Discrete-Time Control Systems

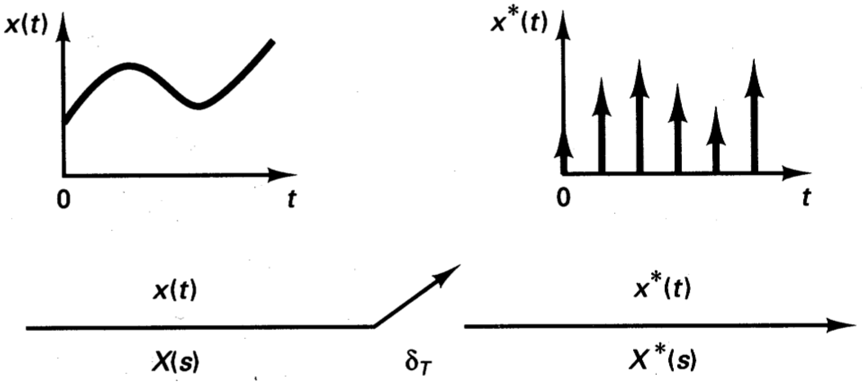
Introduction

- Backgrounds necessary for the analysis and design of discrete-time control systems in the z plane are presented.
  - The main advantage of the z transform method: it enables us to apply conventional continuous-time design methods to discrete-time systems.

- The chapter covers:
  - Mathematical representation of the sampling operation
  - The convolution integral method for obtaining the z transform
  - The sampling theorem based on the fact that the Laplace transform of the sampled signal is periodic
  - Mathematical modeling of digital controllers in terms of pulse transfer function
  - Realization of digital controllers and digital filters

Impulse Sampling and Data Hold

- Impulse Sampling
  - A fictitious sampler
  - The output of the sampler is a train of impulses.



$$x^{*}(t)=\sum_{k=0}^{\infty}x(kT)\delta(t-kT)$$

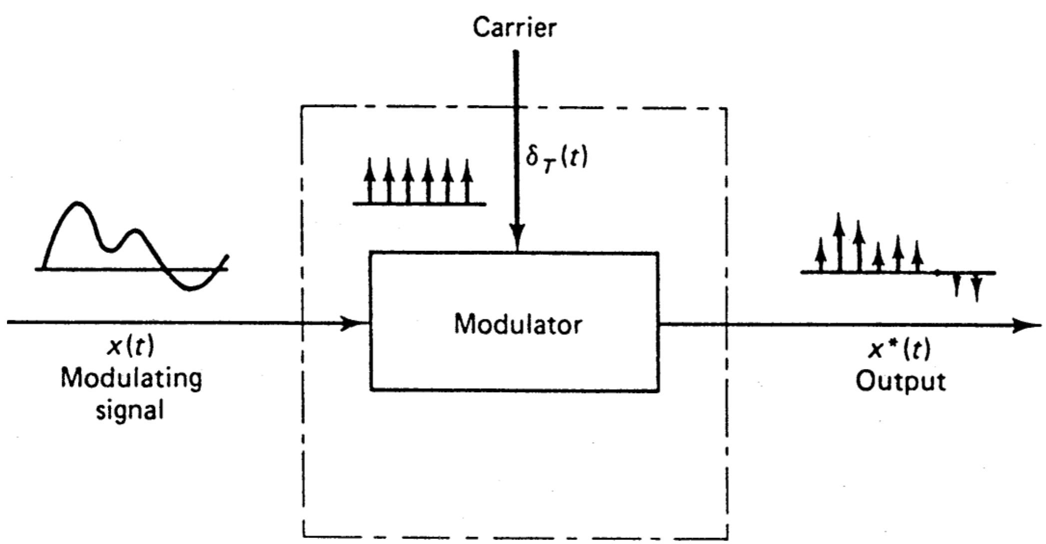
$$x^{*}(t)=x(0)\delta(t)+x(T)\delta(t-T)+\cdots+x(kT)\delta(t-kT)+\cdots$$

Impulse Sampling and Data Hold

- Impulse Sampling (cont.)
  - Let's define a train of unit impulses:

$$\delta_T(t) = \sum_{k=0}^{\infty} \delta(t - kT)$$

  - The sampler may be considered a modulator with
    - The modulating signal: the input $x(t)$
    - The carrier : the train of unit impulses $\delta_T(t)$



Impulse Sampling and Data Hold

- Impulse Sampling (cont.)
  - The Laplace transform of $x^*(t)$

$$X^*(s) = \mathcal{L}\left[x^*(t)\right] = x(0)\mathcal{L}\left[\delta(t)\right] + x(T)\mathcal{L}\left[\delta(t-T)\right] + x(2T)\mathcal{L}\left[\delta(t-2T)\right] + \cdots$$

$$= x(0) + x(T)e^{-Ts} + x(2T)e^{-2Ts} + \cdots$$

$$= \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

  - If we define $e^{Ts} = z$ or $s = \frac{1}{T}\ln z$

$$X^*(s)\Big|_{s=(1/T)\ln z} = X^*\left(\frac{1}{T}\ln z\right) = \sum_{k=0}^{\infty} x(kT)z^{-k} = X(z)$$

