ME – VII SEM

Course Name- Mechanical Vibrations

MULTIDEGREE OF FREEDOM SYSTEMS

- A multi degrees of freedom (dof) system is one, which requires two or more coordinates to describe its motion.
- These coordinates are called generalized coordinates when they are independent of each other and equal in number to the degrees of freedom of the system.
- The *N* dof system differs from the single dof system in that it has *N* natural frequencies, and for each of the natural frequencies there corresponds a natural state of vibration with a displacement configuration known as the normal mode. Mathematical terms associated with these quantities are eigenvalues and eigenvectors.
- Normal mode vibrations are free vibrations that depend only on the mass and stiffness of the system and how they are distributed.

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- Analytical/closed-form solutions can be established for 2 degrees of freedom systems. But for more degree of freedom systems numerical analysis using computer is required to find natural frequencies (eigen-values) and mode shapes (eigen vectors). The equations of motion for these systems in matrix form can be written as
- $$M\ddot{x} + Kx = 0, \qquad (1)$$

where $M \text{ and } K$ are mass and stiffness matrix respectfully. Pre-multiplying M^{-1} , the above equation becomes,

$$I\ddot{x} + Ax = 0 \qquad (2)$$

where . $A = M^{-1}K$

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- Assuming harmonic motion i.e., $x = X \sin \omega t$, equation (2) reduces to
-
- $$[A - \lambda I]X = 0 \qquad (3)$$
- where $\lambda = \omega^2$
- For non-trivial solution of *X* , equation (3) reduces $|A - \lambda I| = 0$ to which is known as the characteristic equation. One may find (*i* : λ_i 1,2, ,n), by finding the roots of this characteristic equation or by finding the eigenvalues of *A* . The eigenvector will correspond to the normal modes of the system. For an *n* degree of freedom system one will get *n* natural frequencies (square root of eigen values) and *n* normal modes (eigen vectors).
- It is possible to find eigenvectors from **adjacent matrix** of the system:

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- Let $B = A - \lambda I$ (4)
-
- $$B^{-1} = \frac{1}{|B|} adj B$$ (5)
- Premultiplying equation (5) by $|B|B$
-
- $$|B|BB^{-1} = \frac{|B|B}{|B|} adjB$$ (6)
- or,
- $$|B|I = BadjB$$ (7)
- Hence, $$|A - \lambda I|I = (A - \lambda I)adj(A - \lambda I)$$ (8)

$for\ \lambda = \lambda_i, |A - \lambda I| = 0,$

- hence,
- $$[A - \lambda I]adj[A - \lambda I] = 0$$ (9)

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- Comparing equation (3) and (9), one gets
-
- $$X_i = adj[A - \lambda I] \qquad (10)$$

- **Hence the normal modes of the system can be obtained by finding the adjoint of** $[A - \lambda I]$.
- **Example 1**
- Find the normal modes for torsional vibration of a shaft with two rotors as shown in figure 1.

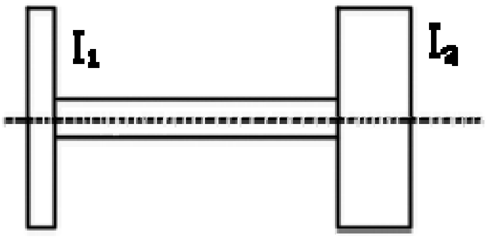

Figure 1 : A rotating shaft with two rotors.

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- **Solution**
- Let θ_1 and θ_2 be the rotation of rotor 1 and 2 respectively.
- The equation of motion of the system can be given by

$$\begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Let us assume $\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sin \omega t$

- So, $A = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}^{-1}\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$

- Hence,

$$\begin{bmatrix} \frac{k}{I_1} - \lambda & -\frac{k}{I_1} \\ -\frac{k}{I_2} & \frac{k}{I_2} - \lambda \end{bmatrix}$$

