



ME – VII SEM

**Course Name-
Mechanical Vibrations**

U-5, Two Degree of Freedom Vibration System

- * The vibrating systems, which require two coordinates to describe its motion, are called two-degrees-of -freedom systems.
- * These coordinates are called generalized coordinates when they are independent of each other and equal in number to the degrees of freedom of the system.
- * Unlike single degree of freedom system, where only one co-ordinate and hence one equation of motion is required to express the vibration of the system, in two-dof systems minimum two co-ordinates and hence two equations of motion are required to represent the motion of the system. For a conservative natural system, these equations can be written by using mass and stiffness matrices.
- * One may find a number of generalized co-ordinate systems to represent the motion of the same system. While using these co-ordinates the mass and stiffness matrices may be coupled or uncoupled. When the mass matrix is coupled, the system is said to be dynamically coupled and when the stiffness matrix is coupled, the system is known to be statically coupled.



Two Degree of Freedom Vibration System

- * The set of co-ordinates for which both the mass and stiffness matrix are uncoupled, are known as principal co-ordinates. In this case both the system equations are independent and individually they can be solved as that of a single-dof system.
- * A two-dof system differs from the single dof system in that it has two natural frequencies, and for each of the natural frequencies there corresponds a natural state of vibration with a displacement configuration known as the normal mode. Mathematical terms associated with these quantities are eigenvalues and eigenvectors.
- * Normal mode vibrations are free vibrations that depend only on the mass and stiffness of the system and how they are distributed. A normal mode oscillation is defined as one in which each mass of the system undergoes harmonic motion of same frequency and passes the equilibrium position simultaneously.
- * The study of two-dof- systems is important because one may extend the same concepts used in these cases to more than 2-dof- systems. Also in these cases one can easily obtain an analytical or closed-form solutions. But for more degrees of freedom systems numerical analysis using computer is required to find natural frequencies (eigenvalues) and mode shapes (eigenvectors).



Two Degree of Freedom Vibration System

- Figure 6.1.1 shows two masses m_1 and m_2 with three springs having spring stiffness k_1 , k_2 and k_3 free to move on the horizontal surface. Let x_1 and x_2 be the displacement of mass respectively.

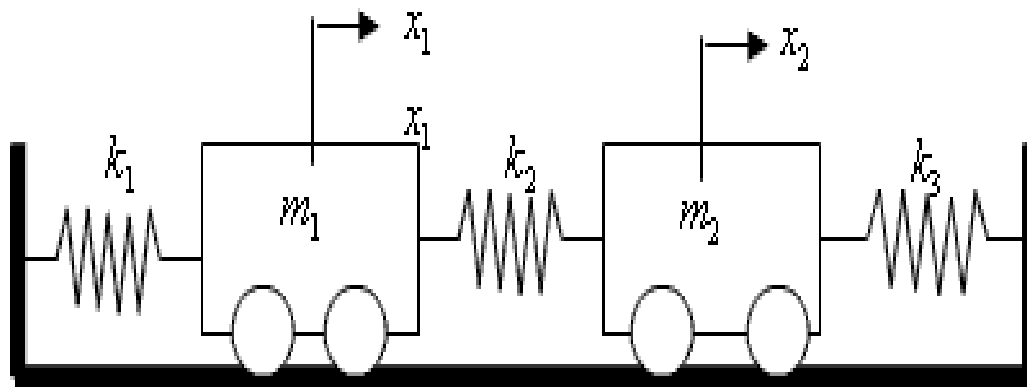


Figure 6.1.1(a)



Two Degree of Freedom Vibration System

- by using d'Alembert principle or the energy principle (Lagrange principle or Hamilton's principle)

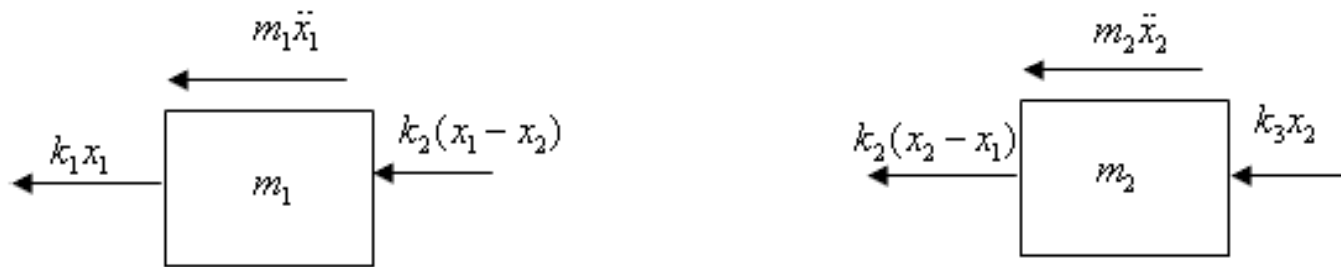


Figure 6.1.1(b): Free body diagrams

- Using d'Alembert principle for mass m_1 from the free body diagram shown in figure 6.1.1(b)

- $$m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0 \quad (6.1.1)$$

- and similarly for mass m_2

- $$m_2 \ddot{x}_2 - k_1 x_1 + (k_2 + k_3)x_2 = 0 \quad (6.1.2)$$



Two Degree of Freedom Vibration System

Important points to remember

- Inertia force acts opposite to the direction of acceleration, so in both the free body diagrams inertia forces are shown towards left.
- For spring m_2 assuming $x_1 > x_2$, The spring will pull mass m_2 towards right by $k_2 (x_2 - x_1)$ and it is stretched by $x_2 - x_1$ (towards right) it will exert a force of $k_2 (x_2 - x_1)$ towards left on mass m_2 .
- Similarly assuming $x_1 < x_2$ the spring get compressed by an amount $x_2 - x_1$ and exert tensile force of $k_2 (x_2 - x_1)$. One may note that in both cases, free body diagram remain unchanged.
- Now if one uses Lagrange principle,

- The Kinetic energy =
$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 \quad (6.1.3)$$

- Potential energy =
$$U = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_1 - x_2)^2 + \frac{1}{2} k_3 x_2^2 \quad (6.1.4)$$



Two Degree of Freedom Vibration System

- So, the Lagrangian

- $$L = T - U = \left(\frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 \right) - \left(\frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_1 - x_2)^2 + \frac{1}{2} k_3 x_2^2 \right) \quad (6.1.5)$$

- The equation of motion for this free vibration case can be found from the Lagrange principle

- $$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0 \quad (6.1.6)$$

- and noting that the generalized co-ordinate $q_1 = x_1$ and $q_2 = x_2$,
- which yields

- $$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0 \quad (6.1.7)$$

- $$m_2 \ddot{x}_2 - k_2 x_1 + (k_2 + k_3) x_2 = 0 \quad (6.1.8)$$



Two Degree of Freedom Vibration System

- Same as obtained before using d'Alembert principle.

- Now writing the equation of motion in matrix form

- $$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (6.1.9)$$

- Here it may be noted that for the present two degree-of-freedom system, the system is dynamically uncoupled but statically coupled.