  - Hence we may write

$$\boxed{X^*(s)\Big|_{s=(1/T)\ln z} = X(z)}$$

Impulse Sampling and Data Hold

- Impulse Sampling (cont.)
  - Summary
    - If the continuous-time signal $x(t)$ is impulse sampled in a periodic manner, the sampled signal may be represented by

$$x^*(t) = \sum_{k=0}^{\infty} x(t)\delta(t - kT)$$

    - The Laplace transform of the impulse-sampled signal $x^*(t)$ has been shown to be the same as the z transform of signal $x(t)$ if

$$e^{Ts} = z$$

Impulse Sampling and Data Hold

- Data-Hold Circuits
  - Data-hold: a process of generating a continuous-time signal $h(t)$ from a discrete-time sequence $x(kT)$.
  - A hold circuit approximately reproduces the signal applied to the sampler.

$$h(kT+\tau) = a_n \tau^n + a_{n-1}\tau^{n-1} + \cdots + a_1 \tau + a_0 \qquad \text{where } 0 \le \tau < T$$

Note that signal $h(kT)$ must equal $x(kT)$:

$$h(kT) = x(kT)$$

$$\Rightarrow \quad h(kT+\tau) = a_n \tau^n + a_{n-1}\tau^{n-1} + \cdots + a_1 \tau + x(kT)$$ n-th order hold

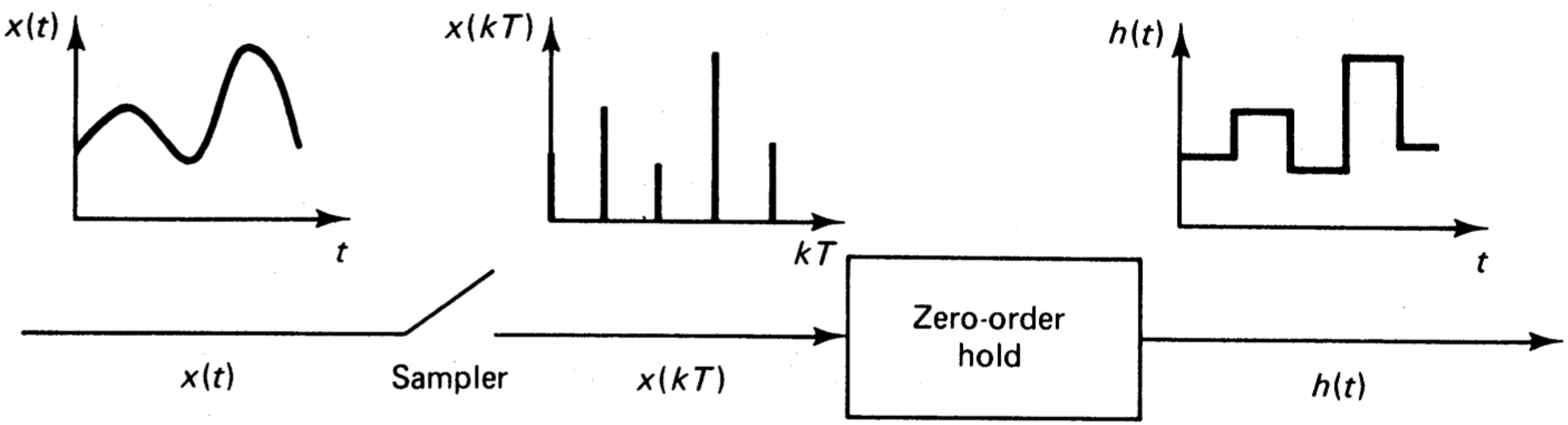
  - The simplest data-hold: zero-order hold (clamper)

$$h(kT+\tau) = x(kT)$$

Impulse Sampling and Data Hold

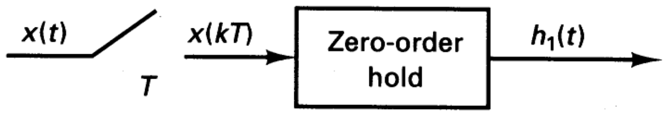
- Zero-Order Hold

$$h(kT+t) = x(kT), \quad \text{for } 0 \le t < T$$



Impulse Sampling and Data Hold

- Zero-Order Hold (cont.)



(a)

A real sampler and zero-order hold

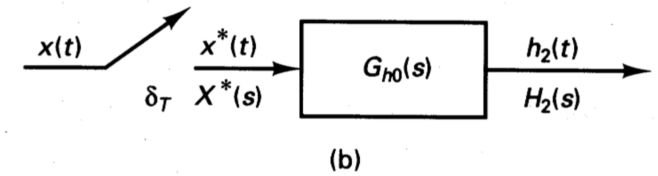
$$h_1(t) = x(0)[1(t) - 1(t-T)] + x(T)[1(t-T) - 1(t-2T)] + x(2T)[1(t-2T) - 1(t-3T)] + \cdots$$
$$= \sum_{k=0}^{\infty} x(kT)[1(t-kT) - 1(t-(k+1)T)]$$

Since $\mathcal{L}[1(t-kT)] = \dfrac{e^{-kTs}}{s}$

$$\Rightarrow \mathcal{L}[h_1(t)] = H_1(s) = \sum_{k=0}^{\infty} x(kT)\frac{e^{-kTs} - e^{-(k+1)Ts}}{s} = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

Impulse Sampling and Data Hold

- Zero-Order Hold (cont.)