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$$[A-\lambda I]=\begin{bmatrix}\frac{k}{I_1}-\lambda & -\frac{k}{I_1}\\ -\frac{k}{I_2} & \frac{k}{I_2}-\lambda\end{bmatrix}$$ and solving $|A-\lambda I|=0$ one will get

$$\lambda\left(\lambda-\left(\frac{k}{I_1}+\frac{k}{I_2}\right)\right)=0$$

X

Hence $\lambda=0, \text{or}, \lambda=\frac{k}{I_1}+\frac{k}{I_2}$ So the system is a degenerate system

Corresponding to $\lambda=0$, i.e, for rigid body motion, one may find the normal mode X .

$$X=adj[A-\lambda I]_{\lambda=0}=\begin{bmatrix}\frac{k}{I_1} & -\frac{k}{I_1}\\ -\frac{k}{I_2} & \frac{k}{I_2}\end{bmatrix}=\begin{bmatrix}\frac{k}{I_2} & \frac{k}{I_1}\\ \frac{k}{I_2} & \frac{k}{I_1}\end{bmatrix}$$

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- One may note that, the normalized value of each column yield the same value and in this case it is equal to the expected value of
- $$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
- i.e., both the rotor will rotate same amount giving rising to rigid-body motion.
- Now to find the normal mode for, $\lambda = \frac{k}{I_1} + \frac{k}{I_2}$ one may find $Adj\,|A - \lambda I|$ is equal to
- $$Adj \begin{bmatrix} -\frac{k}{I_2} & -\frac{k}{I_1} \\ -\frac{k}{I_2} & -\frac{k}{I_1} \end{bmatrix} = \begin{bmatrix} -\frac{k}{I_1} & \frac{k}{I_2} \\ \frac{k}{I_1} & -\frac{k}{I_2} \end{bmatrix} = \begin{bmatrix} -\frac{k}{I_1} & \frac{k}{I_1} \\ \frac{k}{I_2} & -\frac{k}{I_2} \end{bmatrix}$$

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- Hence, $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} -k / I_1 \\ k / I_2 \end{bmatrix}$
- To normalize the value, taking $X_2 = 1$
- $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} -I_2 / I_1 \\ 1 \end{bmatrix}$
- Hence both the rotor rotate in opposite directions and their amplitude ratio is inversely proportional to their inertia ratio.
- Flexibility Coefficient Method :
- Since the elastic behavior of motion may be expressed in terms of the stiffness or flexibility, the equations of motion may be formulated by either the stiffness matrix $[k]$ or the flexibility matrix $[a]$.
- -In stiffness formulation , the force f is expressed in terms of displacement x by

$$\{f\} = [k]\{x\}$$

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- Also one may write $\{x\} = [k]^{-1}\{f\} = [a]\{f\}$, which leads to the **flexibility approach**.
- -The choice as to which approach one should adopt depends on the problem. Some problems are more easily pursued as the basis of stiffness, whereas for others the flexibility approach may be desirable.
- **Flexibility matrix :**
- For a four degree-of-freedom system, the displacement as forces are related by flexibility matrix as
- $$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix}$$
- The **flexibility influence coefficient** a_{ij} defined as the displacement at *i* due to unit force applied at *j* with all other forces equal to zero. Thus the first column represent displacement corresponding to $f_1 = 1, f_2 = f_3 = 0.$ Similarly the second column represents the displacements for $f_2 = 1$ and $f_1 = f_3 = 0$ and so on.

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- **Example 1:** Determine the flexibility matrix for the axial displacement of the spring system shown in figure 3.
- To find a_{11} and a_{21} one has to apply unit force at 1, and no force at 2. So $a_{11} = 1/K_1$, and $a_{21} = 1/K_1$ as the second spring will have a
- rigid body motion. Similarly to find a_{12} and a_{22} one
- has to apply only unit force at 2 .As spring 1 and 2
- are in series, displacement at 1 i.e., $a_{12} = 1/K_1$
- and $a_{22} = K_1K_2/(K_1 + K_2)$
- Hence flexibility matrix can be given by a_{12} and a_{22}
- $$\begin{bmatrix} 1/K_1 & 1/K_1 \\ 1/K_1 & (K_1+K_2)/(K_1K_2) \end{bmatrix}$$

