y(t)=\mathcal{L}^{-1}\left[G(s)X^*(s)\right]=\int_0^t g(t-\tau)x^*(\tau)d\tau=\int_0^t g(t-\tau)\sum_{k=0}^{\infty}x(\tau)\delta(\tau-kT)d\tau$$

$$=\sum_{k=0}^{\infty}\int_0^t g(t-\tau)x(\tau)\delta(\tau-kT)d\tau=\sum_{k=0}^{\infty}g(t-kT)x(kT)$$

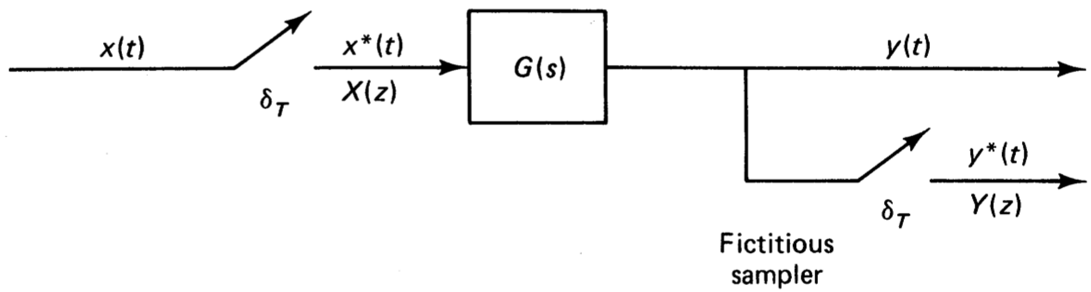
$$Y(z)=\mathcal{Z}\left[y(t)\right]=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty}g(nT-kT)x(kT)\right]z^{-n}=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}g(mT)x(kT)z^{-(k+m)}$$

$$=G(z)X(z)$$

⇒ $$Y^*(s)=G^*(s)X^*(s)$$

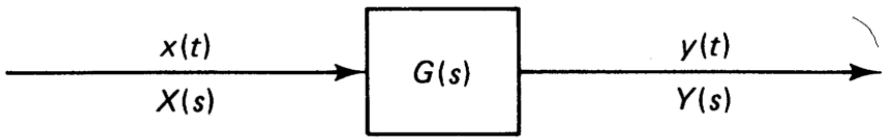
The Pulse Transfer Function

- General Procedures for Obtaining Pulse Transfer Functions



$$\frac{Y(z)}{X(z)} = G(z) = \mathcal{Z}[G(s)]$$

$$Y(s) = G(s)X^*(s) \qquad Y^*(s) = G^*(s)X^*(s)$$



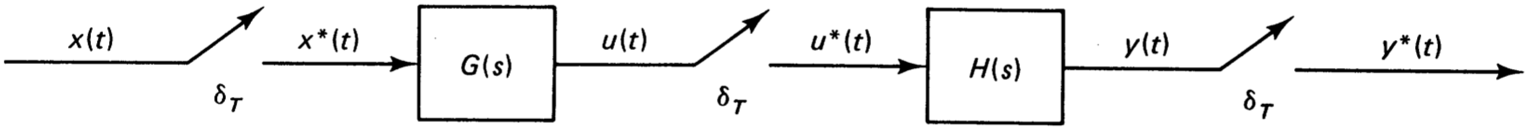
$$\frac{Y(s)}{X(s)} = G(s)$$

$$Y^*(s) = [G(s)X(s)]^* = [GX(s)]^*$$

$$Y(z) = \mathcal{Z}[Y(s)] = \mathcal{Z}[G(s)X(s)] = \mathcal{Z}[GX(s)] = GZ(z) \neq G(z)X(z)$$