Two Degree of Freedom Vibration System

- Consider a lathe machine, which can be modeled as a rigid bar with its center of mass not coinciding with its geometric center and supported by two springs,

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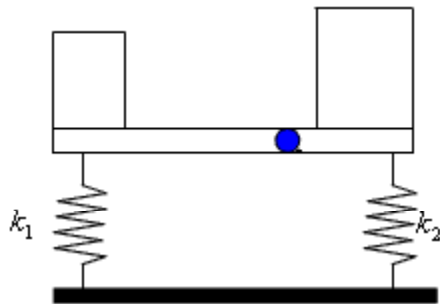


Figure 6.1.2

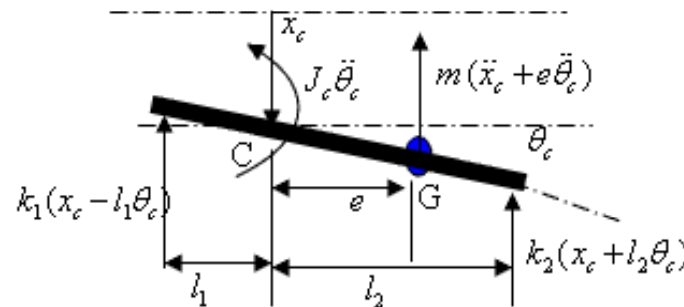


Figure 6.1.3: Free body diagram of the system

- In this example, it will be shown, how the use of different coordinate systems lead to static and or dynamic coupled or uncoupled equations of motion.



Two Degree of Freedom Vibration System

- Clearly this is a two-degree-of freedom system and one may express the co-ordinate system in many different ways. Figure 6.1.3 shows the free body diagram of the system where point G is the center of mass. Point C represents a point on the bar at which we want to define the co-ordinates of this system. This point is at a distance l_1 from the left end and l_2 from right end. Distance between points C and G is e . Assuming x_c is the linear displacement of point C and θ_c the rotation about point C, the equation of motion of this system can be obtained by using d'Alembert's principle. Now summation of all the forces, viz. the spring forces and the inertia forces must be equal to zero leads to the following equation.

- $$m\ddot{x}_c + me\ddot{\theta}_c + k_1(x_c - l_1\theta_c) + k_2(x_c + l_2\theta_c) = 0 \quad (6.1.10)$$

- Again taking moment of all the forces about point C

- $$J_G\ddot{\theta}_c + (m\ddot{x}_c + me\ddot{\theta}_c)e - k_1(x_c - l_1\theta_c)l_1 + k_2(x_c + l_2\theta_c)l_2 = 0 \quad (6.1.11)$$



Two Degree of Freedom Vibration System

- Noting $J_c = J_G + me^2$, the above two equations in matrix form can be written as

$$\begin{bmatrix} m & me \\ me & J_c \end{bmatrix} \begin{bmatrix} \ddot{x}_c \\ \ddot{\theta}_c \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & k_2 l_2 - k_1 l_1 \\ k_2 l_2 - k_1 l_1 & k_1 l_1^2 + k_2 l_2^2 \end{bmatrix} \begin{bmatrix} x_c \\ \theta_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (6.1.12)$$

- Now depending on the position of point C, few cases can be studied below.
- Case 1 : Considering $e = 0$, i.e., point C and G coincides, the equation of motion can be written as

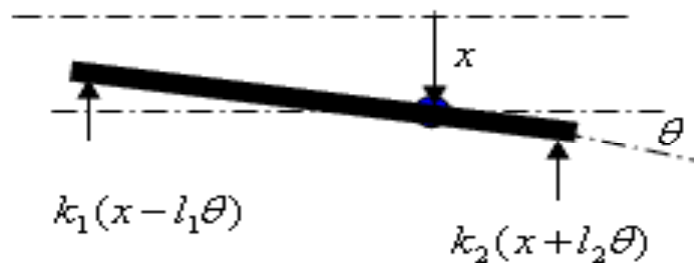


Figure 6.1.4

Two Degree of Freedom Vibration System

- Derivation of Equation of Motion and Coordinate Coupling

$$\begin{bmatrix} m & 0 \\ 0 & J_G \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & k_2 l_2 - k_1 l_1 \\ k_2 l_2 - k_1 l_1 & k_1 l_1^2 + k_2 l_2^2 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (6.1.13)$$

- So in this case the system is statically coupled and if $k_2 l_2 = k_1 l_1$, this coupling disappears, and we obtain uncoupled x and θ vibrations.

- Case 2 : If, $k_2 l_2 = k_1 l_1$, the equation of motion becomes

$$k_1 l_1 = k_2 l_2.$$

$$\begin{bmatrix} m & me \\ me & J_c \end{bmatrix} \begin{bmatrix} \ddot{x}_c \\ \ddot{\theta}_c \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & 0 \\ 0 & k_1 l_1^2 + k_2 l_2^2 \end{bmatrix} \begin{bmatrix} x_c \\ \theta_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (6.1.14)$$

- Hence in this case the system is dynamically coupled but statically uncoupled.



Two Degree of Freedom Vibration System

- Case 3: If we choose $l_1 = 0$, i.e. point C coincide with the left end,
- the equation of motion will become

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$$\begin{bmatrix} m & me \\ me & J_c \end{bmatrix} \begin{pmatrix} \ddot{x}_c \\ \ddot{\theta}_c \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & k_2 l_2 \\ k_2 l_2 & k_2 l_2^2 \end{bmatrix} \begin{pmatrix} x_c \\ \theta_c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (6.1.15)$$

- Here the system is both statically and dynamically coupled.



Two Degree of Freedom Vibration System

○ Normal Mode Vibration

- Again considering the problem of the spring-mass system in figure 6.1.1 with $m_1 = m$, $m_2 = 2m$, $k_1 = k_2 = k_3 = k$, the equation of motion (6.1.9) can be written as

- $$m\ddot{x}_1 + k(x_1 - x_2) + kx_1 = 0$$
- $$2m\ddot{x}_2 - k(x_1 - x_2) + kx_2 = 0 \quad (6.1.16)$$

- We define a normal mode oscillation as one in which each mass undergoes harmonic motion of the same frequency, passing simultaneously through the equilibrium position. For such motion, we let

- $$x_1 = A_1 e^{i\omega t}, x_2 = A_2 e^{i\omega t} \quad (6.1.17)$$

- Hence,

- $$(2k - m\omega^2)A_1 - kA_2 = 0$$
- $$-kA_1 + (2k - 2m\omega^2)A_2 = 0 \quad (6.1.18)$$



Two Degree of Freedom Vibration System

- or, in matrix form

$$\begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - 2m\omega^2 \end{bmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (6.1.19)$$

- Hence for nonzero values of A_1, A_2 ($\lambda^2 - (3\frac{k}{m})\lambda + \frac{3}{2}(\frac{k}{m})^2 = 0$ for non-trivial response)

$$\begin{vmatrix} 2k - m\omega^2 & -k \\ -k & 2k - 2m\omega^2 \end{vmatrix} = 0 \quad (6.1.20)$$

- Now substituting $\omega^2 = \lambda$, equation 6.1. yields

$$\lambda^2 - (3\frac{k}{m})\lambda + \frac{3}{2}(\frac{k}{m})^2 = 0 \quad (6.1.21)$$

