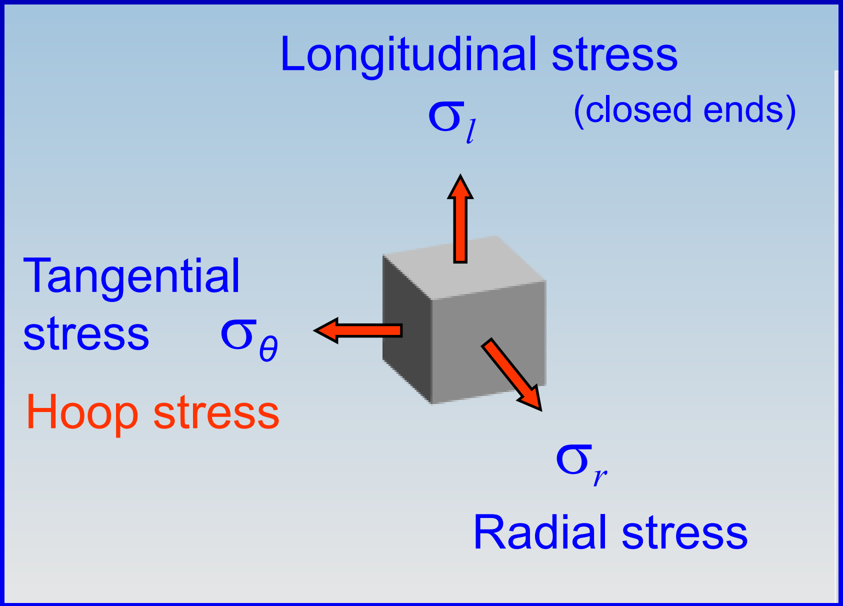
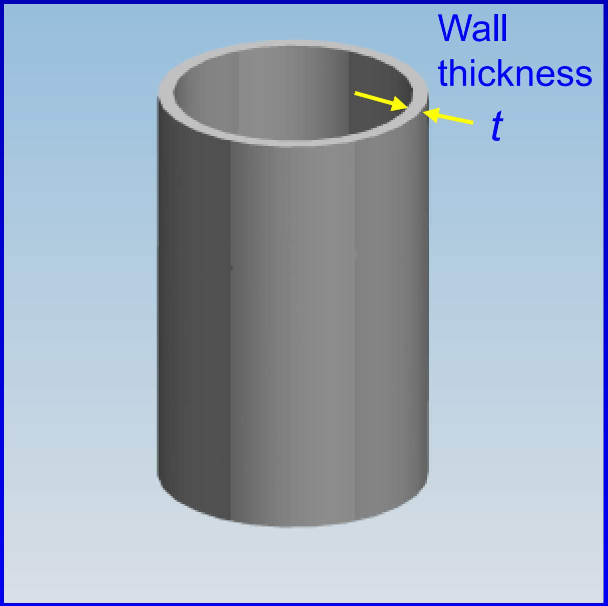
Stresses in Pressurized Cylinders

Cylindrical pressure vessels, hydraulic cylinders, shafts with components mounted on (gears, pulleys, and bearings), gun barrels, pipes carrying fluids at high pressure,….. develop tangential, longitudinal, and radial stresses.

Stress element



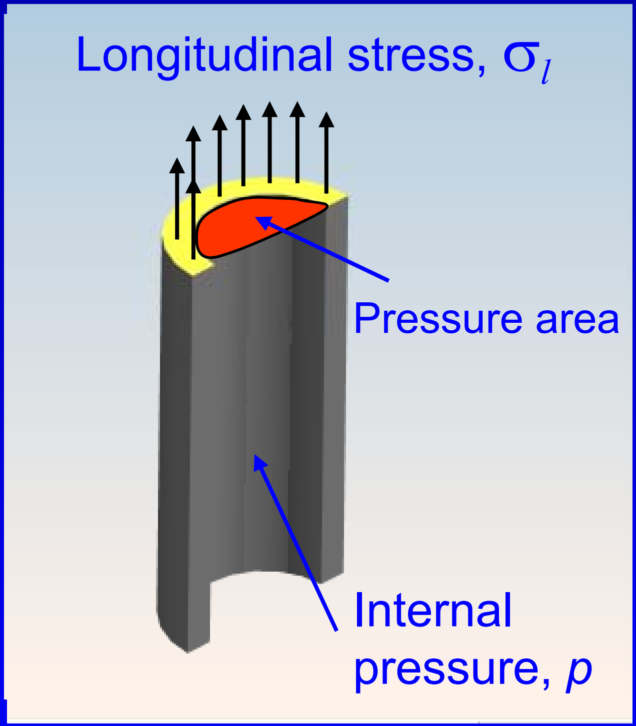


A pressurized cylinder is considered a thin-walled vessel if the wall thickness is less than one-twentieth of the radius.