The Pulse Transfer Function

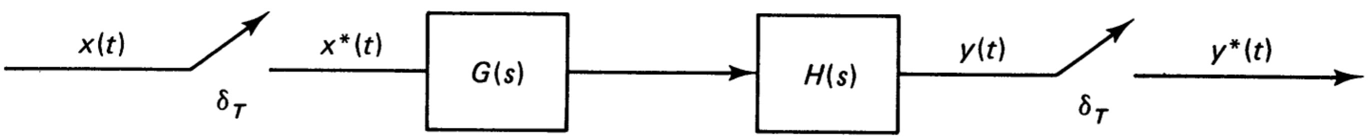
- Pulse Transfer Function of Cascaded Elements



$$U(s) = G(s)X^*(s), \qquad Y(s) = H(s)U^*(s)$$
$$U^*(s) = G^*(s)X^*(s), \qquad Y^*(s) = H^*(s)U^*(s)$$

$\Rightarrow$ $$Y^*(s) = H^*(s)U^*(s) = H^*(s)G^*(s)X^*(s)$$

$\Rightarrow$ $$Y(z) = G(z)H(z)X(z) \Rightarrow \frac{Y(z)}{X(z)} = G(z)H(z)$$

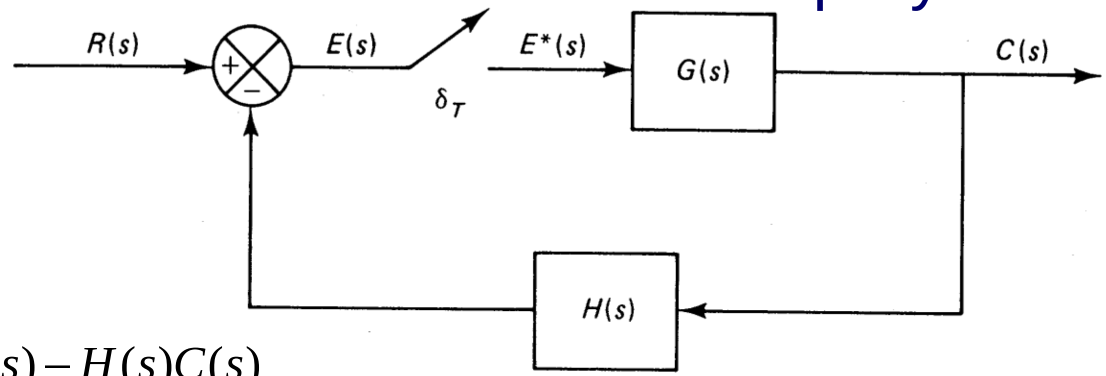


$$Y(s) = G(s)H(s)X^*(s) = GH(s)X^*(s)$$
$$Y^*(s) = [GH(s)]^* X^*(s)$$
$$Y(z) = GH(z)X(z)$$
$$\frac{Y(z)}{X(z)} = GH(z) = Z[GH(s)]$$