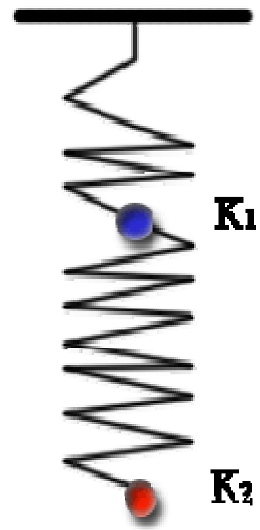

Figure 3

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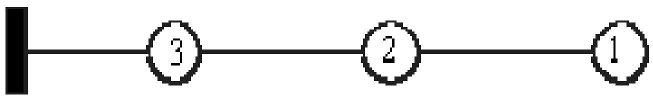
- **Example 2:** Determine the flexibility influence coefficient for the transverse vibration of a cantilever beam with three equal mass placed at equal interval as shown in figure 4. The flexibility of the rod is EI
- 
- Figure 4
- **Solution**
  In this the flexibility influence coefficients can be obtained by finding the required displacement, which can easily be obtained by using moment area method.
  To find the first column of the flexibility matrix, one should apply only unit force at 1. From the bending moment diagram ABC, shown in figure 5(a) .

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$$a_{11} = \frac{1/2(3l).(3l).\frac{2}{3}.3l}{EI} = \frac{9l^3}{EI}$$ (=area moment of the ΔABC/EI about 1)

$$a_{21} = \frac{(2l.l.l) + (\frac{1}{2}.2l.2l.\frac{2}{3}.2l)}{EI} = \frac{14l^3}{3EI}$$ (= area moment of the trapesum 2BCE/EI about 2)

$$a_{31} = \frac{(2l.l)l/2 + (\frac{1}{2}.l.l.\frac{2}{3}.l)}{EI} = \frac{4l^3}{3EI}$$ (= area moment of the trapesum 21BCD/EI about 3)

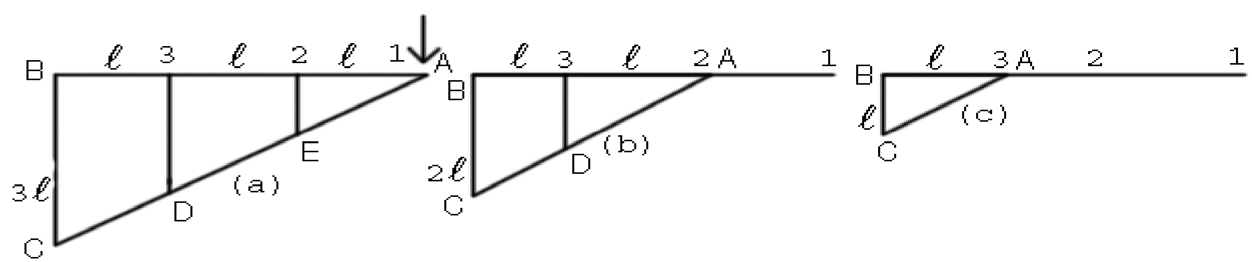

Figure 5: Bending moment diagram considering unit load at point (a) 1, (b) 2 , (c) 3

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- Now to find the second column of the flexibility matrix, one should apply unit force at 2 and draw the corresponding bending moment diagram as shown in Figure 5 (b). From this figure
- $$a_{12} = \frac{\frac{1}{2}(2l.2l)(l + \frac{2}{3}.2l)}{EI} = \frac{14l^3}{3EI} \quad (= \text{area moment of the } \Delta 2BC \text{ about } 1),$$
$$a_{22} = \frac{\frac{1}{2}(2l.2l).\frac{2}{3}(2l)}{EI} = \frac{8l^3}{3EI} (= \text{area moment of the } \Delta 2BC \text{ about } 2),$$
$$a_{32} = \frac{\frac{1}{2}(2l.2l).\frac{2}{3}(2l)}{EI} = \frac{8l^3}{3EI} (= \text{area moment of the 3BCD about } 3),$$
- Now to find the third column of the flexibility matrix, one should apply unit force at 3 and draw the corresponding bending moment diagram as shown in Figure 5 (c). From this figure
- $$a_{31} = \frac{\frac{1}{2}(l.l)(2l + \frac{2}{3}l)}{EI} = \frac{4l^3}{3EI},$$