$$\frac{t}{r} < 1/20 \longrightarrow \text{Thin-walled pressure vessel}$$

Stresses in a Thin-Walled Pressurized Cylinders

In a thin-walled pressurized cylinder the radial stress is much smaller than the tangential stress and can be neglected.



Longitudinal stress

$\Sigma F_y = 0$

$$(\sigma_l)\ \pi/4\ [\ (d_o)^2 - (d_i)^2] = (\ p\ )\ \pi/4\ (d_i)^2$$

$$(\sigma_l)\ [\ (d_i + 2t)^2 - (d_i)^2] = (\ p\ )\ (d_i)^2$$

$4t^2$ is very small,

$$(\sigma_l)\ (4 d_i\, t) = (\ p\ )\ (d_i)^2$$

$$\sigma_l = \frac{p\ d_i}{2\ t}$$

$$\boxed{\sigma_l = \frac{p\ (d_i + t)}{4\ t}}$$

Max. longitudinal stress

Stresses in a Thin-Walled Pressurized Cylinders

Tangential (hoop) stress

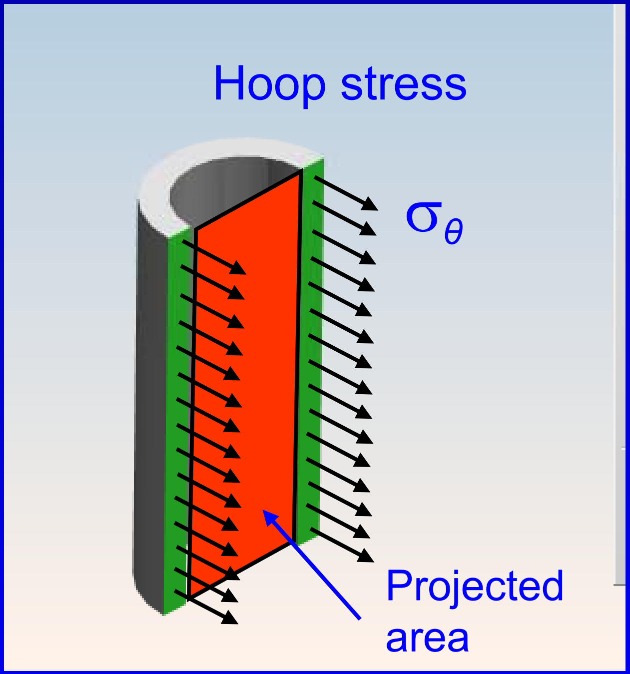
$\Sigma F_x = 0$

$$2(\sigma_\theta)\, t\, (\text{length}) = (\, p\,)\, (d_i)\, (\text{length})$$

$$\sigma_\theta = \frac{p\, d_i}{2\, t}$$

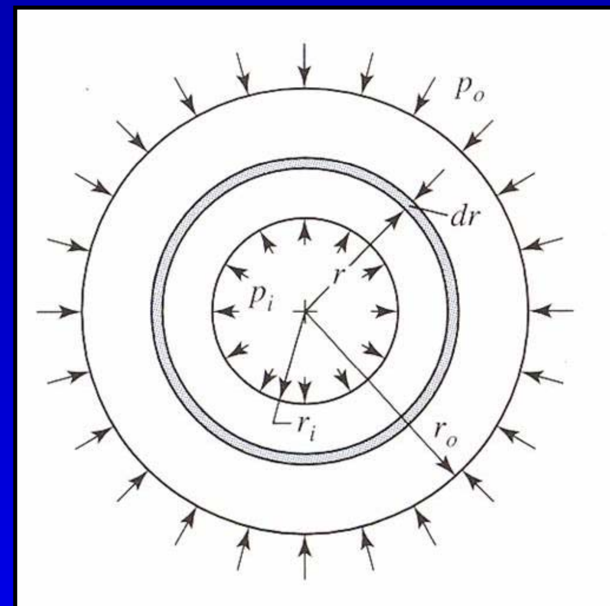
Max. Hoop stress

$$\sigma_l = \tfrac{1}{2}\, \sigma_\theta$$



Stresses in a Thick-Walled Pressurized Cylinders

In case of thick-walled pressurized cylinders, the radial stress, σ_r , cannot be neglected.