- Hence, $\lambda_1 = (\frac{3}{2} - \frac{1}{2}\sqrt{3})\frac{k}{m} = 0.634\frac{k}{m}$ and $\lambda_2 = (\frac{3}{2} + \frac{1}{2}\sqrt{3})\frac{k}{m} = 2.366\frac{k}{m}$



Two Degree of Freedom Vibration System

- So, the natural frequencies of the system are $\omega_1 = \sqrt{\lambda_1} = \sqrt{0.634 \frac{k}{m}}$ and

- $\omega_2 = \sqrt{2.366 \frac{k}{m}}$

- Now from equation 6.1., it may be observed that for these frequencies, as both the equations are not independent, one can not get unique value of A_1 and A_2 . So one should find a normalized value. One may normalize the response by finding the ratio of A_1 to A_2 . From the first equation 6.1. the normalized value can be given by

- $$\frac{A_1}{A_2} = \frac{k}{2k - m\omega^2} = \frac{k}{2k - m\lambda} \quad (6.1.22)$$

- and from the second equation of 6.1., the normalized value can be given by

- $$\frac{A_1}{A_2} = \frac{2k - 2m\omega^2}{k} = \frac{2k - 2m\lambda}{k} \quad (6.1.23)$$



Two Degree of Freedom Vibration System

- Now, substituting $\omega_1^2 = \lambda_1 = 0.634 \frac{k}{m}$ in equation 6.1.22 and 6.1.23 yields the same values, as both these equations are linearly dependent. Here,

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}_{\lambda=\lambda_1} = \frac{0.732}{1} \quad (6.1.24)$$

- and similarly for $\omega_2^2 = \lambda_2 = 2.366 \frac{k}{m}$

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}_{\lambda=\lambda_2} = \frac{-2.73}{1} \quad (6.1.25)$$



Two Degree of Freedom Vibration System

It may be noted

- Equation (6.1.19) gives only the ratio of the amplitudes and not their absolute values, which are arbitrary.
- If one of the amplitudes is chosen to be 1 or any number, we say that amplitudes ratio is normalized to that number.
- The normalized amplitude ratios are called the normal modes and designated by $\phi_n(x)$.
- From equation 6.1.24 and 6.1.25, the two normal modes of this problem are:

- $$\phi_1(x) = \begin{Bmatrix} 0.731 \\ 1.00 \end{Bmatrix} \quad \phi_2(x) = \begin{Bmatrix} -2.73 \\ 1.00 \end{Bmatrix}$$

- In the 1st normal mode, the two masses move in the same direction and are said to be in phase and in the 2nd mode the two masses move in the opposite direction and are said to be out of phase. Also in the first mode when the second mass moves unit distance, the first mass moves 0.731 units in the same direction and in the second mode, when the second mass moves unit distance; the first mass moves 2.73 units in opposite direction.



Two Degree of Freedom Vibration System

- When the system is disturbed from its initial position, the resulting free-vibration of the system will be a combination of the different normal modes. The participation of different modes will depend on the initial conditions of displacements and velocities. So for a system the free vibration can be given by

- $$x = \phi_1 A \sin(\omega_1 t + \psi_1) + \phi_2 B \sin(\omega_2 t + \psi_2)$$

- Here A and B are part of participation of first and second modes respectively in the resulting free vibration and ψ_1 and ψ_2 are the

phase difference. They depend on the initial conditions. This is explained with the help of the following example.

- **Example :**
- Let us consider the same spring-mass problem (figure 6.2.1) for which the natural frequencies and normal modes are determined. We have to determine the resulting free vibration when the system is given an initial displacement



Two Degree of Freedom Vibration System

- Free vibration using normal modes

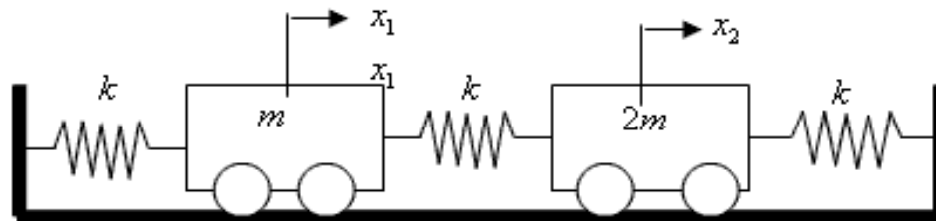


Figure 6.2.1

- Solution : Any free vibration can be considered to be the superposition of its normal modes. For each of these modes the time solution can be expressed as:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0.731 \\ 1 \end{Bmatrix} \sin \omega_1 t$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_2 = \begin{Bmatrix} -2.731 \\ 1.00 \end{Bmatrix} \sin \omega_2 t$$



Two Degree of Freedom Vibration System

- The general solution for the free vibration can then be written as:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = A \begin{Bmatrix} 0.731 \\ 1.00 \end{Bmatrix} \sin(\omega_1 t + \psi_1) + B \begin{Bmatrix} -2.73 \\ 1 \end{Bmatrix} \sin(\omega_2 t + \psi_2).$$

- where A and B allow different amounts of each mode and ψ_1
- ψ_2 and allows the two modes different phases or starting values.

- Substituting:

$$\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} 5 \\ 1 \end{Bmatrix} = A \begin{Bmatrix} 0.731 \\ 1 \end{Bmatrix} \sin \psi_1 + B \begin{Bmatrix} -2.731 \\ 1 \end{Bmatrix} \sin \psi_2.$$

$$\begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \omega_1 A \begin{Bmatrix} 0.731 \\ 1 \end{Bmatrix} \cos \psi_1 + \omega_2 B \begin{Bmatrix} -2.731 \\ 1 \end{Bmatrix} \cos \psi_2.$$

$$\cos \psi_1 = \cos \psi_2 = 0 \Rightarrow \psi_1 = \psi_2 = 90^\circ$$



Two Degree of Freedom Vibration System

- Substituting in 1st set:

$$\begin{Bmatrix} 5 \\ 1 \end{Bmatrix} = A \begin{Bmatrix} 0.731 \\ 1 \end{Bmatrix} + B \begin{Bmatrix} -2.731 \\ 1 \end{Bmatrix}.$$

$$\left. \begin{array}{l} 0.731A - 2.731B = 5 \\ A + B = 1 \end{array} \right\} \begin{array}{l} A = 2.233 \\ B = -1.233 \end{array}$$

-
-
- Hence the resulting free vibration is

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 2.233 \begin{Bmatrix} 0.731 \\ 1.00 \end{Bmatrix} \cos \omega_1 t - 1.233 \begin{Bmatrix} -2.731 \\ 1.000 \end{Bmatrix} \cos \omega_2 t.$$



Two Degree of Freedom Vibration System

- Normal modes from eigenvalues
- The equation of motion for a two-degree-of freedom system can be written in matrix form as

$$M \ddot{x} + K x = 0 \quad (6.2.)$$

- where M and K are the mass and stiffness matrix respectively; x is the vector of generalized co-ordinates. Now pre-multiplying in both side of equation 6.2. one may get

$$I \ddot{x} + M^{-1}K x = 0$$

or, $I \ddot{x} + A x = 0$

- where $A = M^{-1}K$ is known as the dynamic matrix. Now to find the normal modes,