Note that $G(z)H(z) \neq GH(z) = Z[GH(s)]$

The Pulse Transfer Function

Pulse Transfer Function of Closed-Loop Systems



$$E(s) = R(s) - H(s)C(s)$$

$$C(s) = G(s)E^*(s)$$

$$E(s) = R(s) - H(s)G(s)E^*(s)$$

$$E^*(s) = R^*(s) - GH^*(s)E^*(s) \qquad E^*(s) = \frac{R^*(s)}{1 + GH^*(s)}$$

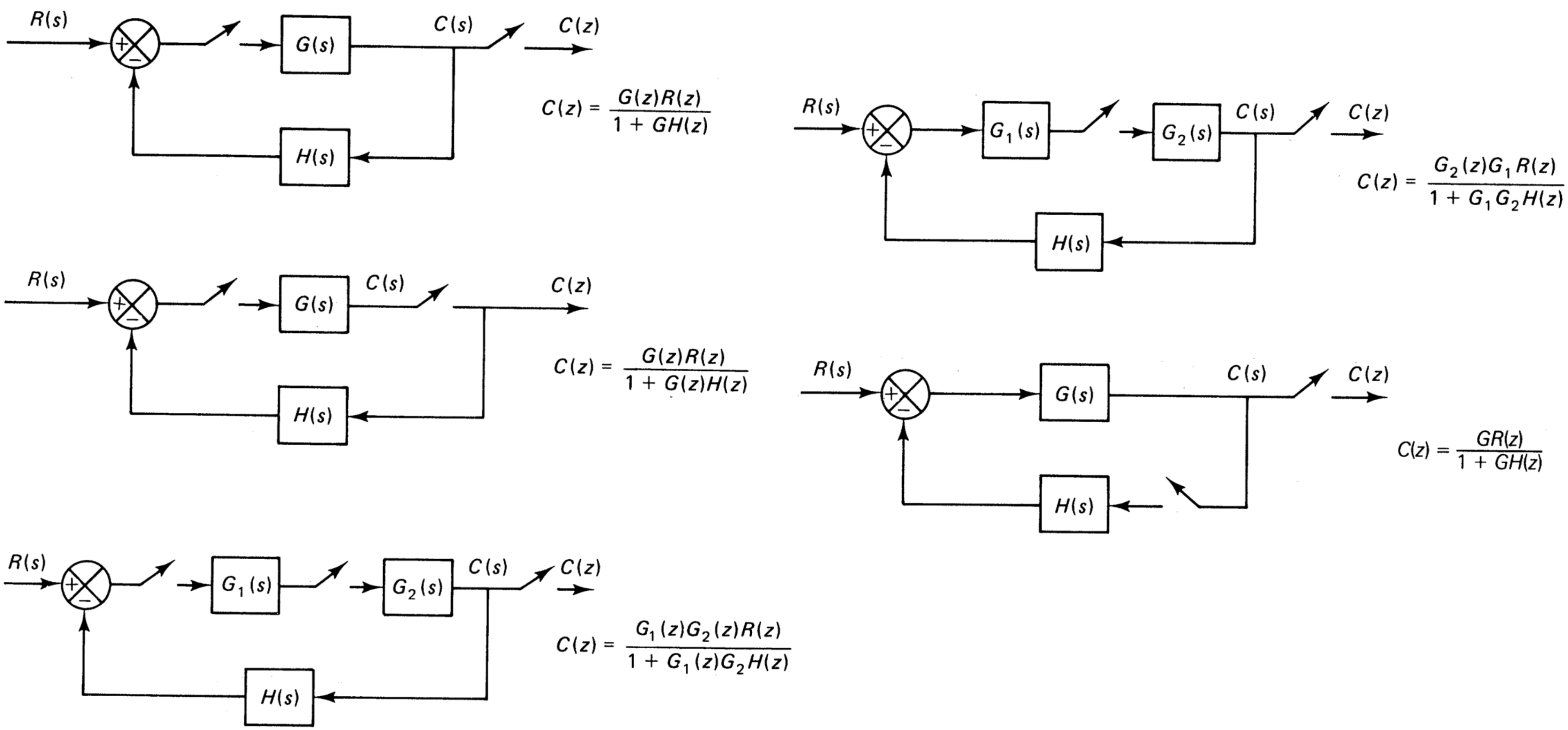
$$C^*(s) = G^*(s)E^*(s)$$

$$C^*(s) = \frac{G^*(s)R^*(s)}{1 + GH^*(s)} \Rightarrow C(z) = \frac{G(z)R(z)}{1 + GH(z)} \Rightarrow \frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}$$

Refer to Table 3-1

The Pulse Transfer Function

- Table 3-1: Five typical configurations for closed-loop discrete-time control systems



The Pulse Transfer Function

- Pulse Transfer Function of a Digital Controller
  - The input to the digital controller is $e(k)$ and the output is $m(k)$

$$m(k)+a_1m(k-1)+a_2m(k-2)+\cdots+a_nm(k-m)$$
$$=b_0e(k)+b_1e(k-1)+\cdots+b_ne(k-n)$$