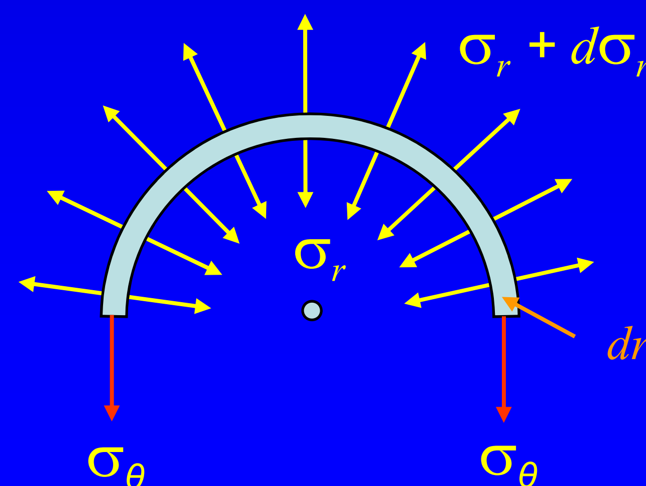


Assumption – longitudinal elongation is constant around the plane of cross section, there is very little warping of the cross section, ε_l = *constant*

l = length of cylinder

$\sum F = 0$

$$2(\sigma_\theta)(dr)(l) + \sigma_r(2rl) - (\sigma_r + d\sigma_r)\left[2(r + dr)l\right] = 0$$



$(d\sigma_r)\,(dr)$ is very small compared to other terms ≈ 0

$$\sigma_\theta - \sigma_r - r\frac{d\sigma_r}{dr} = 0 \qquad (1)$$

Stresses in a Thick-Walled Pressurized Cylinders

Deformation in the longitudinal direction

$$\varepsilon_l = -\mu \frac{\sigma_\theta}{E} - \mu \frac{\sigma_r}{E} \longrightarrow \sigma_\theta + \sigma_r = -\frac{\varepsilon_l E}{\mu} = 2C_1 \qquad (2)$$

(constant: $-\frac{\varepsilon_l E}{\mu}$)

$$\sigma_\theta - \sigma_r - r\frac{d\sigma_r}{dr} = 0 \qquad (1)$$

Subtract equation (1) from (2),

$$\sigma_r + \sigma_r + r\frac{d\sigma_r}{dr} = 2C_1$$

Consider,

$$\frac{d(\sigma_r r^2)}{dr} = r^2\frac{d\sigma_r}{dr} + 2r\sigma_r$$

Multiply the above equation by r

$$2r\sigma_r + r^2\frac{d\sigma_r}{dr} = 2rC_1 \longrightarrow \frac{d(\sigma_r r^2)}{dr} = 2rC_1$$

$$\sigma_r r^2 = r^2 C_1 + C_2$$

$$\boxed{\sigma_r = C_1 + \frac{C_2}{r^2}} \qquad \boxed{\sigma_\theta = C_1 - \frac{C_2}{r^2}}$$

Stresses in a Thick-Walled Pressurized Cylinders

Boundary conditions

$$\begin{cases} \sigma_r = -p_i \text{ at } r = r_i \\ \sigma_r = -p_o \text{ at } r = r_o \end{cases}$$

$$\sigma_\theta = \frac{p_i r_i^2 - p_o r_o^2 - r_i^2 r_o^2 (p_o - p_i) / r^2}{r_o^2 - r_i^2}$$

Hoop stress

$$\sigma_r = \frac{p_i r_i^2 - p_o r_o^2 + r_i^2 r_o^2 (p_o - p_i) / r^2}{r_o^2 - r_i^2}$$

