Linear Programming Models

"Programming" here means "Planning"

Definitions

linear function

$$f\left(x_1, x_2, \ldots x_n\right) = c_0 + \sum_{i=1}^{n} c_i x_i$$

$$= c_0 + c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

examples

$$2x_1 + 5x_2 + x_3 + 1$$

$$x_1 - 3x_3$$

linear inequality

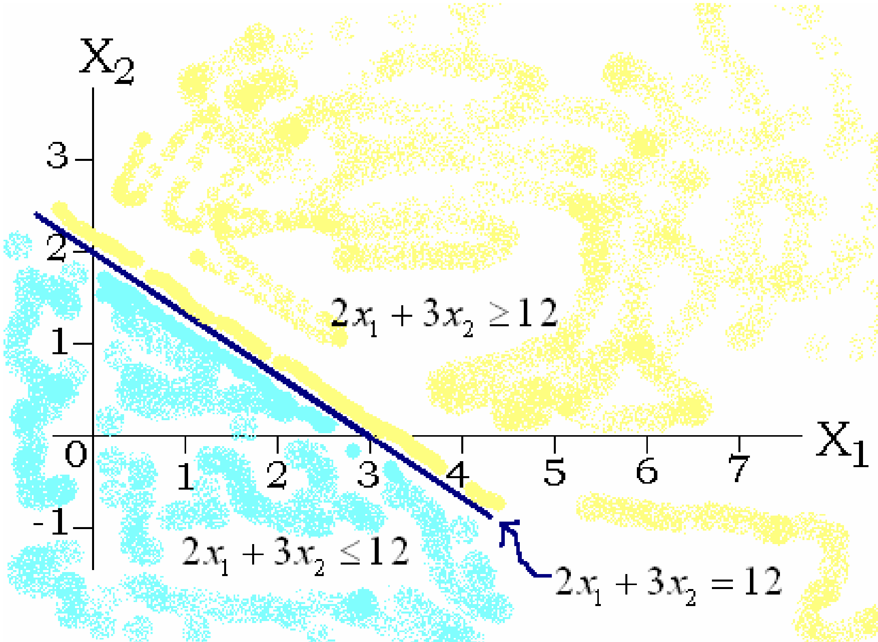
$$\sum_{i=1}^{n} a_i x_i \leq \quad (or \geq) \;\; b$$

examples

$$x_1 - 2x_2 \leq 5$$

$$2x_1 + x_2 - x_3 \geq -10$$

Graphical Representation



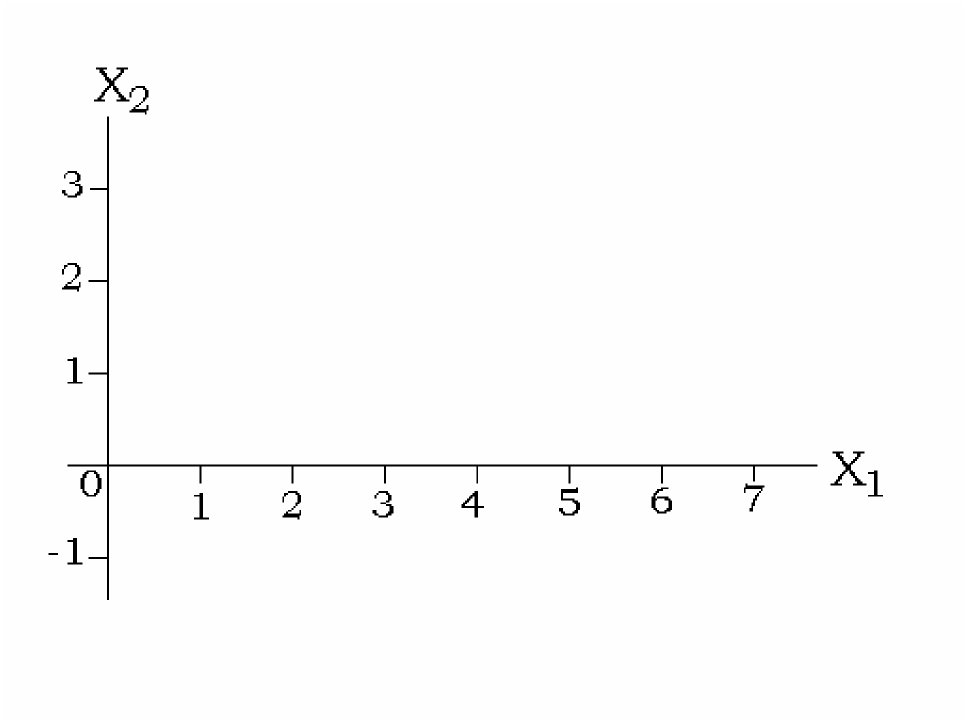
In ***two*** dimensions, the graph of a linear equation is a ***line***, and the graph of a linear inequality is a ***half-space*** *(including the line)*.