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$$a_{32} = \frac{\frac{1}{2}(l.l)(l + \frac{2}{3}l)}{EI} = \frac{5l^3}{6EI}$$

$$a_{33} = \frac{\frac{1}{2}(l.l)\frac{2}{3}l}{EI} = \frac{l^3}{3EI}$$

Hence, the flexibility matrix can be written as

$$a = \frac{l^3}{3EI}\begin{bmatrix} 27 & 14 & 4 \\ 14 & 8 & 2.5 \\ 4 & 2.5 & 1 \end{bmatrix}$$

In this example it may be observed that $a_{ij} = a_{ji}$ which is known as reciprocity theorem.

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- **Properties of Vibrating Systems:**
- Since the elastic behavior of motion may be expressed in terms of the stiffness or flexibility, the equations of motion may be formulated by either the stiffness matrix or the $[k]$ ility matrix . $[a]$
- In stiffness formulation , the force f expressed in terms of displacement by x.
- $$\{f\} = [k]\{x\}. \qquad (11)$$
- Also one may write
- $$\{x\} = [k]^{-1}\{f\} = [a]\{f\}. \qquad (12)$$
- which leads to the **flexibility approach** .
- The choice as to which approach one should adopt depends on the problem. Some problems are more easily pursued as the basis of stiffness, whereas for others the flexibility approach may be desirable.

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- **Flexibility matrix:**
- For a three degree-of-freedom system, the displacement as forces are related by flexibility matrix as
- $$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix}.$$
- The **flexibility influence coefficient** is defined as the displacement at *i* due to unit force applied at *j* with all other forces equal to zero. Thus the first column represent displacement corresponding to Similarly, second co $f_1 = 1, f_2 = f_3 = 0$. he displacements for and so on. $f_2 = 1$ and $f_1 = f_3 = 0$.

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- **Reciprocity theorem :** States that in a linear system $a_{ij} = a_{ji}$
- **Proof :** Consider a linear system and now applying force f_i the work done
- = $\frac{1}{2}$ force X displacement = $\frac{1}{2} f_i (f_i a_{ii}) = \frac{1}{2} f_i^2 a_{ii}$
- Then applying force f_j the work done = $\frac{1}{2} f_j^2 a_{jj}$
- However due to application of force f_j undergoes further displacement, and the additional work done by f_i becomes $(a_{ij} f_j) f_i$
So, total work done

$$= \frac{1}{2} f_i^2 a_{ii} + \frac{1}{2} f_j^2 a_{jj} + a_{ij} f_j f_i \qquad (14)$$

- Now if one reverses the order of application of forces, i.e., first a force f_j acts at j followed by a force f_i acting at *i* , the work done will
- be

$$= \frac{1}{2} f_j^2 a_{jj} + \frac{1}{2} f_i^2 a_{ii} + a_{ji} f_i f_j \qquad (15)$$

- Since the work done in the two cases must be equal
- hence,

$$a_{ij} = a_{ji} \qquad (16)$$

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- **Stiffness matrix :**
- For a three dof system, the force and displacements are related by stiffness matrix as
- $$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \quad (17)$$
- The stiffness k_{ij} is defined, as the force required at point *i* to have unit displacement at point *j*, displacement at other places being zero. So k_{11}, k_{21} and k_{31} forces required at points 1,2,3 respectively to have unit displacement at 1, i.e., $x_1 = 1, x_2 = x_3 = 0$

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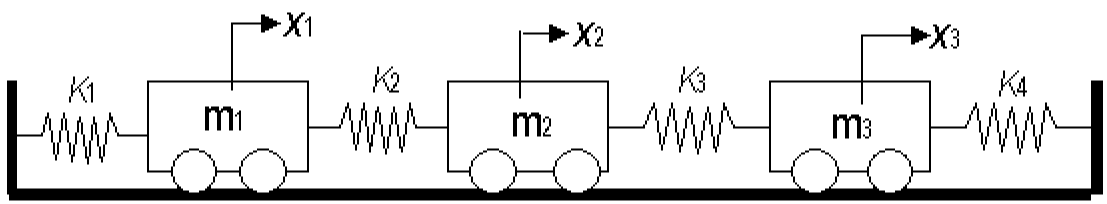
- **Example 3:** Find stiffness matrix for the spring-mass system shown in Figure 6.