Radial stress

$$\sigma_l = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}$$

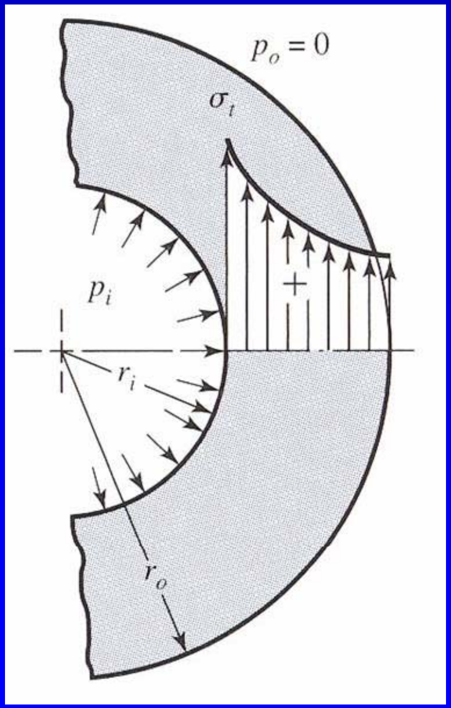
Longitudinal stress

Stresses in a Thick-Walled Pressurized Cylinders

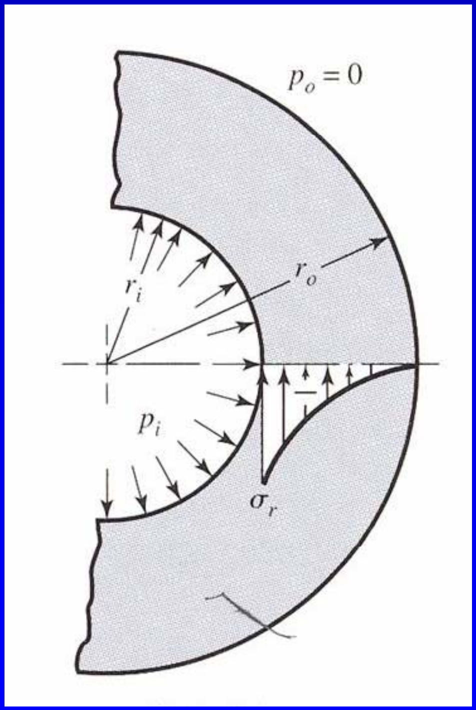
Special case, p_o (external pressure) = 0

$$\sigma_\theta = \frac{p_i r_i^2}{r_o^2 - r_i^2}\left(1 + \frac{r_o^2}{r^2}\right)$$

$$\sigma_r = \frac{p_i r_i^2}{r_o^2 - r_i^2}\left(1 - \frac{r_o^2}{r^2}\right)$$



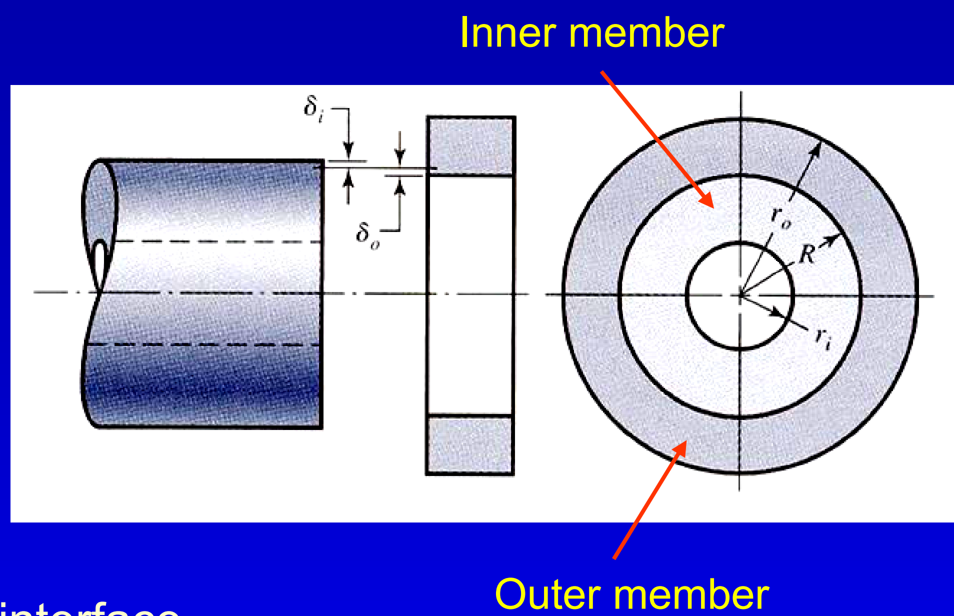
Hoop stress distribution,
maximum at the inner surface



