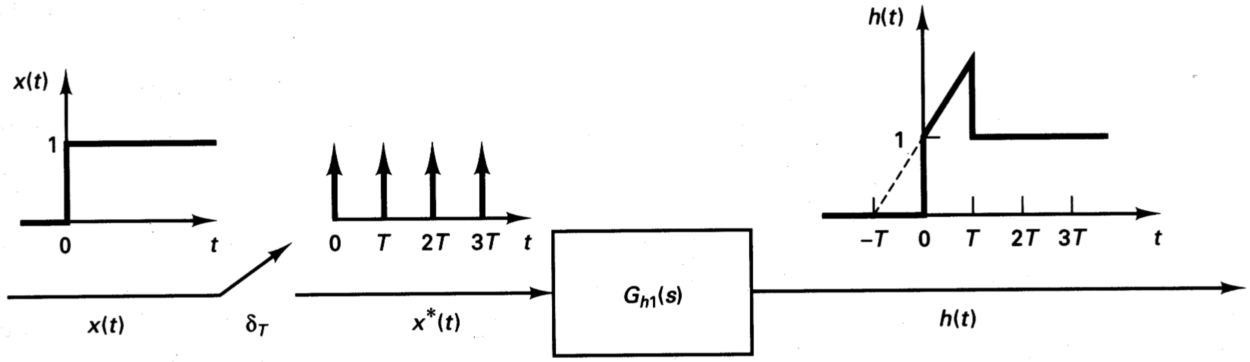
Impulse Sampling and Data Hold

- Transfer function of First-Order Hold (cont.)
  - Derivation of the transfer function (cont.)



$$X^*(s) = \sum_{k=0}^{\infty} 1(kT)e^{-kTs} = \frac{1}{1-e^{-Ts}}$$

The transfer function

$$G_{h1}(s) = \frac{H(s)}{X^*(s)} = \left(1-e^{-Ts}\right)^2 \frac{Ts+1}{Ts^2} = \left(\frac{1-e^{-Ts}}{s}\right)^2 \frac{Ts+1}{T}$$

Reconstructing Original Signals from Sampled Signals

$$X^{*}s(x)\Big|_{s=\frac{1}{T}\ln Z}\ \sum_{K=0}^{\infty} X(KT)Z^{-K} = X(Z)$$

$$x^{*}(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

$$\delta_{T}(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT) = \sum_{n=-\infty}^{\infty} C_{n}e^{jnW_{s}t}$$

$$C_{n} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\delta_{T}(t)e^{-jnW_{s}t}dt = \frac{1}{T}e^{-j0} = \frac{1}{T}$$

$$\delta_{T}(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jnW_{s}t}$$

$$X^{*}(s) = \frac{1}{T}x(t)\sum_{n=-\infty}^{\infty} e^{jnW_{s}t}$$

Reconstructing Original Signals from Sampled Signals

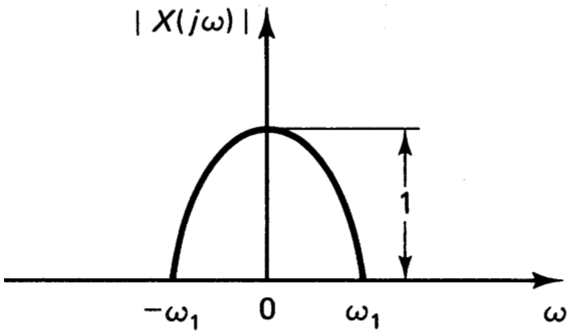
$$X^*(s) = \frac{1}{T}\int_0^\infty x(t)\sum_{n=-\infty}^{\infty} e^{jnW_s t} e^{-st} dt$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty}\int_0^\infty x(t) e^{-(S-jnW_s)t} dt$$

$$X^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(S - jnW_s)$$

Reconstructing Original Signals from Sampled Signals

- Sampling Theorem
  - If the sampling frequency is sufficiently high compared with the highest-frequency component involved in the continuous-time signal, the amplitude characteristics of the continuous-time signal may be preserved in the envelope of the sampled signal.
  - To reconstruct the original signal from a sampled signal, there is a certain minimum frequency that the sampling operation must satisfy.
  - We assume that $x(t)$ does not contain any frequency components above ω_1 rad/sec.



