PART 2

Multibody Kinematics and Dynamics

Chapter 1

1.1 INTRODUCTION

- Determine appropriate movement of the wipers
  - View range
  - Tandem or opposite
  - Wipe angle
  - Location of pivots
- Timing of wipers
- Wiping velocity
- The force acting on the wipers

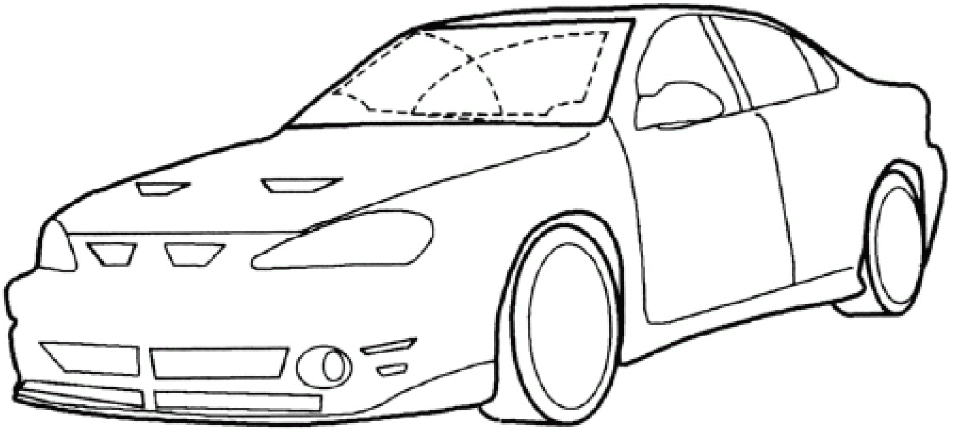


Figure 1.1 Proposed windshield wiper movements.

Kinematics

- Kinematics
  - Deal with the way things move
- Kinematic analysis
  - Determine
    - Position, displacement, rotation, speed, velocity, acceleration
  - Provide
    - Geometry dimensions of the mechanism
    - Operation range
- Dynamic analysis
  - Power capacity, stability, member load
- Planar mechanism – motion in 2D space

1.4 MECHANISM TERMINOLOGY

Mechanism

- Design synthesis is the process of developing mechanism to satisfy a set of performance requirements for the machine.
- Analysis ensures that the mechanism will exhibit motion to accomplish the requirements.

FIGURE 1.3 Elliptical trainer exercise machine (photo from www.precor.com).

- Linkage
- Frame
- Links– rigid body
- Joint
- Primary joint (full joint)
  - Revolute joint (pin or hinge joint)– pure rotation
  - Sliding joint (piston or prism joint)– linear sliding

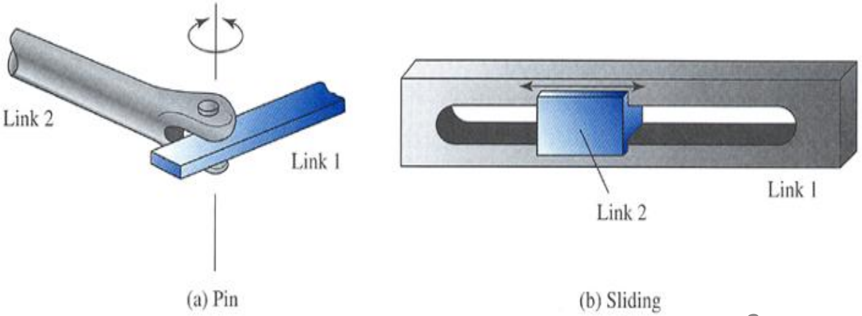
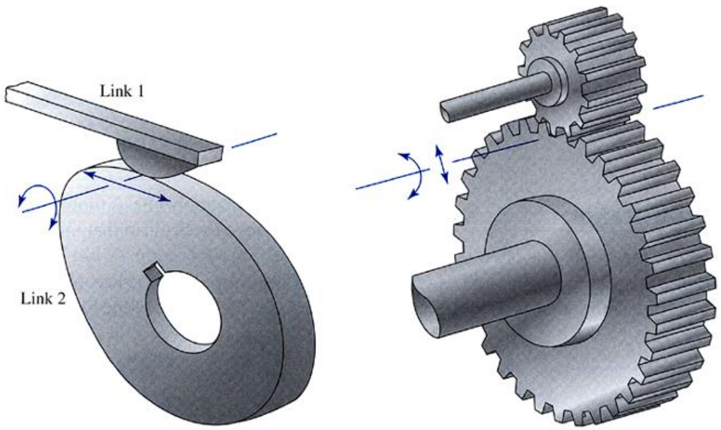


FIGURE 1.4 Primary joints: (a) Pin and (b) Sliding.

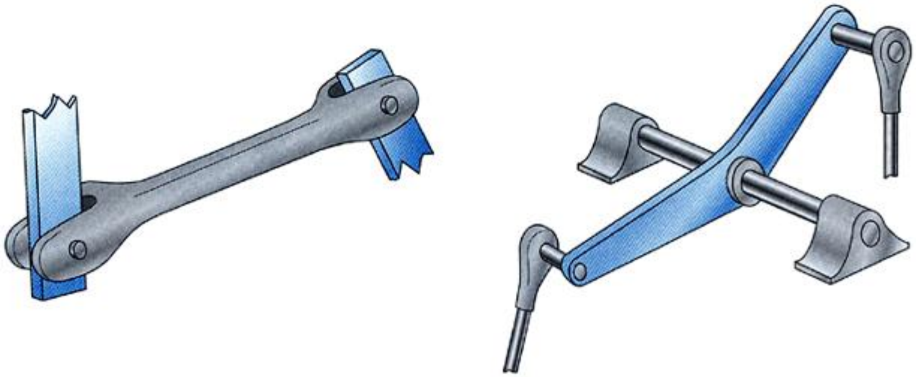
Mechanism Joint

- Higher-order joint (half joint)
  - Allow rotation and sliding
  - Cam joint
  - Gear connection
- Simple link
  - A rigid body contains only two joints
  - Crank
  - Rocker
- Complex link
  - A rigid body contains more than two joints
  - Rocker arm
  - