ME – VII SEM

Course Name-
Mechanical Vibrations

U-II, FREE AND DAMPED VIBRATIONS

FREE VIBRATIONS

- Objective of the present section will be to write the equation of motion of a system and evaluate its natural frequency, which is mainly a function of *mass, stiffness,* and *damping* of the system from its general solution.
- In many pratical situation, damping has little influence on the natural frequency and may be neglected in its calculation.
- In absence of damping, the system can be considered as conservative and principle of conservation of energy offers another approach to the calculation of the natural frequency.
- The effect of damping is mainly evident in diminishing of the vibration amplitude at or near the resonance

FREE AND DAMPED VIBRATIONS

Vibration Model :

- The basic vibration model consists of a mass, spring (stiffness) and damper (damping) as shown in Figure 2.1.

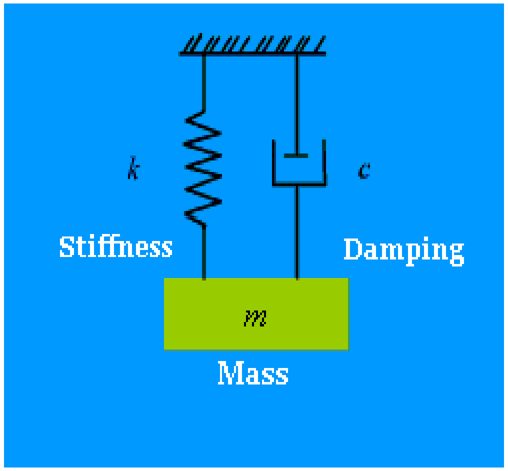

Figure 2.1: Spring-damper vibration model.

The inertia force model is $F_i = m\ddot{x}$ (2.1)

- where m is the mass in kg, $\ddot{x}$ is the acceleration in m/sec^2 and F_i is the inertia force in N.
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FREE AND DAMPED VIBRATIONS

- The linear stiffness force model is

  $F_s = kx$ (2.2)

  where k is the stiffness (N/m), x is the displacement and F_s is the spring force.
- The damping force model for the viscous damping is

  $F_d = c\dot{x}$ (2.3)
- where c is the damping coefficient in N/m/sec, $\dot{x}$ is the velocity in m/sec and F_d is the damping force.
- **Undamped Free Vibration**
- A spring mass system as shown in Figure 2.2 is considered. For simplicity at present the **damping is not considered.**
- The direction of x in the downward direction is positive.
- Also velocity, $\dot{x}$,acceleration, $\ddot{x}$ and force, F, are positive in the downward direction as shown in Figure 2.2

EQUATION OF MOTION (EOM) FOR FREE VIBRATIONS

(2.4)