- $x_1 = X_1 e^{i\omega t}$, $x_2 = X_2 e^{i\omega t}$, the above equation will reduce to

- $[A - \lambda I] X = 0$ where $X = \{x_1 \ x_2\}^T$ and $\lambda = \omega^2$

From equation 6.2. it is apparent that the free vibration problem in this case is reduced to that of finding the eigenvalues and eigenvectors of the matrix A



Damped-free vibration of two-dof systems

- Consider a two degrees of freedom system with damping as shown in figure 6.2.7

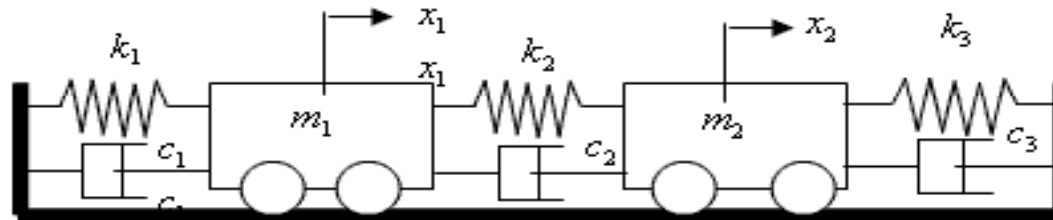


Figure 6.2.7

- Now the equation of motion of this system can be given by

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (6.2.a)$$



Damped-free vibration of two-dof systems

- As in the previous case, here also the solution of the above equations can be written as

$$x_1 = A_1 e^{st} \quad \text{and} \quad x_2 = A_2 e^{st} \quad (6.2.b)$$

- where A_1 , A_2 and s are constants. Substituting 6.2.b in 6.2.a, one may write

$$\begin{bmatrix} m_1 s^2 + (c_1 + c_2)s + k_1 + k_2 & -c_2 s - k_2 \\ -c_2 s - k_2 & m_2 s^2 + (c_2 + c_3)s + k_2 + k_3 \end{bmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Now for a nontrivial response i.e., for non-zero values of A_1 and A_2 , the determinant of their coefficient matrix must vanish. Hence

$$\begin{vmatrix} m_1 s^2 + (c_1 + c_2)s + k_1 + k_2 & -c_2 s - k_2 \\ -c_2 s - k_2 & m_2 s^2 + (c_2 + c_3)s + k_2 + k_3 \end{vmatrix} = 0$$



Damped-free vibration of two-dof systems

- or $(m_1 s^2 + (c_1 + c_2)s + k_1 + k_2)(m_2 s^2 + (c_2 + c_3)s + k_2 + k_3) + (c_2 s + k_2)^2 = 0$
- which is a fourth order equation in s and is known as the characteristic equation of the system. This equation is to be solved to get four roots. The general solution of the system can be given by

$$x_1 = A_{11}e^{s_1 t} + A_{12}e^{s_2 t} + A_{13}e^{s_3 t} + A_{14}e^{s_4 t}$$

$$x_2 = A_{21}e^{s_1 t} + A_{22}e^{s_2 t} + A_{23}e^{s_3 t} + A_{24}e^{s_4 t}$$

- Here $A_{1i}, i = 1, 2, 3, 4$ are four arbitrary constants to be determined from the initial conditions and the coefficients. $A_{2i}, i = 1, 2, 3, 4$ are related to A_{1i} and can be determined as

$$A_{2i}$$

$$\frac{A_{1i}}{A_{2i}} = \frac{c_2 s_i + k_2}{m_1 s_i^2 + (c_1 + c_2)s_i + k_1 + k_2}$$

- For a physical system with A_{2i} die out with time. For a stable system, all the four roots must be either real negative numbers or complex number with negative real parts. It may be recalled that, if the roots contain complex conjugate numbers, the motion will be oscillatory.



Damped-free vibration of two-dof systems(SEMI-DEFINITE SYSTEMS)

The systems with have one of their natural frequencies equal to zero are known as semi-definite or degenerate systems. One can show that the following two systems are degenerate systems.

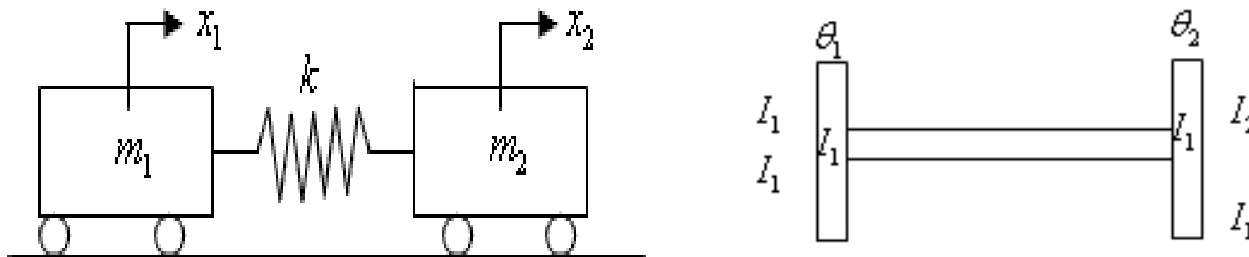


Figure 6.2.8

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- From figure 6.2.8, the equation of motion of the system is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



Damped-free vibration of two-dof systems(SEMI-DEFINITE SYSTEMS)

- Assuming the solution $x_1 = A_1 e^{i\omega t}$ and $x_2 = A_2 e^{i\omega t}$

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$$\begin{bmatrix} k - m_1 \omega^2 & -k \\ -k & k - m_2 \omega^2 \end{bmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- So for non-zero values of A_1, A_2

-

$$\begin{vmatrix} k - m_1 \omega^2 & -k \\ -k & k - m_2 \omega^2 \end{vmatrix} = 0$$

- or $(k - m_1 \omega^2)(k - m_2 \omega^2) - k^2 = 0$

- or $k^2 - k(m_1 + m_2)\omega^2 + m_1 m_2 \omega^4 - k^2 = 0$

- or $\omega^2 (m_1 m_2 \omega^2 - k(m_1 + m_2)) = 0$



Damped-free vibration of two-dof systems(SEMI-DEFINITE SYSTEMS)

- $\Rightarrow \omega_1 = 0$: and $\omega_2 = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}}$
- Hence, the system is a semi-definite or degenerate system. Corresponding to the first mode frequency, i.e $\omega_1 = 0, A_1 = A_2$ the system will have a
- rigid-body motion. For the second mode frequency

$$\frac{A_1}{A_2} = \frac{k}{k - m_1 \omega^2} = \frac{k m_1 m_2}{k m_1 m_2 - m_1 k (m_1 + m_2)} = \frac{k m_1 m_2}{-m_1 k m_1} = -\frac{m_2}{m_1}$$

- amplitude ratio is inversely proportional to the mass ratio the system. Similarly one may show for the two-rotor system,

$$\frac{\theta_1}{\theta_2} = -\frac{I_2}{I_1}$$

- the ratio of angle of rotation inversely proportional to the moment of inertia of the rotors.



Two Degree of Freedom Forced Harmonic Vibration System

- Consider a system excited by a harmonic force $F_1 \sin \omega t$ expressed by the matrix equation

-
- $$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix} \sin \omega t \quad (6.3.1)$$

- Since the system is undamped, the solution can be assumed as

-
- $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sin \omega t \quad (6.3.2)$$
-



Two Degree of Freedom Forced Harmonic Vibration System

○ Substituting equation 6.3.2 in equation 6.3.1, one obtains

○ Hence

$$\begin{bmatrix} k_{11} - m_{11}\omega^2 & k_{12} - m_{12}\omega^2 \\ k_{21} - m_{21}\omega^2 & k_{22} - m_{22}\omega^2 \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sin \omega t = \begin{pmatrix} F_1 \\ 0 \end{pmatrix} \sin \omega t$$

○ or

$$\begin{bmatrix} k_{11} - m_{11}\omega^2 & k_{12} - m_{12}\omega^2 \\ k_{21} - m_{21}\omega^2 & k_{22} - m_{22}\omega^2 \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{bmatrix} k_{11} - m_{11}\omega^2 & k_{12} - m_{12}\omega^2 \\ k_{21} - m_{21}\omega^2 & k_{22} - m_{22}\omega^2 \end{bmatrix}^{-1} \begin{pmatrix} F \\ 0 \end{pmatrix}$$

$$= \frac{\begin{bmatrix} k_{22} - m_{22}\omega^2 & -k_{12} + m_{12}\omega^2 \\ -k_{21} + m_{21}\omega^2 & k_{11} - m_{11}\omega^2 \end{bmatrix} \begin{pmatrix} F \\ 0 \end{pmatrix}}{\begin{vmatrix} k_{11} - m_{11}\omega^2 & k_{12} - m_{12}\omega^2 \\ k_{21} - m_{21}\omega^2 & k_{22} - m_{22}\omega^2 \end{vmatrix}}$$



Two Degree of Freedom Forced Harmonic Vibration System

- Hence

$$X_1 = \frac{(k_{22} - m_{22}\omega^2) F}{|Z(\omega)|}$$

-

- where

-

$$[Z(\omega)] = \begin{bmatrix} k_{11} - m_{11}\omega^2 & k_{12} - m_{12}\omega^2 \\ k_{21} - m_{21}\omega^2 & k_{22} - m_{22}\omega^2 \end{bmatrix}$$

$$X_2 = \frac{(k_{21} - m_{21}\omega^2) F}{|Z(\omega)|}$$



Two Degree of Freedom Forced Harmonic Vibration System

- Two vectors x_1 and x_2 are normal if $x_1^T x_1 = 1$ and $x_2^T x_2 = 1$
- Two vectors x_1 and x_2 are orthogonal if $x_1^T x_2 = 0$
- If x_1 and x_2 normal and orthogonal, they are called orthonormal, in that case

$$x_i^T x_j = \delta_{ij} \quad i = 1, 2, \quad j = 1, 2$$

- where δ_{ij} is the Kronecker delta, defined by

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$



Two Degree of Freedom Forced Harmonic Vibration System

- Example 6.3.1

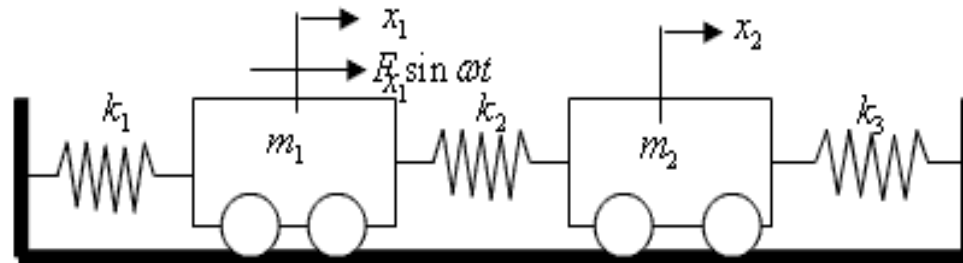


Figure 6.3.1

- Consider the system shown in figure 6.3.1 where the mass m_1 is subjected to a force $F \sin \omega t$. Find the response of the system
- when $m_1 = m_2$.
- and $k_1 = k_2 = k_3$.



Two Degree of Freedom Forced Harmonic Vibration System

○ Solution :

- The equation of motion of this system can be written as

- $$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F \sin \omega t \\ 0 \end{pmatrix}$$

- $$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F \sin \omega t \\ 0 \end{pmatrix}$$

- So assuming the solution
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sin \omega t$$
- and proceeding as explained before



Two Degree of Freedom Forced Harmonic Vibration System

$$[Z(\omega)] = \begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{bmatrix}$$

$$|Z(\omega)| = (2k - m\omega^2)^2 - k^2 = m^2\omega^4 - 4mk\omega^2 + 3k^2 = m^2 \left(\omega^4 - 4\frac{k}{m}\omega^2 + 3\frac{k^2}{m^2} \right)$$

or

$$|Z(\omega)| = m^2 \left(\omega^2 - \frac{k}{m} \right) \left(\omega^2 - 3\frac{k}{m} \right) = m^2 (\omega^2 - \omega_1^2) (\omega^2 - \omega_2^2)$$

where, $\omega_1^2 = \frac{k}{m}$ and $\omega_2^2 = 3\frac{k}{m}$ are normal mode frequencies of this system.

Hence,

○

$$X_1 = \frac{(2k - m\omega^2)F}{m^2(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)} \quad X_2 = \frac{kF}{m^2(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)}$$



Two Degree of Freedom Forced Harmonic Vibration System

- So it may be observed that the system will have maximum vibration when $\omega = \omega_1$ or $\omega = \omega_2$. So it may be observed that $X_1 = 0$ when

$$\omega^2 = 2 \frac{k}{m}$$



TUNED VIBRATION ABSORBER

- Consider a vibrating system of mass m_1 , stiffness k_1 , subjected to a force $F \sin \omega t$. As studied in case of forced vibration of single-degree of freedom system, the system will have a steady state response given by

- $$x = \frac{F \sin \omega t}{m(\omega_1^2 - \omega^2)}, \text{ where } \omega_1 = \sqrt{k_1 / m_1}$$

- which will be maximum when $\omega = \omega_1$.
- Now to absorb this vibration, one may add a
- secondary spring and mass system as shown
- in figure 13.

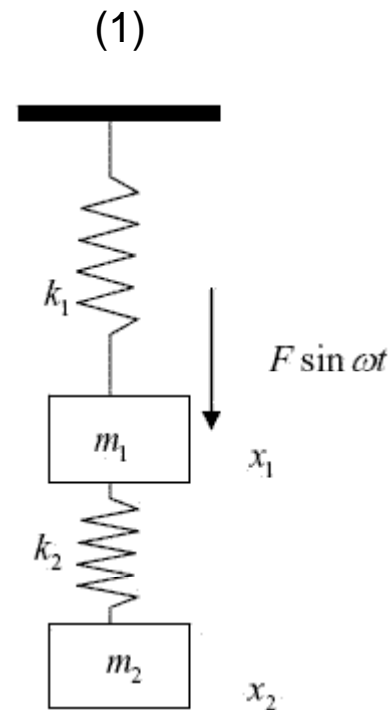


Fig.13



TUNED VIBRATION ABSORBER

- The equation of motion for this system can be given by

-
- $$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F \sin \omega t \\ 0 \end{pmatrix} \quad (2)$$

- As we know for steady state vibration, the system will vibrate with a frequency of the external excitation; we can assume the solution to be

- $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sin \omega t \quad (3)$$

- Substituting Equation (3) in equation (2) one may write

- $$\begin{pmatrix} k_1+k_2-m_1\omega^2 & -k_2 \\ -k_2 & k_2-m_2\omega^2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sin \omega t = \begin{pmatrix} F \\ 0 \end{pmatrix} \sin \omega t \quad (4)$$



VIBRATION ABSORBER

○ Or,
$$\begin{pmatrix} k_1 + k_2 - m_1 \omega^2 & -k_2 \\ -k_2 & k_2 - m_2 \omega^2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix} \quad (5)$$

○ Using Cramer's rule one may write

○
$$X_1 = \frac{\begin{vmatrix} F & -k_2 \\ 0 & k_2 - m_2 \omega^2 \end{vmatrix}}{\begin{vmatrix} k_1 + k_2 - m_1 \omega^2 & -k_2 \\ -k_2 & k_2 - m_2 \omega^2 \end{vmatrix}} = \frac{(k_2 - m_2 \omega^2) F}{|Z(\omega)|} \quad (6)$$

○
$$X_2 = \frac{\begin{vmatrix} k_1 + k_2 - m_1 \omega^2 & F \\ -k_2 & 0 \end{vmatrix}}{\begin{vmatrix} k_1 + k_2 - m_1 \omega^2 & -k_2 \\ -k_2 & k_2 - m_2 \omega^2 \end{vmatrix}} = \frac{-k_2 F}{|Z(\omega)|} \quad (7)$$



VIBRATION ABSORBER

- where

$$Z(\omega) = \begin{pmatrix} k_1 + k_2 - m_1 \omega^2 & -k_2 \\ -k_2 & k_2 - m_2 \omega^2 \end{pmatrix} \quad (9)$$

- Now

-

$$|Z(\omega)| = \begin{vmatrix} k_1 + k_2 - m_1 \omega^2 & -k_2 \\ -k_2 & k_2 - m_2 \omega^2 \end{vmatrix} = k_1 k_2 - m_1 k_2 \omega^2 - k_2 m_2 \omega^2 + m_1 m_2 \omega^4 \quad (10)$$

- Here λ_1 and λ_2 are the roots of the characteristic equation. One may note that these roots are the normal mode frequency for this two-degrees of freedom system. These free-vibration frequencies can be given by

$$\lambda_{1,2} = 0.5 \left\{ \left(\frac{k_1}{m_1} + \frac{k_2}{m_2} + \frac{k_2}{m_1} \right) \pm \sqrt{\left(\frac{k_1}{m_1} + \frac{k_2}{m_2} + \frac{k_2}{m_1} \right)^2 - 4 \frac{k_1 k_2}{m_1 m_2}} \right\} \quad (11)$$



VIBRATION ABSORBER

- From equation (6), it is clear that,

$$X_1 = 0, \text{ when } (k_1 - m_1 \omega^2) = 0, \text{ or, when } \omega^2 = \frac{k_1}{m_1}.$$

- Hence, if a system called the primary system with a stiffness k_1 and mass m_1 is subjected to an exciting force or base motion to vibrate, it is possible to completely eliminate the vibration of the primary system by suitably designing an attached spring-mass system (secondary system) with stiffness k_2 and mass m_2 such that the natural frequency of the secondary system coincide with the exciting frequency.

- .

$$\omega = \sqrt{\frac{k_2}{m_2}}$$

(12)

- This is the principle of dynamic vibration absorber.
- From equation (1) it may be noted that the primary system will have resonance when the natural frequency of the primary system coincide with that of the excitation frequency.



VIBRATION ABSORBER

- Hence to reduce the vibration at resonance of the primary system one should design the secondary system such that the natural frequency of both the components coincides.

$$\omega_n^2 = \frac{k_2}{m_2} = \frac{k_1}{m_1}$$

- For this condition

$$\begin{aligned} X_1 &= \frac{(k_2 - m_2 \omega^2) F}{(k_1 + k_2 - m_1 \omega^2)(k_2 - m_2 \omega^2) - k_2^2} \\ &= \frac{(k_2 - m_2 \omega^2) F}{m_1 m_2} \\ &= \frac{\left(\frac{k_1}{m_1} + \frac{k_2}{m_1} - \omega^2 \right) \left(\frac{k_2}{m_2} - \omega^2 \right) - \frac{k_2}{m_2} \frac{k_2}{m_1}}{m_1 m_2} \end{aligned}$$



VIBRATION ABSORBER

- Substituting $\mu = m_2 / m_1$ and $r = \omega / \omega_n$, the above equation reduces to

$$X_1 = \frac{(1 - r^2)(F / k_1)}{(1 + \mu - r^2)(1 - r^2) - \mu}$$

- or,

$$\frac{k_1 X_1}{F} = \frac{(1 - r^2)}{(1 + \mu - r^2)(1 - r^2) - \mu}$$



VIBRATION ABSORBER

○ For, $\omega^2 = k_2 / m_2$,

○

$$|Z(\omega)| = k_1 k_2 - m_1 k_2 \frac{k_2}{m_2} - k_1 m_2 \frac{k_2}{m_2} - k_2 m_2 \frac{k_2}{m_2} + m_1 m_2 \frac{k_2}{m_2} \frac{k_2}{m_2}$$

○

$$= k_1 k_2 - m_1 \frac{k_2^2}{m_2} - k_1 k_2 - k_2^2 + m_1 \frac{k_2^2}{m_2} = -k_2^2$$

○

(61)

$$X_1 = 0$$

○ and

○

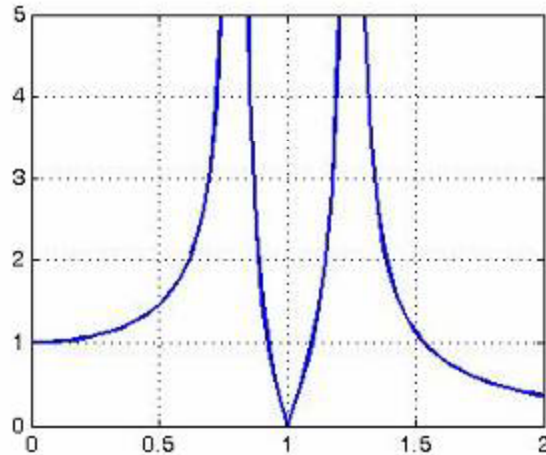
$$X_2 = \frac{-k_2 F}{-k_2^2} = \frac{F}{k_2}$$

○

(62)



VIBRATION ABSORBER



To keep the displacement of secondary mass small, the stiffness of the secondary spring should be very large.

To have this the secondary mass should also be large which is not desirable from practical point of view.

Hence a compromise is usually made between the amplitude and the mass ratio. The mass ratio is usually kept between 0.05 and 0.25.

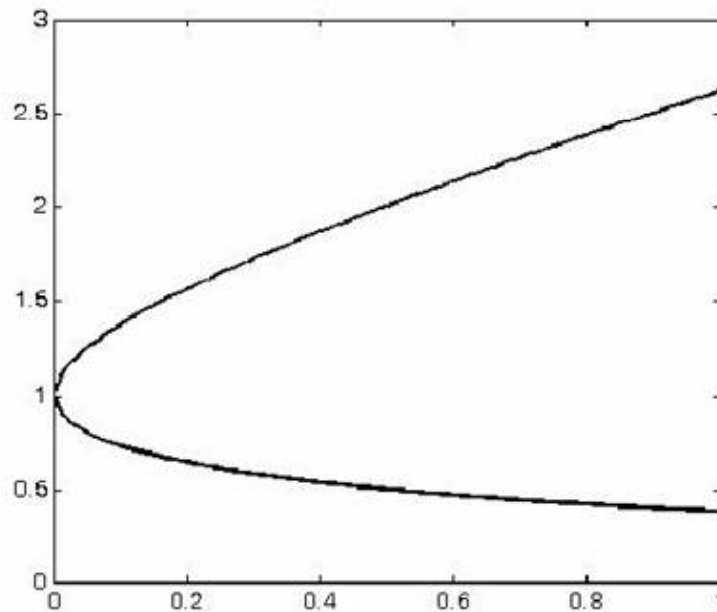


VIBRATION ABSORBER

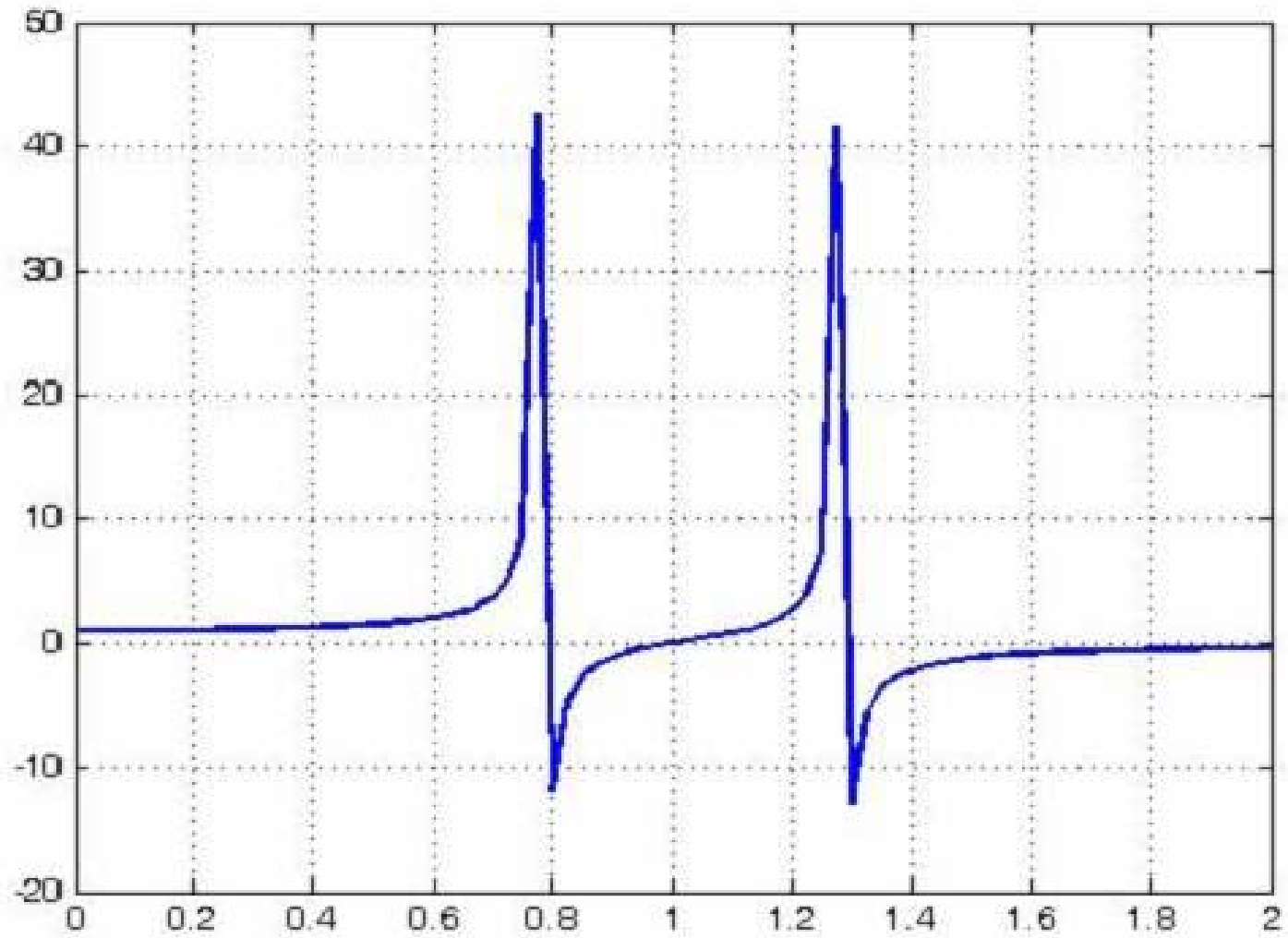
- Resonant frequency of the vibration absorber

- $$\frac{\omega^4}{\omega_2^4} - (2 + \mu) \left(\frac{\omega}{\omega_2} \right)^2 + 1 = 0$$

- $$\text{or. } \left(\frac{\omega}{\omega_2} \right)^2 = \left((2 + \mu) \pm \sqrt{(2 + \mu)^2 - 4} \right) / 2$$
- $$= \left(1 + \frac{\mu}{2} \right) \pm \sqrt{\mu + \frac{\mu^2}{4}}$$



VIBRATION ABSORBER

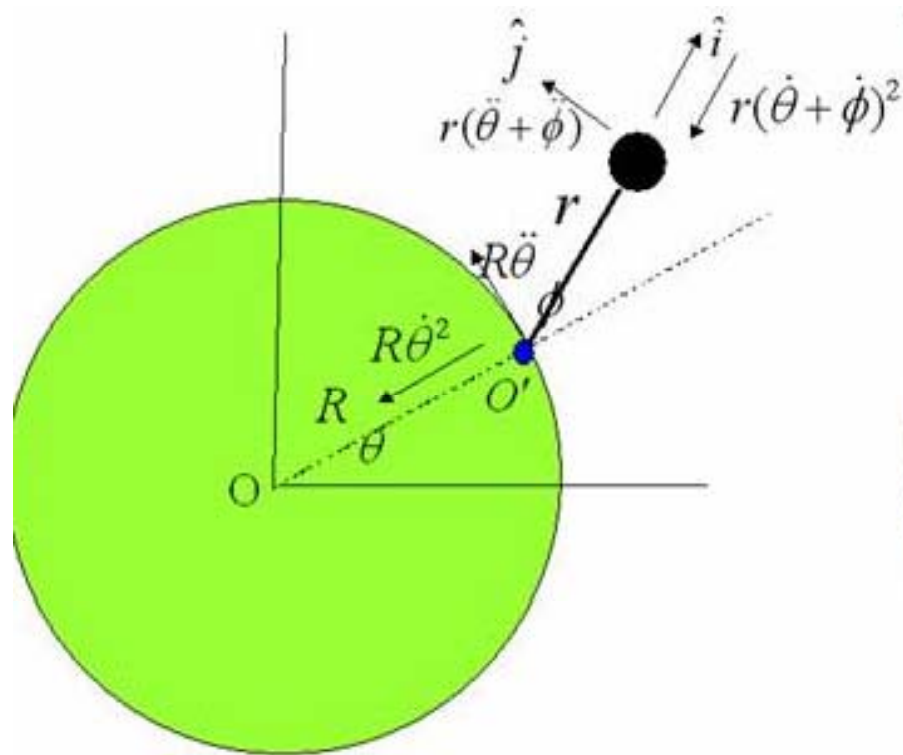


CENTRIFUGAL PENDULUM VIBRATION ABSORBER

- The centrifugal pendulum vibration absorber was devised and patented in France about 1935 and at the same time it was independently conceived and put into practice by E. S. Taylor. Its purpose was to overcome serious torsional vibration problem inherent in geared radial aircraft-engine propeller system.
- Later it was modified and incorporated into automobile IC engines in order to reduce the torsional vibrations of the crankshaft. This was done by integrating the absorber mass with crankshaft counter balance mass.
- The tuned vibration absorber is only effective when the frequency of external excitation equals to the natural frequency of the secondary spring and mass system. But in many cases, for example in case of an automobile engine, the exciting torques are proportional to the rotational speed n' which may vary over a wide range.
- For the absorber to be effective, its natural frequency must also be proportional to the speed. The characteristics of the centrifugal pendulum are ideally suited for this purpose.
- Placing the coordinates through point O' , parallel and normal to r , the line r rotates with angular velocity ($\dot{\theta} + \dot{\phi}$)



CENTRIFUGAL PENDULUM VIBRATION ABSORBER



CENTRIFUGAL PENDULUM VIBRATION ABSORBER

- acceleration of mass

- $$a_m = \left[-R\ddot{\theta}^2 \cos \phi + R\ddot{\theta} \sin \phi - r(\dot{\theta} + \dot{\phi})^2 \right] \hat{i} + \left[R\dot{\theta}^2 \sin \phi + R\ddot{\theta} \cos \phi + r(\ddot{\theta} + \ddot{\phi})^2 \right] \hat{j} \quad (63)$$

- Since the moment about O is zero,

- $$M_O = m \left[R\dot{\theta}^2 \sin \phi + R\ddot{\theta} \cos \phi + r(\ddot{\theta} + \ddot{\phi})^2 \right] r = 0 \quad (64)$$

- Assuming ϕ to be small, $\cos \phi = 1, \sin \phi = \phi$, so

- $$\ddot{\phi} + \left(\frac{R}{r} \dot{\theta}^2 \right) \phi = - \left(\frac{R+r}{r} \right) \ddot{\theta} \quad (65)$$

- If we assume the motion of the wheel to be a steady rotation plus a small sinusoidal oscillation of frequency ω , one may write

- $$\theta = \omega t + \theta_0 \sin \omega t \quad (66)$$



CENTRIFUGAL PENDULUM VIBRATION ABSORBER

-
- $$\dot{\theta} = \omega t + \omega \theta_0 \cos \omega t \cong \omega$$
 (67)

-
- $$\ddot{\theta} = -\theta_0 \omega^2 \sin \omega t$$
 (68)

- Substituting the above equations in equation (65) yields,

- $$\ddot{\phi} + \left(\frac{R}{r} \omega^2 \right) \phi = \left(\frac{R+r}{r} \right) \omega^2 \theta_0 \sin \omega t$$
 (69)

- Hence the natural frequency of the pendulum is

- $$\omega_n = \omega \sqrt{\frac{R}{r}}$$
 (70)

- and its steady-state solution is

- $$\phi = \frac{(R+r)/r}{-\omega^2 + (R\omega^2/r)} \omega^2 \theta_0 \sin \omega t$$
 (71)



CENTRIFUGAL PENDULUM VIBRATION ABSORBER

- It may be noted that the same pendulum in a gravity field would have a natural frequency of

$$\sqrt{\frac{g}{r}}$$

- So it may be noted that for the centrifugal pendulum the gravity field is replaced by the centrifugal field

$$R\omega^2$$

Torque exerted by the pendulum on the wheel

- With the \hat{j} component of \mathbf{a}_m equal to zero, the pendulum force is a tension along \hat{r} , given by m times the \hat{r} component of

$$\mathbf{a}_m$$

- $$T = (R \cos \phi \hat{i} + R \sin \phi \hat{j}) \times m [-R\ddot{\theta} \cos \phi + R\dot{\theta}^2 \sin \phi - r(\ddot{\theta} + \dot{\phi}^2)] \hat{i} \quad (72)$$

$$= -mR\phi [-R\omega^2 \sin \alpha \sin \phi - R\omega^2 - r\ddot{\phi} - 2r\dot{\theta}\dot{\phi}]$$

- Now assuming small angle of rotation

-
-

$$T = -m(R+r)\omega^2 R\phi \quad (73)$$



CENTRIFUGAL PENDULUM VIBRATION ABSORBER

- Now substituting the (73) in (72),

$$T = \frac{-mR(R+r)^2 n^2 / r}{(Rn^2 / r) - \omega^2} \omega^2 \theta_0 \sin \omega t \quad (74)$$

$$= \left[\frac{m(R+r)^2}{1 - r\omega^2 / Rn^2} \right] \ddot{\theta} = J_{\theta} \ddot{\theta}$$

- Hence the effective inertia can be written as

$$J_{\theta} = - \left[\frac{m(R+r)^2}{1 - r\omega^2 / Rn^2} \right] = - \frac{m(R+r)^2}{1 - (\omega/\omega_n)^2} \quad (75)$$

- which can be at its natural frequency. This possesses some difficulties in the design of the pendulum. For example to suppress a disturbing torque of frequency equal to four times the natural speed n , the pendulum must meet the requirement

$$\omega^2 = (4n)^2 = n^2 R / r$$



CENTRIFUGAL PENDULUM VIBRATION ABSORBER

- Hence, as the length of the pendulum $r = R/16$ becomes very small it will be difficult to design it. To avoid this one may go for Chilton bifilar design.

- **DESIGN CONSIDERATION**

- To suppress a disturbing torque of frequency equal to 5 times the torsional frequency n , the pendulum must meet the requirement

- $$\omega^2 = (5n)^2$$

- $$\text{or, } 25n^2 = \frac{Rn^2}{r} \Rightarrow \frac{R}{r} = 25$$