- Figure-6

- **Solution:** From the definition of the element of stiffness matrix, the first column $K_{i1}(i=1,2,3)$ can be obtained by finding the forces at mass 1,2, and 3 respectively to have unit displacement for mass 1 and zero displacement for mass 2 and 3, i.e., $x_1=1, x_2=x_2=0$ which is depicted in figure 7.

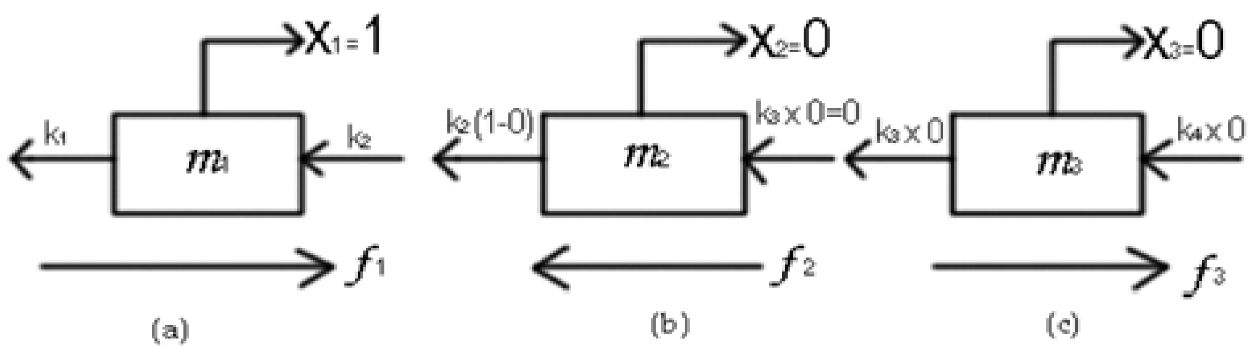


Figure 7 : Freebody diagram considering $x_1=1, x_2=x_3=0$

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- From the freebody diagram figure 7(a), to have the above-mentioned displacements, mass 1 will be subjected to a spring force of .Hence to $K_1 + K_2$ ome this a force is re $f_1 = K_1 + K_2$ 1.
- Similarly from Figure 7(b) it can be observed that a force (negative sign indicate the force to be applied to the left) $f_2 = -K_2$.
- and from figure 7(c) it may be noted that no force is required at mass 3 i.e.,

$$f_3 = 0.$$

- Hen $K_{13} = f_1 = 0$ $K_{23} = f_2 = -K_3$ $K_{33} = f_3 = K_3 + K_4$.

$$K_{11} = f_1 = K_1 + K_2 \qquad K_{21} = f_2 = -K_2 \qquad K_{31} = f_3 = 0$$

Similarly drawing freebody diagram to have $x_1 = 0, x_2 = 1, x_3 = 0$ (Figure 8)

$K_{12} = f_1 = -K_2$, $K_{22} = f_2 = K_2 + K_3$, and $K_{32} = f_3 = -K_3$

From freebody diagrams shown in figure 9 to have

$x_1 = 0, x_2 = 0, x_3 = 1$, $K_{23} = f_2 = -K_3$, and $K_{33} = f_3 = K_3 + K_4$