Reconstructing Original Signals from Sampled Signals

- Sampling Theorem

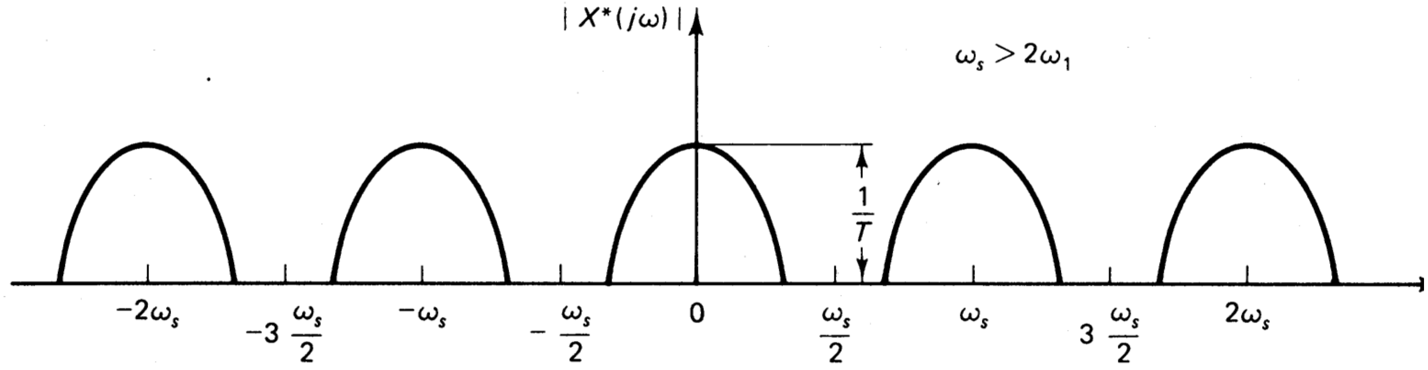
If ω_s, defined as $2\pi/T$ is greater than $2\omega_1$, where ω_1 is the highest-frequency component present in the continuous-time signal $x(t)$, then the signal $x(t)$ can be reconstructed completely from the sampled signal $x^*(t)$.

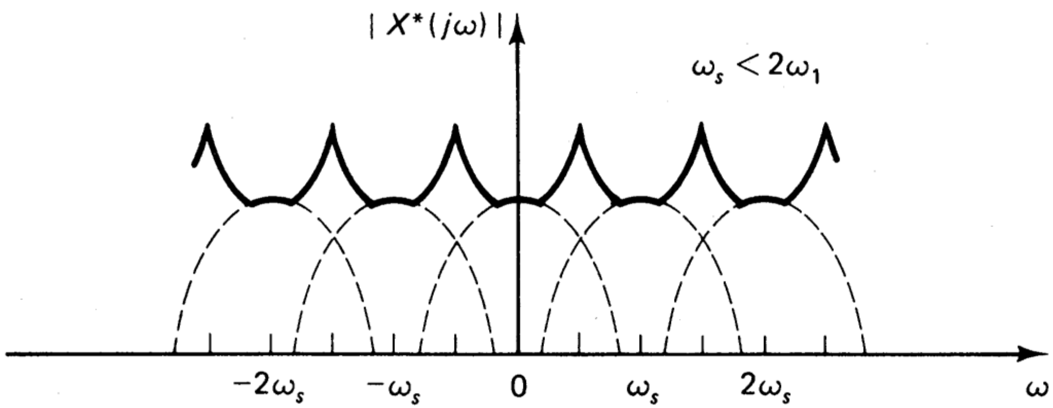
  – The frequency spectrum:

$$X^*(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{k=\infty} X(j\omega + j\omega_s k)$$

$$= \cdots + \frac{1}{T}X(j(\omega-\omega_s)) + \frac{1}{T}X(j\omega) + \frac{1}{T}X(j(\omega+\omega_s)) + \cdots$$

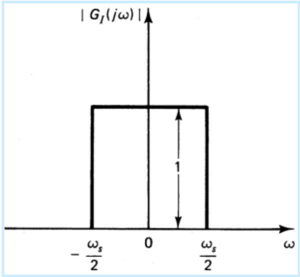
Reconstructing Original Signals from Sampled Signals