Mathematical model: an impulse sampler and transfer function

$$\mathcal{L}\left[h_2(t)\right] = H_2(s) = H_1(s)$$

$$H_2(s) = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

From the figure $H_2(s) = G_{h0}(s)X^*(s)$

Since $X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$

$$\Rightarrow \quad H_2(s) = \frac{1-e^{-Ts}}{s}X^*(s)$$

$$\Rightarrow \quad G_{h0}(s) = \frac{1-e^{-Ts}}{s}$$

Impulse Sampling and Data Hold

- Transfer function of First-Order Hold

$$h(kT+\tau)=a_n\tau^n+a_{n-1}\tau^{n-1}+\cdots+a_1\tau+x(kT)$$ n-th order hold

1-st order hold $h(kT+\tau)=a_1\tau+x(kT)$

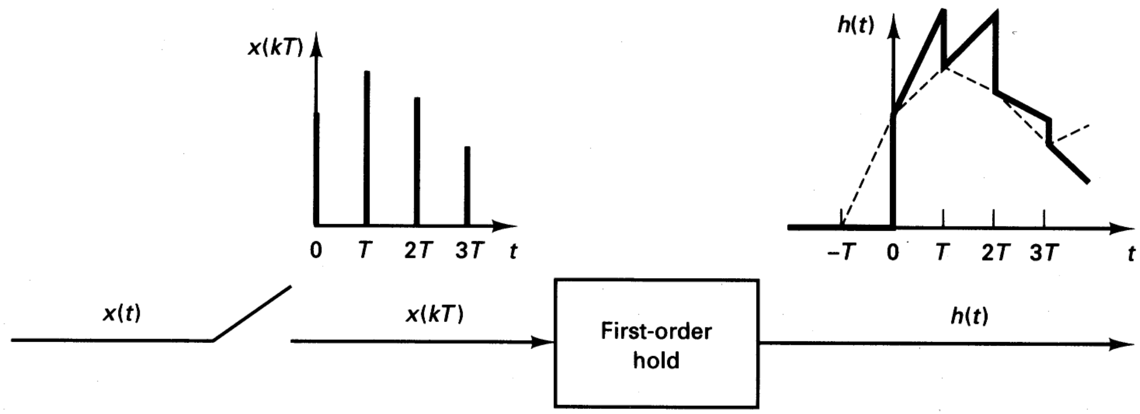
By applying the condition that $h((k-1)T)=x((k-1)T)$

$$h((k-1)T)=-a_1T+x(kT)=x((k-1)T)$$

$$a_1=\frac{x(kT)-x((k-1)T)}{T}$$

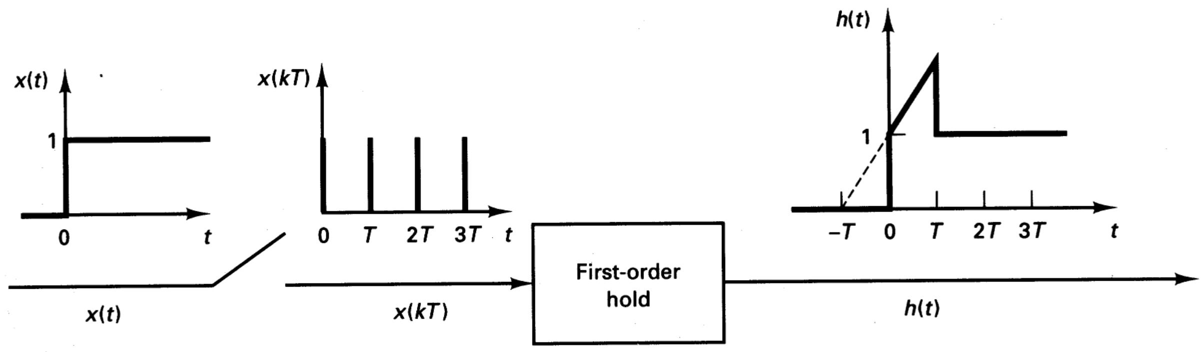
⇒

$$h(kT+\tau)=x(kT)+\frac{x(kT)-x((k-1)T)}{T}\tau$$



Impulse Sampling and Data Hold

- Transfer function of First-Order Hold (cont.)
  - Derivation of the transfer function



$$h(t) = \left(1 + \frac{t}{T}\right)1(t) - \frac{t-T}{T}1(t-T) - 1(t-T)$$

Taking the Laplace transform

$$H(s) = \left(\frac{1}{s} + \frac{1}{Ts^2}\right) - \frac{1}{Ts^2}e^{-Ts} - \frac{1}{s}e^{-Ts}$$

$$= \frac{1-e^{-Ts}}{s} + \frac{1-e^{-Ts}}{Ts^2} = \left(1-e^{-Ts}\right)\frac{Ts+1}{Ts^2}$$