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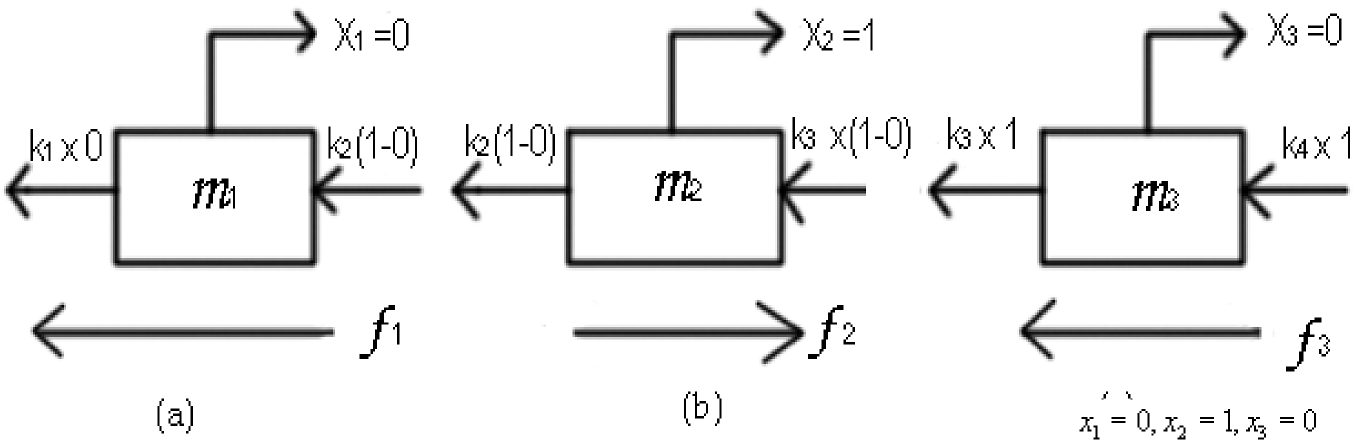

Figure 8: Freebody diagram considering $x_1 = 0, x_2 = 1, x_3 = 0$

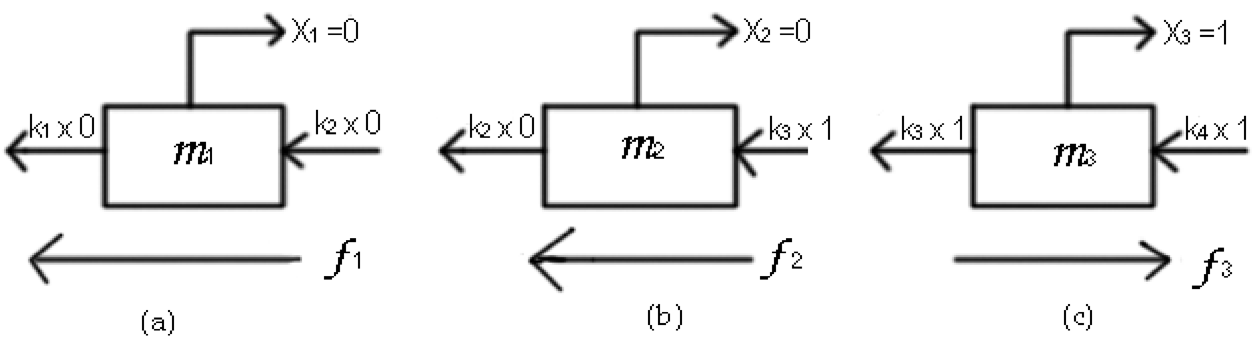

Figure 9: Freebody diagram considering $x_1 = 0, x_2 = 0, x_3 = 1$

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- Hence the stiffness matrix can be given by

$$K = \begin{bmatrix} K_1 + K_2 & -K_2 & 0 \\ -K_2 & K_2 + K_3 & -K_3 \\ 0 & -K_3 & K_3 + K_4 \end{bmatrix}$$

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- **Orthogonal Properties of the Eigen Vectors:**
- stem can be shown to be orthogonal with respect to the mass and stiffness matrices. In equation (1) $x = X \sin \omega t$ the equation for the *i* th mode be
-
- $$KX_i = \lambda_i MX_i \qquad (18)$$
- Premultiplying by the transpose of mode j,
- $$X_j' KX_i = X_j' \lambda_i MX_i = \lambda_i (X_j MX_i) \qquad (19)$$
- Now start with the equation for the jth mode and premultiplying by tc X_i' obtain,
- $$X_i' KX_j = \lambda_i (X_i' MX_j) \qquad (20)$$
- Since K and M are symmetric matrices
- $$X_j' MX_i = X_i' MX_j \text{ and } X_j' KX_i = X_i' KX_j \qquad (21)$$

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- Subtracting (20) from (19),

$$(\lambda_i - \lambda_j) X_2{}' M X = 0 \quad (22)$$

$$\text{As } \lambda_i \neq \lambda_j, X_2{}' M X_j = 0 \text{ and } X_2{}' K X_j = 0 \quad (23)$$

- Equation (23) shows the orthogonal character of the normal modes.
- If i = j,

$$X_2{}' M X_i = M_i$$
$$X_2 K X_i = K_i \quad (24)$$

- M_i and K_i are known as the generalized mass and generalized stiffness of the ith mode.