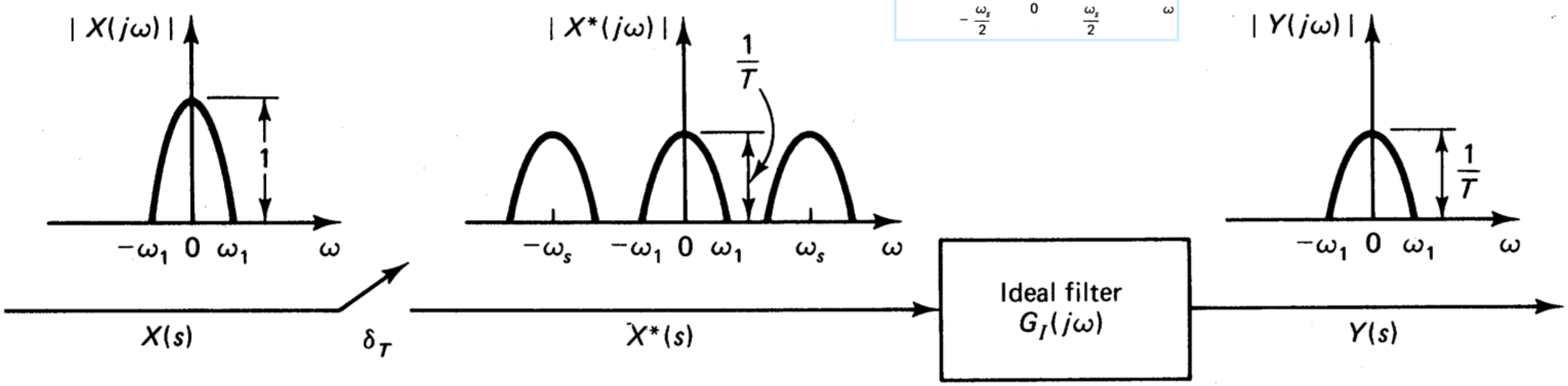




Reconstructing Original Signals from Sampled Signals

- Ideal Low-pass filter
  - The ideal filter attenuates all complementary components to zero and will pass only the primary component.
  - If the sampling frequency is less than twice the highest-frequency component of the original continuous-time signal, even the ideal filter cannot reconstruct the original continuous-time signal.



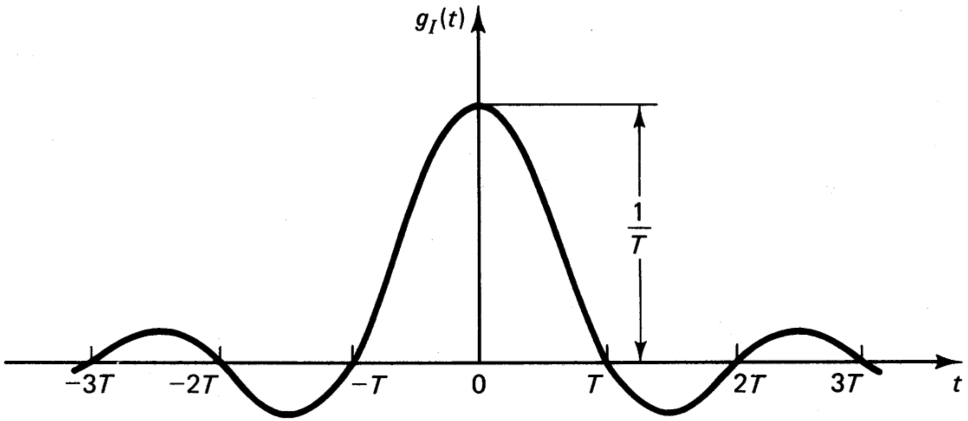


Reconstructing Original Signals from Sampled Signals

- Ideal Low-pass filter Is NOT Physically Realizable
  - For the ideal filter an output is required prior to the application of the input to the filter – *physically not realizable*.

$$G_I(j\omega) = \begin{cases} 1, & -\frac{1}{2}\omega_s \le w \le \frac{1}{2}\omega_s \\ 0, & elsewhere \end{cases}$$

$$\begin{aligned} g_I(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G_I(j\omega)e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_s/2}^{\omega_s/2} e^{j\omega t}d\omega \\ &= \frac{1}{2\pi jt}\left(e^{(1/2)j\omega_s t} - e^{-(1/2)j\omega_s t}\right) \\ &= \frac{1}{\pi t}\sin\frac{\omega_s t}{2} \\ &= \frac{1}{T}\frac{\sin(\omega_s t/2)}{\omega_s t/2} \end{aligned}$$