To draw the graph of a linear inequality, first draw the graph of the equation, and then decide which side is the correct half-space by testing whether (0,0) is feasible.

In ***three*** dimensions, the graph of a linear equation is a ***plane***.

In ***n*** dimensions, the graph of a linear equation is a ***hyperplane***.

Exercise

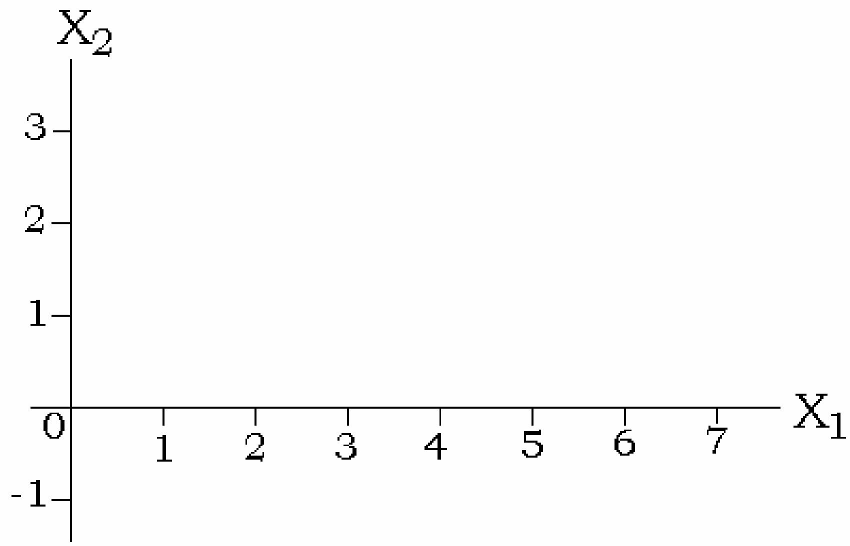


Graph the linear inequality:

$X_1 - X_2 \leq 2$

*(Shade the region representing points which are feasible in **both** inequalities.)*

(exercise, continued)



Graph the solutions of the pair of linear inequalities:

$$\begin{cases} X_1 - X_2 \le 2 \\ X_1 + 3X_2 \ge 6 \end{cases}$$

*(Shade the region representing points which are feasible in **both** inequalities.)*

linear programming (LP): an optimization problem for which

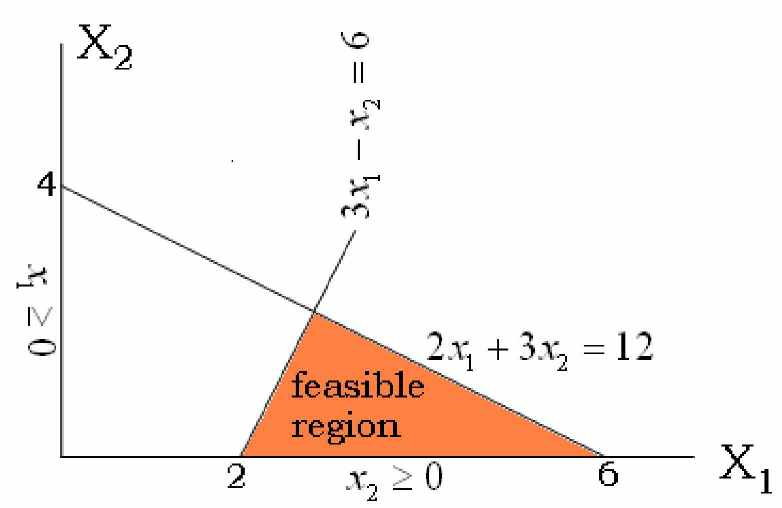
- we ***maximize*** or ***minimize*** a linear function of the ***decision variables*** (this function is called the ***objective*** function)
- the values of the decision variables must satisfy a set of ***constraints***, each consisting of a linear equation or linear inequality
- a sign restriction, i.e., usually ***nonnegativity*** $(x_i \geq 0)$ but perhaps ***nonpositivity*** $(x_i \leq 0)$, may be associated with each decision variable.

example:

$$\begin{aligned} \text{maximize} \quad & 2x_1 + x_2 \\ \textit{subject to} \quad & 3x_1 - x_2 \geq 6 \\ & 2x_1 + 3x_2 \leq 12 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

Graphical Representation

$$\begin{aligned} \text{maximize} \quad & 2x_1 + x_2 \\ \textit{subject to} \quad & 3x_1 - x_2 \geq 6 \\ & 2x_1 + 3x_2 \leq 12 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

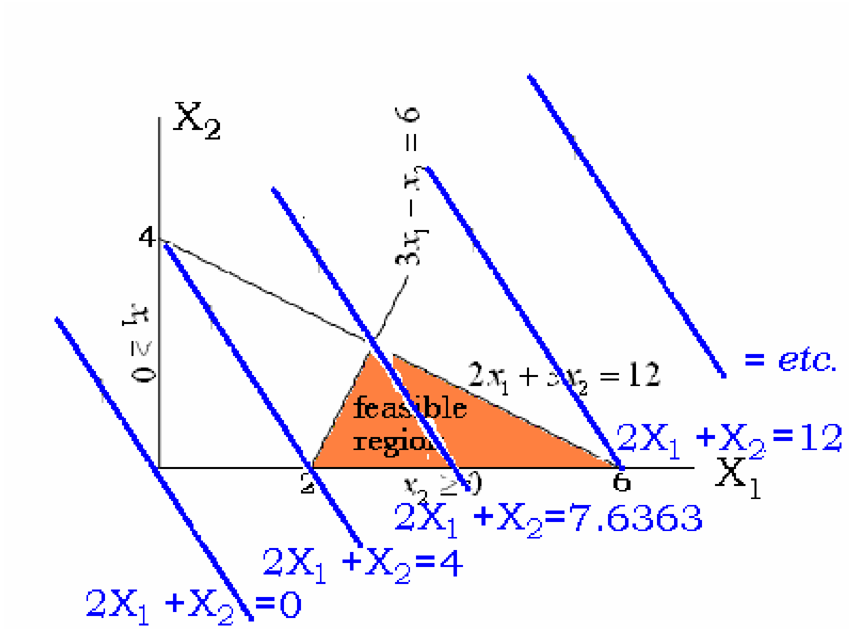


Each point in the shaded ***feasible region*** satisfies ***all four*** inequality constraints (including nonnegativity) and represents a possible solution of the problem.

The ***optimal solution*** is the feasible solution for which the objective function is largest.

By graphing the linear equations

$$2X_1 + X_2 = 0,\ \ 2X_1 + X_2 = 4,\ 2X_1 + X_2 = 12,\ etc.,$$



we see that the ***slope remains the same***,

but the line is ***shifted to the right***.

How far to the right can the line be shifted while still including a feasible solution of the set of inequalities?

The optimal solution is the corner farthest to the right, $(X_1, X_2) = (6,0)$.

In fact, an optimal solution of an LP problem can always be found at a corner point!

Example:

- A manufacturer can make two products: P and Q.
- Each product requires processing time on each of four machines: A, B, C, and D.
- Each machine is available 24 hours per day = 1440 minutes per day.
- The profit per unit of products P and Q are $45 and $60, respectively.
- Maximum demand for products P and Q are 100/day and 40/day, respectively.

	Unit Processing	Time (minutes)	
Machine \Product:	P	Q	Available (min.)
A	20	10	1440
B	12	28	1440
C	15	6	1440
D	10	15	1440
Profit/unit	45	60	

How much of each product should be manufactured each day in order to maximize profits?

Define the decision variables

P = *number of units/day of product P*

Q = *number of units/day of product Q*

Objective: *Maximize* $45P + 60Q$ ($/day)

Constraints: *do not exceed the available processing time on each machine:*

$$20P + 10Q \leq 1440$$

$$12P + 28Q \leq 1440$$

$$15P + 6Q \leq 1440$$

$$10P + 15Q \leq 1440$$

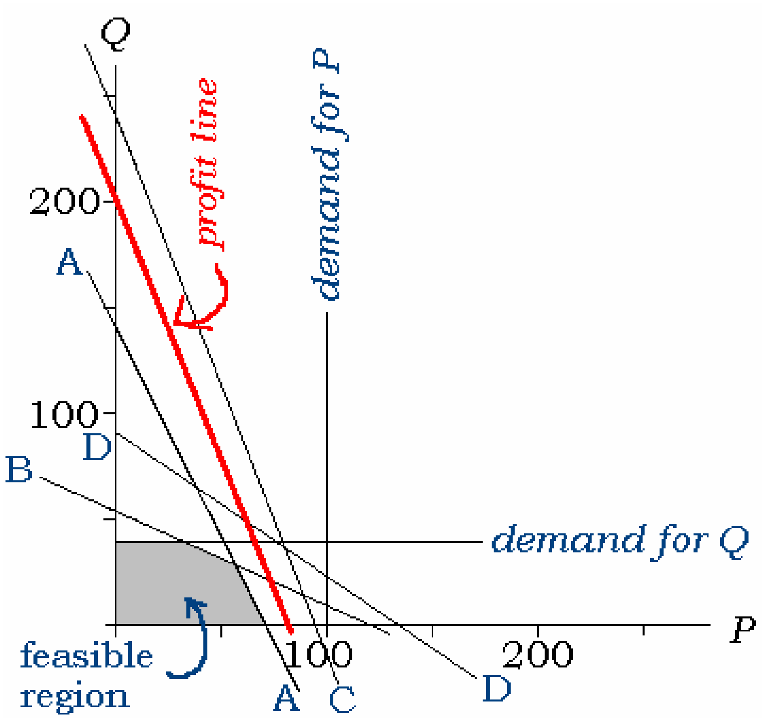
do not produce more than the demand for the products:

$$P \leq 100$$

$$Q \leq 40$$

a negative quantity of product is meaningless:

$$P \geq 0, Q \geq 0$$



The maximum profit is obtained at the corner point $(P,Q)=(58.9,26.2)$

Note: I clearly erred in drawing the isoquant line for the profit!

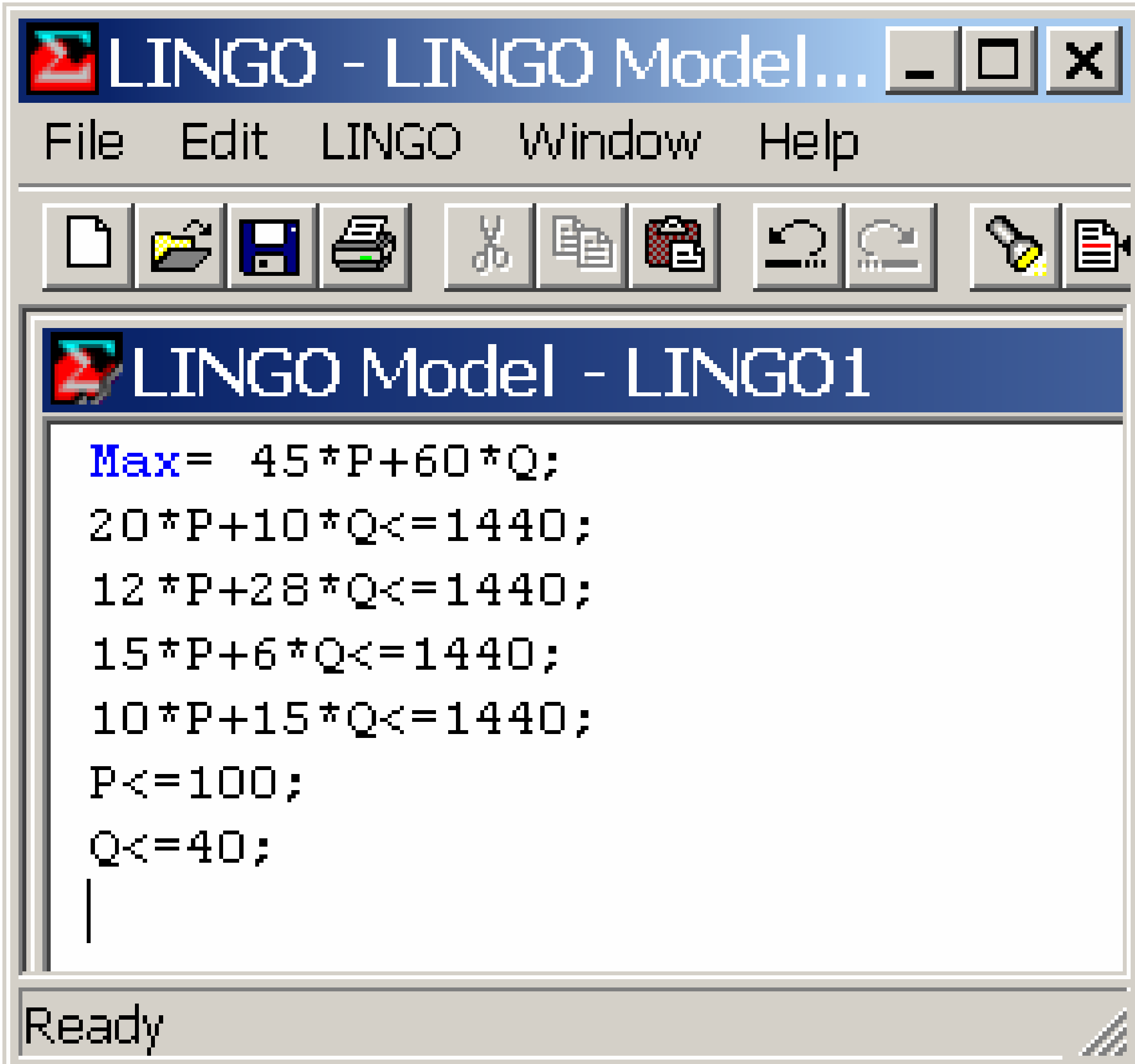
The **graphical method** for solving an LP problem is useless for problems with more than 2 (possibly 3) decision variables…