  - The z transform of the equation

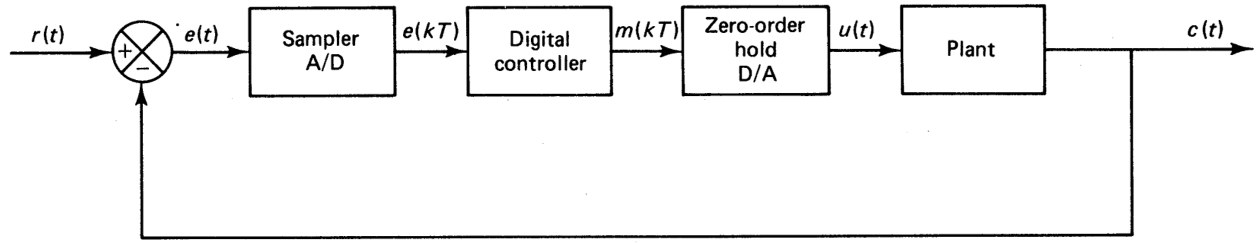
$$M(z)+a_1z^{-1}M(z)+a_2z^{-2}M(z)+\cdots+a_nz^{-n}M(z)$$
$$=b_0E(z)+b_1z^{-1}E(z)+\cdots+b_nz^{-n}E(z)$$

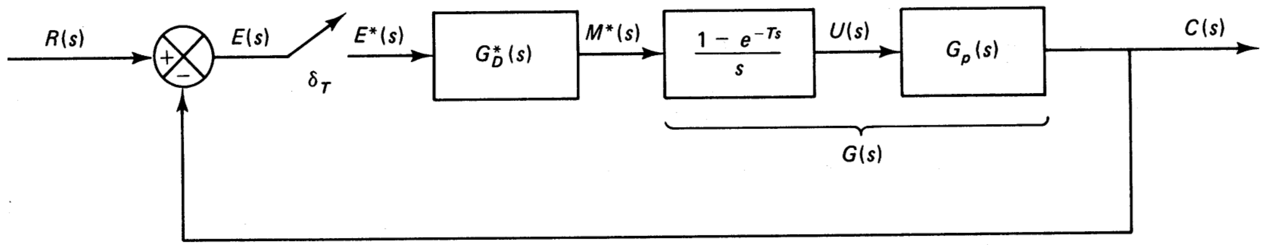
$$\Rightarrow (1+a_1z^{-1}+a_2z^{-2}+\cdots+a_nz^{-n})M(z)=(b_0+b_1z^{-1}+\cdots+b_nz^{-n})E(z)$$

$$\Rightarrow G_D(z)=\frac{M(z)}{E(z)}=\frac{b_0+b_1z^{-1}+\cdots+b_nz^{-n}}{1+a_1z^{-1}+a_2z^{-2}+\cdots+a_nz^{-n}}$$

The Pulse Transfer Function

- Closed-loop Pulse Transfer Function of a Digital Control System





$$\frac{1-e^{-Ts}}{s}G_p(s)=G(s) \qquad C(s)=G(s)G_D^*E^*(s) \qquad C^*(s)=G^*(s)G_D^*(s)E^*(s)$$

$$C(z)=G(z)G_D(z)E(z)$$

$$E(z)=R(z)-C(z)$$

$$C(z)=G_D(z)G(z)[R(z)-C(z)] \quad \Rightarrow \quad \frac{C(z)}{R(z)}=\frac{G_D(z)G(z)}{1+G_D(z)G(z)}$$

The Pulse Transfer Function

- Pulse Transfer Function of a Digital PID Controller
  - The PID control action in analog controllers

$$m(t) = K\left[e(t) + \frac{1}{T_i}\int_0^t e(t) + T_d \frac{de(t)}{dt} \right]$$