Reconstructing Original Signals from Sampled Signals

- Frequency-Response Characteristics of the ZOH

Transfer function of ZOH $$G_{h0}(s) = \frac{1-e^{-Ts}}{s}$$

$$G_{h0}(j\omega) = \frac{1-e^{-Tj\omega}}{j\omega}$$

$$= \frac{2e^{-(1/2)Tj\omega}\left(e^{(1/2)Tj\omega} - e^{-(1/2)Tj\omega}\right)}{2j\omega}$$

$$= T\,\frac{\sin(\omega T/2)}{\omega T/2}\,e^{-(1/2)Tj\omega}$$

$$\left|G_{h0}(j\omega)\right| = T\left|\frac{\sin(\omega T/2)}{\omega T/2}\right|$$

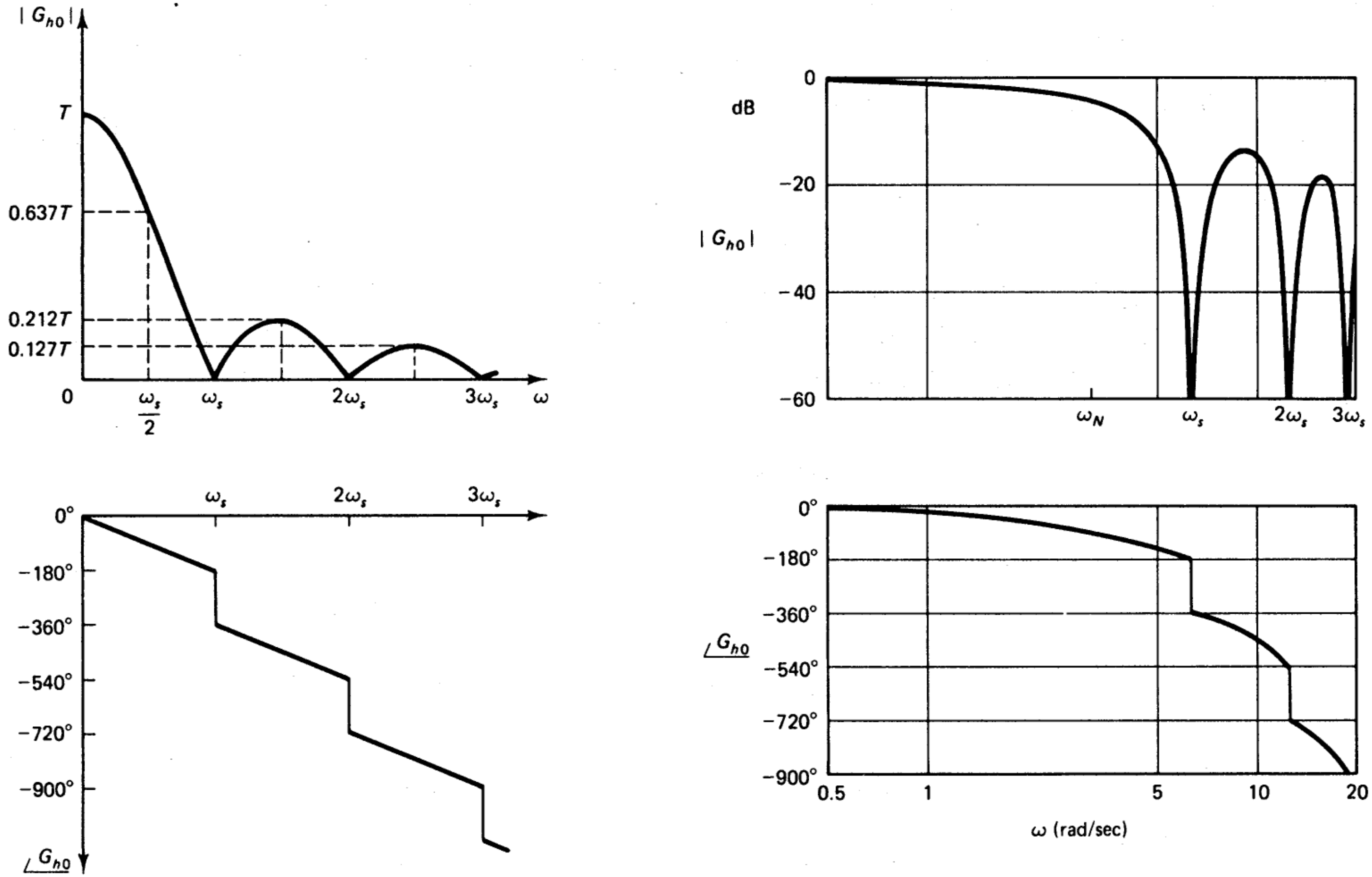
$$\angle G_{h0}(j\omega) = \angle T\,\frac{\sin(\omega T/2)}{\omega T/2}\,e^{-(1/2)Tj\omega}$$