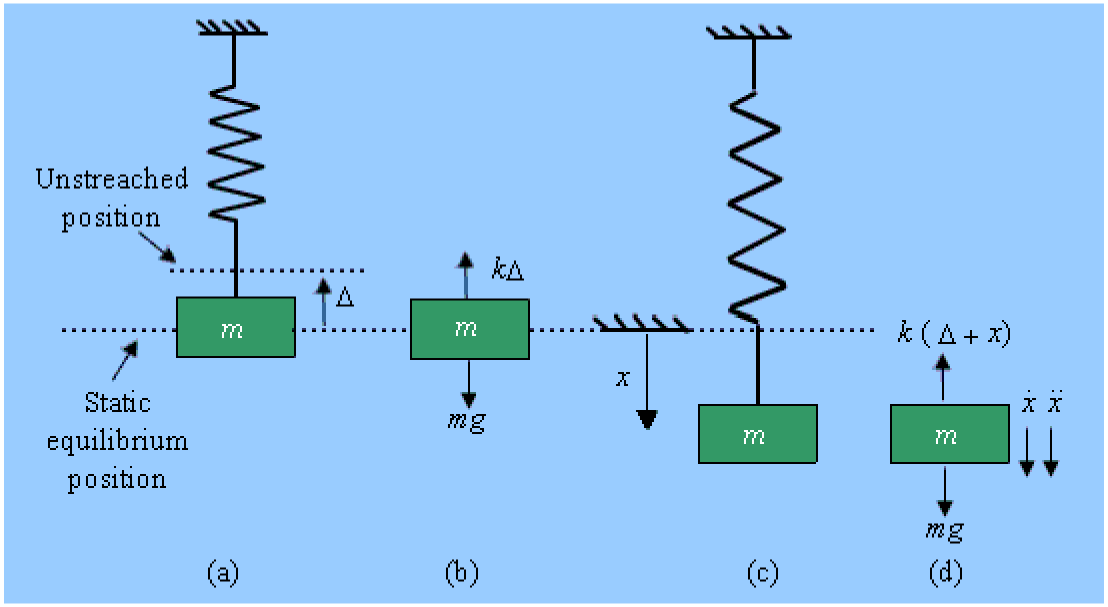


Figure 2.2

From Figure 2.2(d) on application of Newton's second law, we have

$$m\ddot{x} = \sum F \qquad \text{or} \qquad m\ddot{x} = mg - k(\Delta + x)$$

From Figure 2.2(b), $k\Delta = mg$

$$m\ddot{x} + kx = 0 \qquad \ddot{x} + \omega_n^2 x = 0 \qquad \ddot{x} = -\omega_n^2 x$$

$$\omega_n^2 = \frac{k}{m}$$

where ω_n is the natural frequency (in rads/sec).

SOLUTION OF EOM-
FREE VIBRATIONS

Solution of EOM :

- The general solution of equation (2.5) can be written as

$$x = A\sin\omega_n t + B\cos\omega_n t \qquad (2.7)$$

where *A* and *B* are two arbitrary constants, which depend upon initial conditions i.e. *x* (0) and $\dot{x}(0)$. Equation (2.5) can be differentiated to give

$$\dot{x} = A\omega_n\cos\omega_n t - B\omega_n\sin\omega_n t \qquad (2.8)$$

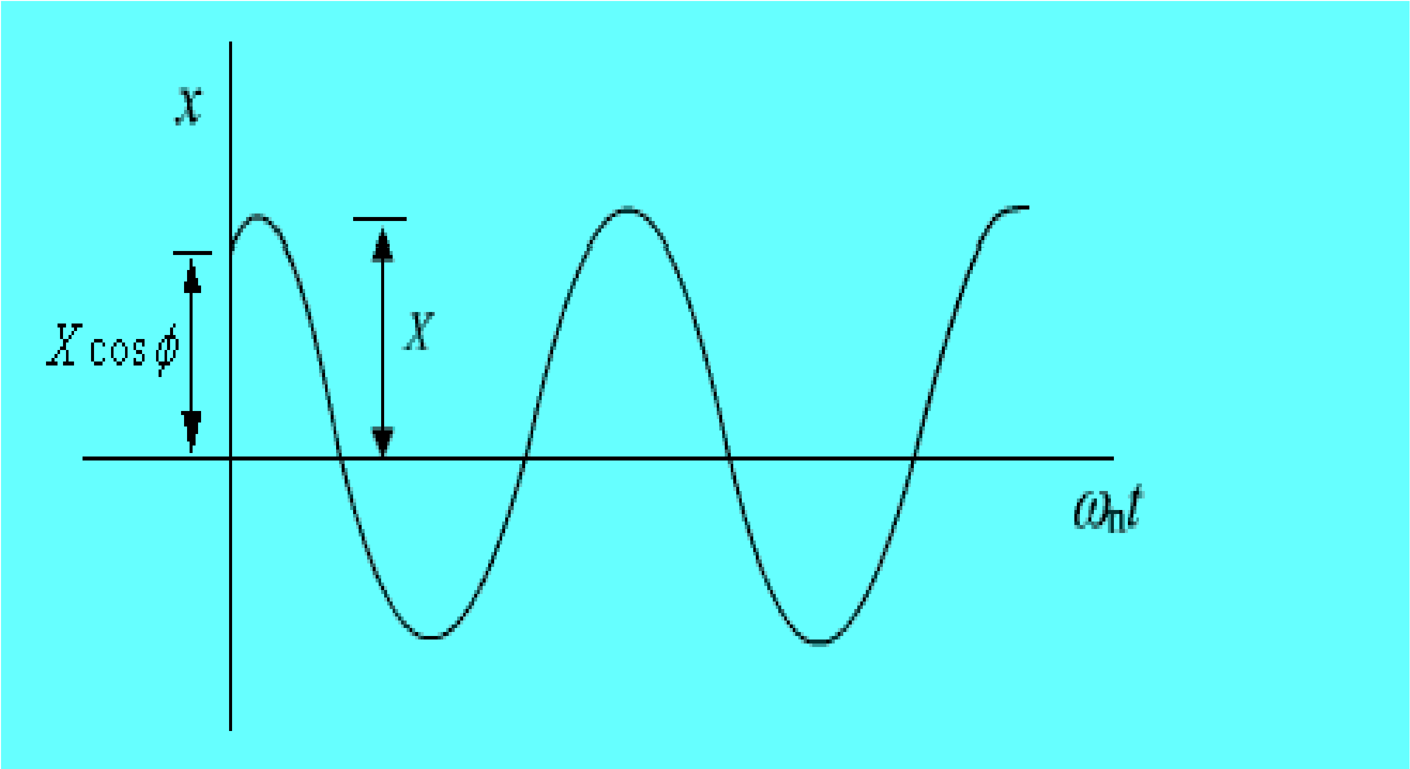
- On application of initial conditions in equation (2.7) and (2.8), we get
- *x* (0) = *B* and $\dot{x} = A\omega_n$
- or $A = \dfrac{\dot{x}(0)}{\omega_n}$ and $B = x(0)$ (2.9)

$$x = \frac{\dot{x}(0)}{\omega_n}\sin\omega_n t + x(0)\cos\omega_n t = X\cos(\omega_n t - \phi)$$

$$X = \sqrt{\{x(0)\}^2 + \left\{\frac{\dot{x}(0)}{\omega_n}\right\}^2} \qquad \tan\phi = \frac{\dot{x}(0)}{\omega_n x(0)}$$

SOLUTION OF EOM- FREE VIBRATIONS

- where X is the amplitude, ω_n is the circular frequency and ϕ is the phase. The undamped free vibration executes the simple harmonic motion as shown in Figure 2.3.



(2.13)

SOLUTION OF EOM- FREE VIBRATIONS

- Since sine & cosine functions repeat after 2 π radians (i.e. Frequency $\times$ Time period = 2 π), we have

$$\omega_n T = 2\pi \qquad T = 2\pi\sqrt{\frac{m}{k}}$$

The natural frequency (in rad /sec or Hertz) can be written as

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad k\Delta = mg \Rightarrow \frac{k}{m} = \frac{g}{\Delta}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta}}$$

Here T , f , ω_n are dependent upon mass & stiffness of the system, which are properties of the system.
Above analysis is valid for all kind of SDOF system including beam or torsional members.
For torsional vibrations the mass may be replaced by the mass moment of inertia and stiffness by stiffness of torsional spring. For stepped shaft an equivalent stiffness can be taken or for distributed mass an equivalent lumped mass can be taken.
The undamped free response can also be written as

$$x = ae^{i\omega_n t} + be^{-i\omega_n t}$$
$$= a[\cos\omega_n t + i\sin\omega_n t] + b[\cos\omega_n t - i\sin\omega_n t]$$
$$= (a+b)\cos\omega_n t + i(a-b)\sin\omega_n t$$
$$= A\cos\omega_n t + B\sin\omega_n t$$

where A & B are constants to be determined from initial conditions, which is same as equation (2.7).

Equivalent Stiffness of Springs

- Equivalent Stiffness of Series and Parallel Springs :
- For this system having springs connected in series or parallel, equation (2.13) is still valid with the equivalent stiffness as shown in Figures 2.4 and 2.5.

$$k_{eq} = \frac{1}{1/k_1 + 1/k_2}$$

Figure 2.4

$$k_{eq} = k_1 + k_2$$

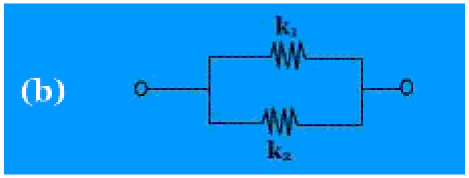
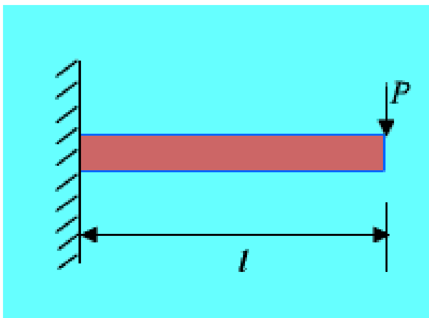


Figure 2.5

Equivalent Stiffness of a Cantilever Beam

The deflection of a cantilever beam as shown inFig. 2.6



$$\delta = \frac{Pl^3}{3EI}$$

The equivalent stiffness is given as

$$k = \frac{P}{\delta} = \frac{3EI}{l^3}$$

δ is the deflection,
E = Young's modulous,
I = mass moment of inertia,
l = length of the beam,
P is the load.

Energy Method :

- ***Energy method*** **:**
- In a conservative system (i.e. with no damping) the total energy is constant, and differential equation of motion can also be established by the principle of conservation of energy.
- For the free vibration of undamped system:
- Energy=(partly kinetic energy + partly potential energy).
- Kinetic energy T is stored in mass by virtue of its velocity.
- Potential energy U is stored in the form of strain energy in elastic deformation or work done in a force field such as gravity, magnetic field etc.

$$T + U = \text{constant} \qquad (2.15)$$

- Hence

$$\frac{d}{dt}(T + U) = 0 \qquad (2.16)$$

Energy Method

- Our interest is to find natural frequency of the system, writing equation (2.15) for two positions i.e.

$T1 + U1 = T2 + U2 = \text{constant}$ (2.17)

where, 1 & 2 represents two instants of time.

Let 1 represents a static equilibrium position (choosing this as the reference point of potential energy, here $U1=0$)

and 2 represents the position corresponding to maximum displacement of mass and at this position velocity of mass will be zero and hence $T2 = 0$. Equation (2.17) reduces to

$T1 + 0 = 0 + U2$ (2.18)

If mass is undergoing harmonic motion then $T1$ & $U2$ are maximum values.

$T\text{max} = U\text{max}$ (2.19)

Damped System

Vibration systems may encounter damping of following types:

- Internal molecular friction.
- Sliding friction
- Fluid resistance

Generally mathematical model of such damping is quite complicated and not suitable for vibration analysis.

- Simplified mathematical model (such as viscous damping or dash-pot) have been developed which leads to simplified formulation.
- A mathematical model of damping in which force is proportional to displacement i.e
- $F_d = cx$ is not possible because with cyclic motion this model will encounter an area of magnitude equal to zero.
- So dissipation of energy is not possible with this model.

VISCOUSLY DAMPED FREE VIBRATION

- Viscous damping force is expressed as,
- $$F_d = c\dot{x} \qquad (3.16)$$
- *c* is the constant of proportionality and it is called damping co-efficient.
- Figure 3.2 shows spring-damper-mass system with free body diagram.
- From free body diagram, we have