Radial stress distribution,
maximum at the inner surface

Press and Shrink Fits

d_s (shaft) > d_h (hub)

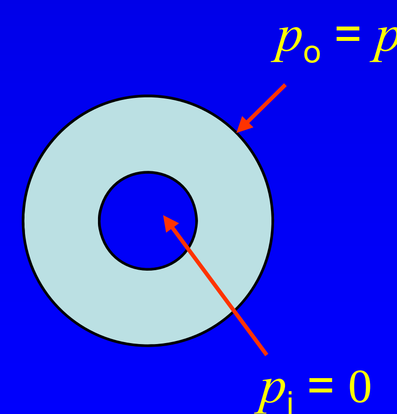


p (contact pressure) = pressure at the outer surface of the inner member or pressure at the inner surface of the outer member.

Consider the inner member at the interface

$$\sigma_{i\theta} = \frac{p_i r_i^2 - p_o r_o^2 - r_i^2 r_o^2 (p_o - p_i) / r^2}{r_o^2 - r_i^2}$$

$$r_i = r_i$$
$$r_o = R$$



$$\begin{cases} \sigma_{i\theta} = \dfrac{-p_o R^2 - r_i^2 R^2 \ \ p / R^2}{R^2 - r_i^2} = -p \dfrac{R^2 + r_i^2}{R^2 - r_i^2} \\ \sigma_{ir} = -p_o = -p \end{cases}$$

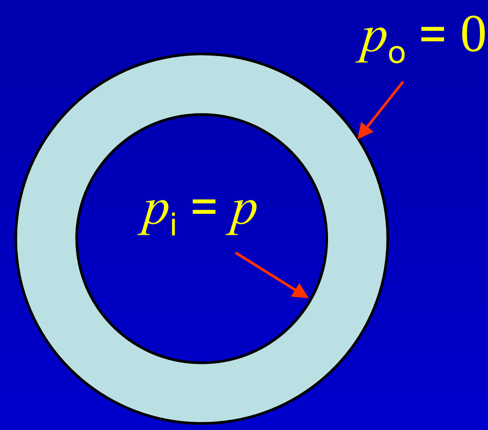
Press and Shrink Fits

Consider the outer member at the interface

$$\begin{cases} \sigma_{o\theta} = p \dfrac{R^2 + r_o^{\,2}}{r_o^{\,2} - R^2} \\ \sigma_{or} = -p \end{cases}$$

$r_i = R$

$r_o = r_o$



δ_o = increase in the inner radius of the outer member

δ_i = decrease in the outer radius of the inner member

Tangential strain at the inner radius of the outer member

$$\varepsilon_{ot} = \frac{\sigma_{o\theta}}{E_o} - \mu_o \frac{\sigma_{or}}{E_o}$$

$$\varepsilon_{ot} = \frac{\delta_o}{R}$$

$$\longrightarrow \quad \delta_o = \frac{Rp}{E_o} \left(\frac{R^2 + r_o^{\,2}}{r_o^{\,2} - R^2} + \mu_o \right)$$

Press and Shrink Fits

Tangential strain at the outer radius of the inner member

$$\varepsilon_{it} = \frac{\sigma_{i\theta}}{E_i} - \mu_i \frac{\sigma_{ir}}{E_i}$$

$$\varepsilon_{it} = -\frac{\delta_i}{R}$$

$$\longrightarrow \quad \delta_i = -\frac{Rp}{E_i}\left(\frac{R^2 + r_i^2}{R^2 - r_i^2} + \mu_i\right)$$

Radial interference, $\delta = \delta_o - \delta_i$

$$\delta = \frac{Rp}{E_o}\left(\frac{R^2 + r_o^2}{r_o^2 - R^2} + \mu_o\right) + \frac{Rp}{E_i}\left(\frac{R^2 + r_i^2}{R^2 - r_i^2} + \mu_i\right)$$

If both members are made of the same material then, $E = E_o = E_i$ and $\mu = \mu_o = \mu_i$

$$p = \frac{E\,\delta}{R}\left[\frac{(r_o^2 - R^2)\,(R^2 - r_i^2)}{2R^2\,(r_o^2 - r_i^2)}\right]$$