Problems occurring in "the real world" may involve

a million decision variables and

thousands of constraints!

We will study computational methods for solving linear programming problems.



LINGO is a software package for solving LP problems….

(by default, variables are assumed to be nonnegative.)

SOLUTION:

```
Global optimal solution found at step:            2
 Objective value:                           4221.818


                       Variable           Value        Reduced Cost
                              P        58.90909           0.0000000
                              Q        26.18182           0.0000000

                            Row    Slack or Surplus      Dual Price
                              1        4221.818            1.000000
                              2       0.0000000            1.227273
                              3       0.0000000            1.704545
                              4        399.2727           0.0000000
                              5        458.1818           0.0000000
                              6        41.09091           0.0000000
                              7        13.81818           0.0000000
```

That is, the manufacturer will maximize profits by producing 58.9 units of P and 26.18 units of Q each day (assuming fractional units are possible).

This plan will yield a profit of $4221.818/day.

Row	Slack or Surplus	Dual Price
1	4221.818	1.000000
2	0.0000000	1.227273
3	0.0000000	1.704545
4	399.2727	0.0000000
5	458.1818	0.0000000
6	41.09091	0.0000000
7	13.81818	0.0000000

This plan will use all of the available time on machines A and B, i.e.,

$$S_A = S_B = 0$$

but unused time on machines C & D will be 399.27 and 458.18, respectively, that is, $S_C = 399.2727$ and $S_D = 458.1818$.

Computational Methods for Solving LPs

It is more convenient to work with linear **equations** rather than linear inequalities.

Define **“slack” variables** S_A, S_B, S_C & S_D to be the **unused** processing time on machines A, B, C & D, respectively.

Then, for example, the inequality constraint for machine A is equivalent to the linear equation and nonnegativity restriction:

$$20P + 10Q \le 1440 \quad \Leftrightarrow \quad 20P + 10Q + S_A = 1440 \;\&\; S_A \ge 0$$

Thus we obtain the **system of equations** *(& simple bounds on the variables)*:

$$
\begin{cases}
20P + 10Q \le 1440 \\
12P + 28Q \le 1440 \\
15P + 6Q \le 1440 \\
10P + 15Q \le 1440
\end{cases}
\iff
\begin{cases}
20P + 10Q + S_A = 1440 \\
12P + 28Q + S_B = 1440 \\
15P + 6Q + S_C = 1440 \\
10P + 15Q + S_D = 1440 \\
0 \le P \le 100,\ 0 \le Q \le 40, \\
S_A \ge 0, S_B \ge 0, S_C \ge 0, S_D \ge 0
\end{cases}
$$

Next we will review computational methods for solving systems of linear equations!