$$\sum F = m\ddot{x} \qquad (3.17)$$

$$-kx - c\dot{x} = m\ddot{x}$$

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad (3.18)$$

VISCOUSLY DAMPED FREE VIBRATION

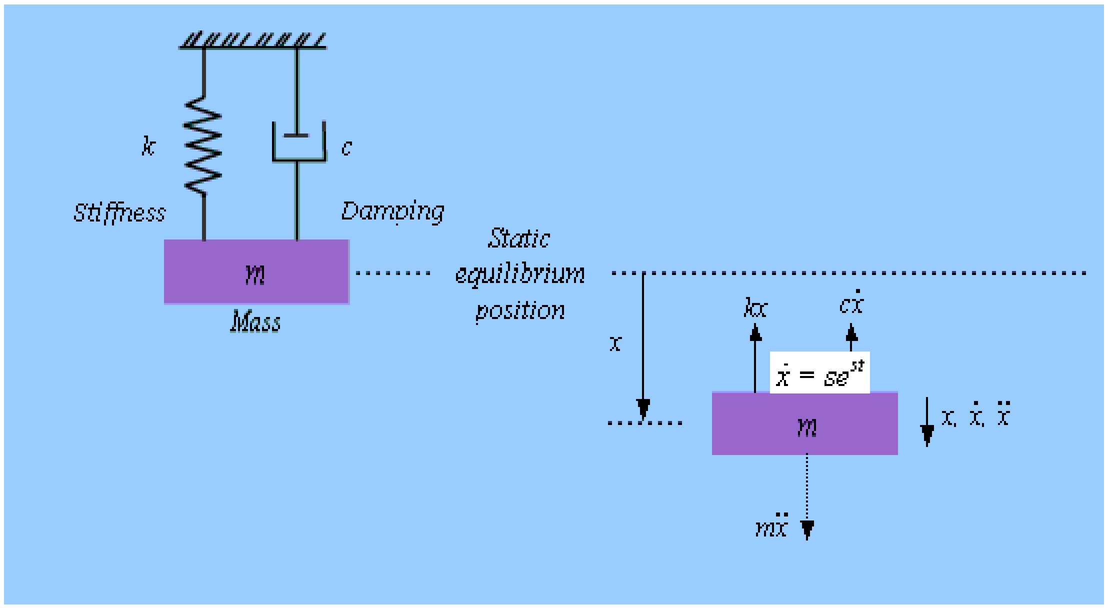


Figure 3.2: Spring-damper-mass system

Let us assume a solution of equation(3.18) of the following form

$$x = e^{st}$$

where *s* is a constant (can be a complex number) and *t* is time.

So that $\dot{x} = se^{st}$ and $\ddot{x} = s^2 e^{st}$

on substituting in equation (3.18), we get,

$$(ms^2 + cs + k)e^{st} = 0 \qquad s^2 + \frac{c}{m}s + \frac{k}{m} = 0$$

a characteristic equation (Frequency equation)

VISCOUSLY DAMPED FREE VIBRATION

$$ax^2 + bx + c = 0$$

solution of which is given as
$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Equation (3.20) has the following form

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \qquad (3.21)$$

Hence the general solution of equation (3.18) from equations (3.19) and (3.21) is given by the equation

$$x = Ae^{s_1 t} + Be^{s_2 t} \qquad (3.22)$$

Substituting equation (3.21) into equation (3.22).

$$x(t) = e^{-(c/2m)t}\left[Ae^{\sqrt{(c/2m)^2 - k/m}\,t} + Be^{-\sqrt{(c/2m)^2 - k/m}\,t}\right] \qquad (3.23)$$

VISCOUSLY DAMPED FREE VIBRATION

- The term outside the bracket in RHS is an exponentially decaying function. The term

$$\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

  can have three cases.
- (i) $\left(\frac{c}{2m}\right)^2 > \frac{k}{m}$: exponents in equation (3.23) will be real numbers.
- No oscillation is possible as shown in Figure 3.3.
- This is an overdamped system (Figure 3.3).

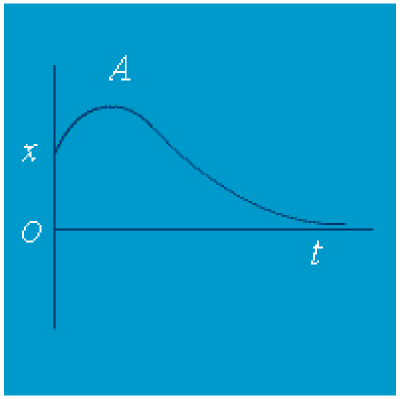


Figure 3.3: Overdamped system

VISCOUSLY DAMPED FREE VIBRATION

- ii) $\left(\frac{c}{2m}\right)^2 < \frac{k}{m}$: exponents in equation (3.23) are
- imaginary numbers : $\pm j\sqrt{\frac{k}{m}-\left(\frac{c}{2m}\right)^2}$

  we can write $e^{\pm j\sqrt{\frac{k}{m}-\left(\frac{c}{2m}\right)^2}t} = \cos\sqrt{\frac{k}{m}-\left(\frac{c}{2m}\right)^2}\,t \pm \sin\sqrt{\frac{k}{m}-\left(\frac{c}{2m}\right)^2}\,t$
- the equation (3.23) takes the following form where

$$x = e^{-\left(\frac{c}{2m}\right)t}\left[(A+B)\cos\sqrt{\frac{k}{m}-\left(\frac{c}{2m}\right)^2}\,t + j(A-B)\sqrt{\frac{k}{m}-\left(\frac{c}{2m}\right)^2}\,t\right]$$

- Let $a = (A+B) = X\cos\phi$ and $b = j(A-B) = X\sin\phi$, equation (3.23) can be written as
- $x = Xe^{\left(\frac{c}{2m}\right)t}\cos\left[\sqrt{\frac{k}{m}-\left(\frac{c}{2m}\right)^2}\,t-\phi\right]$ where $\phi = \tan^{-1}(b/a);\quad X = \sqrt{a^2+b^2}$ (3.24)

VISCOUSLY DAMPED FREE VIBRATION

- iii) Critical case between oscillatory and non-oscillatory motion
- $\left(\frac{c}{2m}\right)^2 = \frac{k}{m}$:Damping corresponding to this case is called critical damping, , ***cc***

$$c_c = 2m\sqrt{k/m} = 2m\omega_n = 2\sqrt{km} \qquad (3.25)$$

- Any damping can be expressed in terms of the critical damping by a non-dimensional number ζ alled the damping ratio.
- $\zeta = c / c_c$ (3.26)
- Response corresponding to the critical damping case is shown in Figure 3.4 for various initial conditions

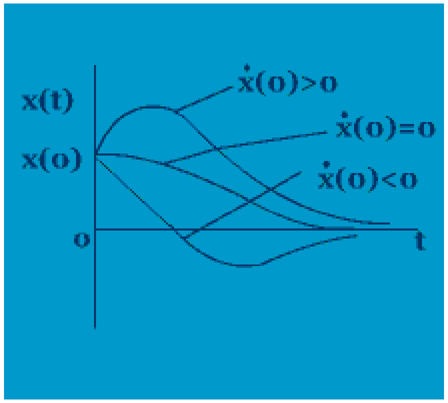


Figure 3.4: Critical damping

VISCOUSLY DAMPED FREE VIBRATION

- Equation of motion for damped system can be expressed in terms of ω_n and ζ as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \qquad (3.27)$$

- This form of equation is usef … equency and damping of system.

- The roots of characteristic equation (3.20) can be written as

$$s_{1,2} = \left[-\zeta \pm \sqrt{(\zeta^2 - 1)}\right]\omega_n \qquad (3.28)$$

- with $\frac{c}{m} = \frac{\zeta c_c}{m} = \frac{\zeta 2m\omega_n}{m} = 2\zeta\omega_n$

- Depending upon value of damping ratio we can have the following cases
- $\zeta > 1$, overdamped condition (Figure 3.3)
- $\zeta < 1$, underdamped condition (Figure 3.7)
- $\zeta = 1$, critical damping (Figure 3.4)
- $\zeta = 0$, undamped system (Figure 3.8)
- Equation (3.28) is shown in complex plane Figure 3.5.
- On the Figure 3.5 various points are described as follows.

VISCOUSLY DAMPED FREE VIBRATION

- For $\zeta = 0$, $\frac{s_{1,2}}{\omega_n} = \pm j,$ $\left|\frac{s_{1,2}}{\omega_n}\right| = \sqrt{0 + 1^2} = 1$
- for $0 < \zeta < 1,$ $\frac{s_{1,2}}{\omega_n} = -\zeta \pm j\sqrt{1-\zeta^2}$
- *s*1 and *s*2 are complex conjugate
- points on a circular arc,

$$\left|\frac{s_{1,2}}{\omega_n}\right| = \sqrt{(-\zeta)^2 + \left(\sqrt{1-\zeta^2}\right)^2} = 1$$

$$\zeta = 1 \ , \qquad \frac{s_{1,2}}{\omega_n} = -1 \pm i \times 0 = -1$$

$$\zeta > 1 \ , \qquad \frac{s_{1,2}}{\omega_n} = -\zeta \pm \sqrt{\zeta^2 - 1}$$