$$= \angle \sin\frac{\omega T}{2} + \angle e^{-(1/2)Tj\omega}$$

$$= \angle \sin\frac{\omega T}{2} - \frac{\omega T}{2}$$

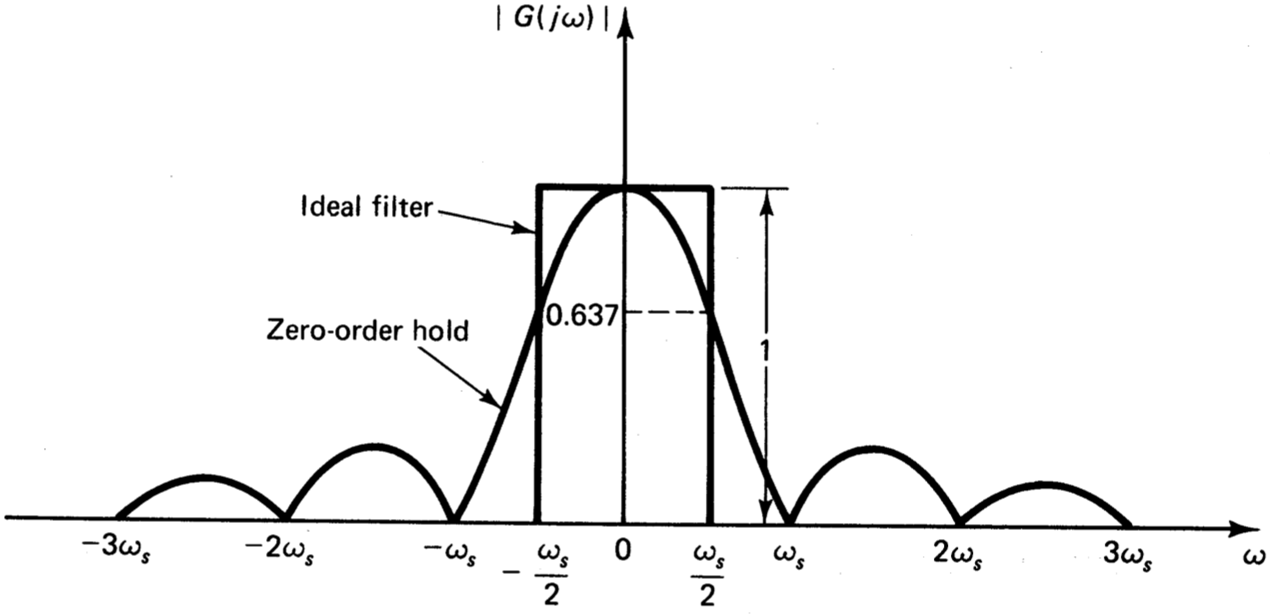
Reconstructing Original Signals from Sampled Signals

- Frequency-Response Characteristics of the ZOH(cont.)



Reconstructing Original Signals from Sampled Signals

- Frequency-Response Characteristics of the ZOH(cont.)
  - The comparison of the ideal filter and the ZOH.
  - ZOH is a low-pass filter, although its function is not quite good.
  - The accuracy of the ZOH as an extrapolator depends on the sampling frequency.

