Review: Duality Theory

Weak duality property: If **x** is a feasible solution for the primal problem and **y** is a feasible solution for the dual problem, then $\mathbf{cx} \leq \mathbf{yb}$.

Strong duality property: If **x*** is an optimal solution for the primal problem and **y*** is an optimal solution for the dual problem, then $\mathbf{cx}^* = \mathbf{y}^*\mathbf{b}$.

Complementary solutions property: At each iteration, the simplex method simultaneously identifies a CPF solution **x** for the primal πroblem and a **complementary solution y** for the dual problem (found in row 0, the coeff of the slack variables), where $\mathbf{cx} = \mathbf{yb}$. If **x** is *not optimal* for the primal, **y** is infeasible.

Complementary optimal solutions property: At the final iteration, the simplex method simultaneously identifies an optimal solution **x*** for the primal problem and a **complementary optimal solution y*** for the dual problem (found in row 0, the coeff of the slack variables), where $\mathbf{cx}^* = \mathbf{y}^*\mathbf{b}$. In this solution, **y*** gives the shadow prices for the primal problem.

Symmetry property: The dual of the dual is the primal.

Find the corresponding complementary basic solutions based on the simplex tableaux:

Final Tableaux

Basic	**Coeff**			
Var	**Z**	**original**	**slack**	**RHS**
Z	1	$\mathbf{c_B B^{-1} A - c}$	$\mathbf{c_B B^{-1}}$	$\mathbf{c_B B^{-1} b}$
$\mathbf{x}_B$	**0**	$\mathbf{B^{-1} A}$	$\mathbf{B^{-1}}$	$\mathbf{B^{-1} b}$

Complementary basic solutions property: Each *basic* solution in the *primal problem* has a **complementary basic solution** in the *dual problem*, where their respective objective function values (Z and W) are equal.

Complementary optimal basic solutions property: Each *optimal* basic solution in the *primal problem* has a **complementary optimal basic solution** in the dual problem, where their respective objective function values (Z and W) are equal.

Complementary slackness property: The variables in the primal basic solution and the complementary dual basic solution satisfy the **complementary slackness** as shown:

Primal variable	Dual variable
Basic	Non-basic (m variables)
Non-basic	Basic (n variables)

Futhermore, the relationship is symmetric.

Check for the complementary slackness property:

Iteration	Basic Var	Coeff Z	x_1	x_2	x_3	x_4	x_5	RHS
0	Z	1	-30	-15	0	0	0	**0**
	x_3	0	1	0	1	0	0	4
	x_4	0	0	2	0	1	0	12
	x_5	0	3	2	0	0	1	18
1	Z	1	0	-15	30	0	0	**120**
	x_1	0	1	0	1	0	0	4
	x_4	0	0	2	0	1	0	12
	x_5	0	0	2	-3	0	1	6
2	Z	1	0	0	7.5	0	7.5	**165**
	x_1	0	1	0	1	0	0	4
	x_4	0	0	0	3	1	-1	6
	x_2	0	0	1	-1.5	0	0.5	3

Relationships between

solutions

Primal Basic Solution		Complementary Dual Basic Solution	
State	Feasible?	State	Feasible?
suboptimal			
optimal			
superoptimal			
neither feasible nor superoptimal			

Duality Theorem: Possible scenarios…

(1) Feasible solutions exist and objective function is bounded, then same is true for other problem.

(2) Feasible solutions exist and objective function is unbounded, then other problem is infeasible.

(3) No feasible solutions exist then other problem is either infeasible or has unbounded objective fn.

Example (Problem 6.1-7):

Consider the following problem.

$$\begin{aligned} \text{Max} \quad Z &= x_1 + 2x_2 \\ \text{s.t.} \quad -x_1 + x_2 &\leq -2 \\ 4x_1 + x_2 &\leq 4 \\ x_1 \geq 0, \; x_2 &\geq 0 \end{aligned}$$

(a) Demonstrate graphically that this problem has no feasible solutions.

(b) Construct the dual problem.

(c) Demonstrate graphically that the dual problem has an unbounded objective function.

Shortcut for conversion between primal and dual:

Primal	**Dual**
Max Z	Min W
constraint i: $\le$ $=$ $\ge$	variable i: $y_i \ge 0$
variable x_j: $x_j \ge 0$	constraint j:

Examle: Quidditch Problem

$$
\begin{aligned}
\text{Max } -Z &= -0.4\,x_1 - 0.5\,x_2 \\
\text{s.t.} \quad & 0.3\,x_1 + 0.1\,x_2 \le 2.7 \\
& 0.5\,x_1 + 0.5\,x_2 = 6 \\
& 0.6\,x_1 + 0.4\,x_2 \ge 6 \\
& x_1 \ge 0, \quad x_2 \ge 0
\end{aligned}
$$

APPLICATIONS:

- The *dual* problem can be solved directly to identify an opt soln for the primal.
- If **x** is a feasible soln, and **y** a feasible soln for the dual is found easily such that **cx** = **yb**, then what can we say about **x**?
- Use in the dual simplex method.
- Plays a central role in sensitivity analysis.
- Use in the economic interpretation of the dual problem and the resulting insights for analyzing primal.

Special request: two problems similar to 5.2-2 and 5.3-5.

5.2-4 Work through the revised simplex method step by step to solve the model given in Prob. 4.1-5:

$$\begin{aligned} \text{Max} \quad Z &= x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + 3x_2 \le 8 \\ & x_1 + x_2 \le 4 \\ & x_1 \ge 0, \; x_2 \ge 0 \end{aligned}$$

Let's discuss how one might set up 5.3-5.

5.3-5 Consider the following problem.

$$\text{Max} \quad Z = c_1x_1 + c_2x_2 + c_3x_3$$

$$\begin{aligned} \text{s.t.} \quad x_1 + 2x_2 + x_3 &\le b \\ 2x_1 + x_2 + 3x_3 &\le 2b \\ x_1 \ge 0,\ x_2 \ge 0,\ x_3 &\ge 0 \end{aligned}$$

Note that values have not been assigned to te coefficients in the objective function (c_1, c_2, c_3), and that the only specification for the right-hand side of the functional constraints is that the second one (2b) be twice as large as the first (b).

Now suppose that your boss has inserted her best estimate of the values of (c_1, c_2, c_3), and b without informing you and then has run the simplex method. You are given the resulting final simplex tableau below (where x_4 and x_5 are the slack variables for the respective functional constraints), but you are unable to read the value of Z*.

Iteration	Basic Var	Coeff						RHS
		Z	**x_1**	**x_2**	**x_3**	**x_4**	**x_5**	
0	Z	1	7/10	0	0	3/5	4/5	**Z***
	x_2	0	1/5	1	0	3/5	−1/5	1
	x_3	0	3/5	0	1	−1/5	2/5	3

(a) Use the fundamental insight to identify the value of (c_1, c_2, c_3) that was used.

(b) Use the fundamental insight to identify the value of b that was used.

(c) Calculate the value of Z* in two ways, where one way uses your result from (a) and the other your result from (b). Show your two methods for finding Z*.