always real numbers

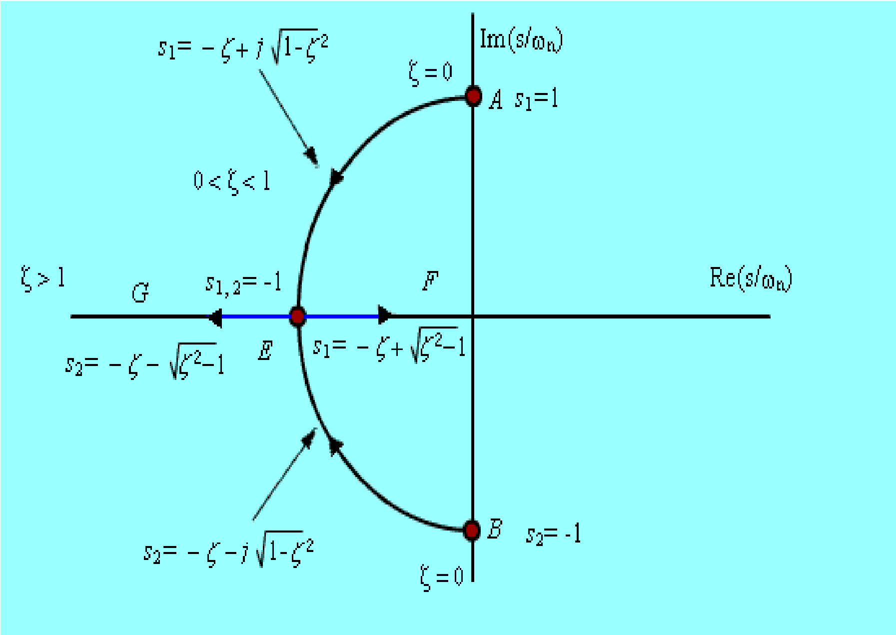


Figure 3.5: Frequency equation in complex plane

VISCOUSLY DAMPED FREE VIBRATION

- In Figure 3.5 various points are as follows
- At A and B, $\zeta = 0$ **: undamped**
- Between A and E and between B and E **: underdamped**
- At E, $\zeta = 1$ **: critical damping**
- Between E to F and E to G, $\zeta > 1$ **: overdamped**
- **1)** Oscillatory motion **:** $\zeta < 1.0$[, underdamped case]

General solution equation (3.19) becomes:

$$x = e^{-\zeta\omega_n t}\left[Ae^{j\sqrt{1-\zeta^2}\omega_n t} + Be^{-j\sqrt{1-\zeta^2}\omega_n t}\right] \quad (3.29)$$

$$= e^{-\zeta\omega_n t}\left[(A+B)\cos\sqrt{1-\zeta^2}\,\omega_n t + j(A-B)\sin\sqrt{1-\zeta^2}\,\omega_n t\right] \quad (3.30)$$

VISCOUSLY DAMPED FREE VIBRATION

$$= e^{-\zeta\omega_n t}\left[C\cos\sqrt{1-\zeta^2}\,\omega_n t + D\sin\sqrt{1-\zeta^2}\,\omega_n t\right]$$

$$= Xe^{-\zeta\omega_n t}\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi\right) \qquad (3.31)$$

From equation (3.30), we have where $C = X\sin\phi$ and $D = X\cos\phi$ and $X = \sqrt{C^2+D^2}$,

$\phi = \tan^{-1}(C/D)$ where C & D and X, ϕ are arbitrary constants (to be determined from initial conditions, x (0) and $\dot{x}$(0).

$$\dot{x} = e^{-\zeta\omega_n t}\left[C\cos\sqrt{1-\zeta^2}\,\omega_n t + D\sin\sqrt{1-\zeta^2}\,\omega_n t\right]$$

$$\dot{x} = (-\zeta\omega_n)e^{-\zeta\omega_n t}\left[C\cos\sqrt{1-\zeta^2}\,\omega_n t + D\sin\sqrt{1-\zeta^2}\,\omega_n t\right]$$

$$+e^{-\zeta\omega_n t}\left[C\left\{-\sin\sqrt{1-\zeta^2}\,\omega_n t\right\}\sqrt{1-\zeta^2}\,\omega_n + D\left\{\cos\sqrt{1-\zeta^2}\,\omega_n t\right\}\sqrt{1-\zeta^2}\,\omega_n\right]$$

On application of initial conditions, we get $x(0)=C$ and

$\dot{x} = -\zeta\omega_n C + D\sin\sqrt{1-\zeta^2}\,\omega_n$ which gives $D = \dfrac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_n\sqrt{1-\zeta^2}}$

VISCOUSLY DAMPED FREE VIBRATION

- Hence, equation (3.30), becomes
- $$x = e^{-\zeta\omega_n t}\left[\frac{\dot{x}(0)+\zeta\omega_n x(0)}{\omega_n\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\,\omega_n t + x(0)\cos\sqrt{1-\zeta^2}\,\omega_n t\right] \qquad (3.32)$$

Equation (3.32) indicates that the frequency of damped system is equal to,

$$\omega_d = \frac{2\pi}{T_d} = \omega_n\sqrt{1-\zeta^2} \qquad (3.33)$$

It should be noted that for small ζ

(which is the case of most engineering systems))

$$\omega_d = \omega_n$$