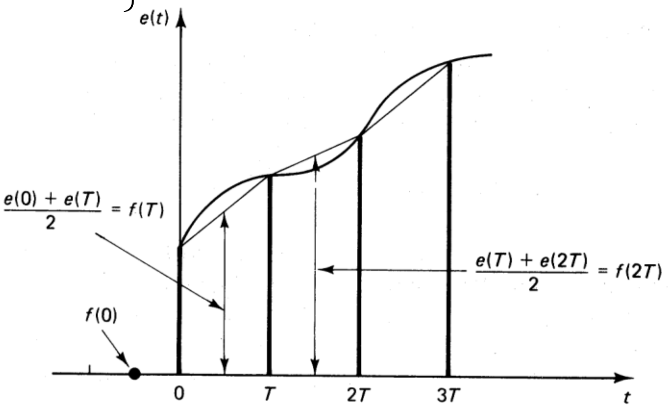
  - Discretization of the equation to obtain the pulse transfer function

$$m(kT) = K\left\{ e(kT) + \frac{T}{T_i}\left[\frac{e(0)+e(T)}{2} + \frac{e(T)+e(2T)}{2} + \cdots + \frac{e((k-1)T)+e(kT)}{2} \right] + T_d \frac{e(kT)-e((k-1)T)}{T} \right\}$$

$$m(kT) = K\left\{ e(kT) + \frac{T}{T_i}\sum_{h=1}^{k} \frac{e((h-1)T)+e(hT)}{2} + \frac{T_d}{T}\left[e(kT)-e((k-1)T) \right] \right\}$$

Define $\frac{e((h-1)T)+e(hT)}{2} = f(hT), \qquad f(0) = 0$

$$\sum_{h=1}^{k} \frac{e((h-1)T)+e(hT)}{2} = \sum_{h=1}^{k} f(hT)$$



The Pulse Transfer Function

- Pulse Transfer Function of a Digital PID Controller(cont.)

$$\mathcal{Z}\left[\sum_{h=1}^{k}\frac{e((h-1)T)+e(hT)}{2}\right]=\mathcal{Z}\left[\sum_{h=1}^{k}f(hT)\right]=\frac{1}{1-z^{-1}}\left[F(z)-f(0)\right]=\frac{1}{1-z^{-1}}F(z)$$

$$F(z)=\mathcal{Z}\left[f(hT)\right]=\frac{1+z^{-1}}{2}E(z)$$

$$\mathcal{Z}\left[\sum_{h=1}^{k}\frac{e((h-1)T)+e(hT)}{2}\right]=\frac{1+z^{-1}}{2(1-z^{-1})}E(z)$$

$$m(kT)=K\left\{e(kT)+\frac{T}{T_i}\sum_{h=1}^{k}\frac{e((h-1)T)+e(hT)}{2}+\frac{T_d}{T}\left[e(kT)-e((k-1)T)\right]\right\}$$

$$\Rightarrow\quad M(z)=K\left[1+\frac{T}{2T_i}\frac{1+z^{-1}}{1-z^{-1}}+\frac{T_d}{T}(1-z^{-1})\right]E(z)$$

$$=K\left[1-\frac{T}{2T_i}+\frac{T}{T_i}\frac{1}{1-z^{-1}}+\frac{T_d}{T}(1-z^{-1})\right]E(z)$$

$$=\underbrace{\left[K_P+\frac{K_I}{1-z^{-1}}+K_D(1-z^{-1})\right]}_{G_D(z)}E(z)$$

The Pulse Transfer Function

- Obtaining response between consecutive sampling instants
  - Laplace transform method
  - Modified z transform method
  - State-space method
- Laplace Transform Method

$$C(s) = G(s)E^*(s) = G(s)\frac{R^*(s)}{1+GH^*(s)}$$

$$c(t) = \mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}\left[G(s)\frac{R^*(s)}{1+GH^*(s)}\right]$$