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- **Example 4: Determine the normal modes for the following system and show that the modes are orthogonal**
- $$\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
- **Solution:**
- **Here** $$A = M^{-1}K = \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}^{-1} \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix} = \begin{bmatrix} \frac{3}{2}\frac{k}{m} & -\frac{k}{2m} \\ -\frac{k}{m} & \frac{k}{m} \end{bmatrix}$$
- **Solving** $|A - \lambda I| = 0$ **one will get** $\lambda_1 = \frac{1}{2}\frac{k}{m}, \quad \lambda_2 = 2\frac{k}{m}$
- **To obtain the normal modes, one may find the *adj*** $|A - \lambda I|$ **Here,**

$$adj[A - \lambda I] = \begin{bmatrix} \left(\frac{k}{m} - \lambda_i\right) & \frac{k}{2m} \\ \frac{k}{m} & \frac{3}{2}\frac{k}{m} - \lambda_i \end{bmatrix}$$

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$$Substituting\ \lambda_i = \lambda_1,$$

$$adj[A - \lambda I] = \begin{bmatrix} 0.5 & 0.5 \\ 1.0 & 1.0 \end{bmatrix} \frac{k}{m},$$

One may note that the normalized value of both the columns of the above matrix are same.

Hence, $X_1 = \begin{Bmatrix} 0.5 \\ 1 \end{Bmatrix}$. Similarly substituting $\lambda_i = \lambda_2$

$$adj[A - \lambda I] = \begin{bmatrix} -1 & 0.5 \\ 1 & 0.5 \end{bmatrix} \frac{k}{m}$$

Hence, $X_2 = \begin{Bmatrix} -1.0 \\ 1.0 \end{Bmatrix}$

Here X_1 and X_2 are normal modes

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- To verify that the normal modes of this system are orthogonal,
- $X_1{}'MX_2 = (0.5 \quad 1)\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = -m + m = 0$ Similarly $X_2{}'MX_1 = 0$
- **Modal Participation in Free Vibration**
- To find how much of each mode will be present in the resulting free vibration, one may write the free vibration $u(o)$ as
- $$u(0) = C_1X_1 + C_2X_2 + \cdots + C_iX_i + \cdots \quad (25)$$
- Where X_i is the *i* th normal mode and the coefficient C_i present the amount of *i* th mode present in the free vibration. Premultiplying equation (25) by $X_2{}'M$ and ta[ke n]ote of the orthogonal property one gets,
- $$X_i{}'Mu(0) = 0 + 0 + \cdots + (X_i{}'MX_i)C_i + 0 = M_iC_i \quad (26)$$
- So,
$$C_i = \frac{X_i{}'Mu(0)}{X_i{}'MX_i} \quad (27)$$

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- **Modal matrix P:**
- The modal matrix for a three dof system can be given by

$$P = \left[\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_1 \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_2 \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_3 \right] = [X_1 X_2 X_3] \qquad (28)$$

$$P' = \begin{bmatrix} (x_1 & x_2 & x_3)_1 \\ (x_1 & x_2 & x_3)_2 \\ (x_1 & x_2 & x_3)_3 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \qquad (29)$$

- $P'MP$ and $P'KP$ will be diagonal matrices since the off diagonal terms simply expresses the orthogonality relation and are zero.

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- For a 2DOF:

$$P'MP = [X_1 \ X_2]'[M][X_1 \ X_2] = \begin{bmatrix} X_1'MX_1 & X_1'MX_2 \\ X_2'MX_1 & X_2'MX_2 \end{bmatrix} = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \qquad (30)$$

- Similarly

$$P'KP = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \qquad (31)$$

- **Weighted Model Matrix** : If each column of the model matrix P is divided by the square root of the generalized mass M_i , the new matrix is called weighted model matrix and designated as $\tilde{\tilde{P}}$.
- So $\tilde{\tilde{P}}'M\tilde{\tilde{P}} = I$ and $\tilde{\tilde{P}}'K\tilde{\tilde{P}} = \boldsymbol{\lambda}$ (32)
- Here I is the unity matrix and matrix $\boldsymbol{\lambda}$ a diagonal matrix with eigenvalues in the diagonal elements.