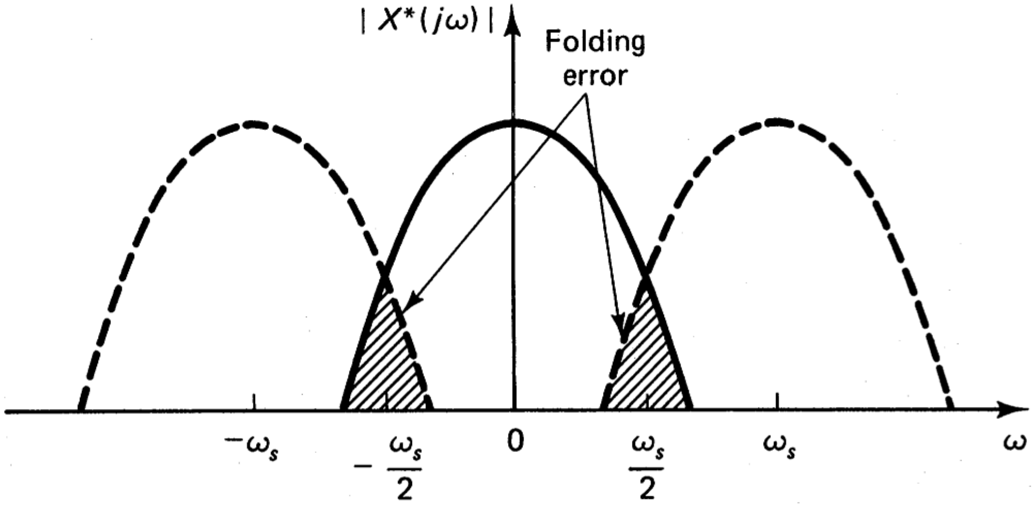


Reconstructing Original Signals from Sampled Signals

- Folding
  - The phenomenon of the overlap in the frequency spectra.
  - The folding frequency (Nyquist frequency): ω_N

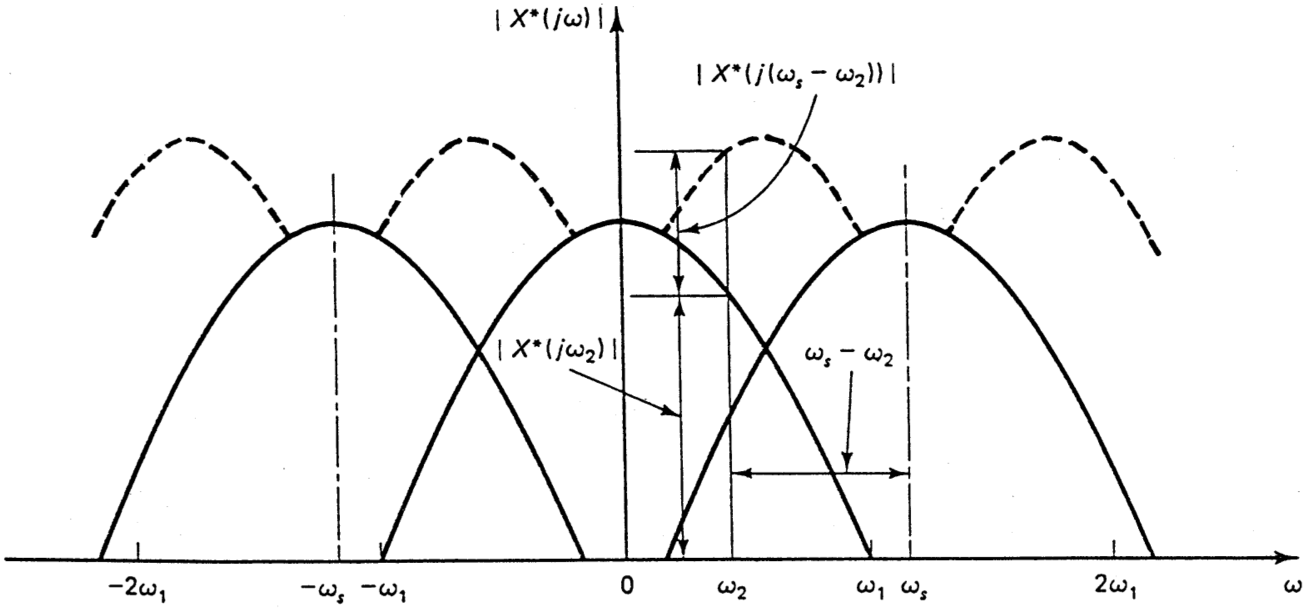
$$\omega_N = \frac{1}{2}\omega_s = \frac{\pi}{T}$$

  - In practice, signals in control systems have high-frequency components, and some folding effect will almost always exist.



Reconstructing Original Signals from Sampled Signals

- Aliasing
  - The phenomenon that the frequency component $n\omega_s \pm \omega_2$ shows up at frequency ω_2 when the signal $x(t)$ is sampled.
  - To avoid aliasing, we must either choose the sampling frequency high enough or use a prefilter ahead of the sampler to reshape the frequency spectrum of the signal before the signal is sampled.



Reconstructing Original Signals from Sampled Signals

- Hidden Oscillation
  - An oscillation existing in $x(t)$ between the sampling periods.

For example, if the signal

$$x(t) = x_1(t) + x_2(t) = \sin t + \sin 3t$$

is sampled at t=0, 2π/3, 4π /3,…, then the sampled signal will not show the frequency component with ω=3 rad/sec.