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- **Example 5**. Find modal and weighted modal matrices for the system

$$\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- **Solution :** The modal matrix

$$P = \begin{bmatrix} 0.5 & -1 \\ 1 & 1 \end{bmatrix}$$

- Also,

$$X_1' M X_1 = \begin{bmatrix} 0.5 & 1 \end{bmatrix}\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = 1.5m$$

$$X_2' M X_2 = \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 3m$$

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- $$X_1' K X_1 = \begin{bmatrix} 0.5 & 1 \end{bmatrix} \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = 0.75k$$

$$X_2' K X_2 = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 6k$$

$$\tilde{P} = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.5/\sqrt{1.5} & -1/\sqrt{3} \\ 1/\sqrt{1.5} & 1/\sqrt{3} \end{bmatrix} = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.408 & -0.5774 \\ 0.8165 & 0.5774 \end{bmatrix}$$

- so

$$\tilde{P}' M \tilde{P} = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.4082 & -0.5774 \\ 0.8165 & 0.5774 \end{bmatrix}' \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \frac{1}{\sqrt{m}} \begin{bmatrix} 0.4082 & -0.5774 \\ 0.8165 & 0.5774 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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- Similarly,

$$\tilde{P}'K\tilde{P} = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.4082 & -0.5774 \\ 0.8165 & 0.5774 \end{bmatrix}' \begin{bmatrix} 3k & -1 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{m}}\begin{bmatrix} 0.4082 & -0.5774 \\ 0.8165 & 0.5774 \end{bmatrix} = \frac{k}{m}\begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix}$$

- It may be noted that the diagonal elements of $\tilde{P}K\tilde{P}$ 'e the eigenvalues of the system.
- **To develop the uncoupled equation of motion by use of modal matrix P**

- For a given system (33)

$$M\ddot{X} + KX = F$$

- Substituting
- (34)

$$X = PY$$

- one will obtain
- (35)

$$MP\ddot{Y} + KPY = F$$

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- Premultiplying P' n equation (35) yields
-
- $$(P'MP)\ddot{Y} + (P'KP)Y = P'F \qquad (36)$$
- As $(P'MP)$ and $(P'KP)$ are diagonal matrices equation (36) represents a set of uncoupled equation which can be easily solved by using the principles used for single dof system.

It can also be reduced to a simpler form by using weighted modal matrix. Considering

$$X = \tilde{P}Y \qquad (37)$$

equation (33) will reduce to

$$M\tilde{P}\ddot{Y} + K\tilde{P}Y = F \qquad (38)$$

Now premultiplying $\tilde{P}'$ in equation (38) one will get

$$\tilde{P}'M\tilde{P}\ddot{Y} + \tilde{P}'K\tilde{P}Y = \tilde{P}'F \qquad (39)$$

But from equation (32) it is known that $\tilde{P}'M\tilde{P} = I$ and $\tilde{P}'K\tilde{P} = \boldsymbol{\lambda}$ so the above equation reduces to

$$I\ddot{Y} + \lambda Y = \tilde{P}'F \qquad (40)$$

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- As this equation is a set of uncoupled equations, hence, each equation is similar to that of a single degree of freedom system and the solution of this system of equations can be easily obtained. Here is known as **principal coordinates.**
- **Summary**
- In this chapter the following concepts are learned
- Flexibility influence coefficients
- Reciprocity theorem
- Stiffness matrix approach
- Flexibility matrix approach
- Normal mode of vibration
- Eigenvalues and eigenvectors
- Determination of normal modes by adjoint matrix method
- Free vibration using normal modes
- Orthogonality principle of normal modes
- Modal matrix
- Weighted modal matrix
- Reduction of coupled equation of motion to uncoupled equation of motion.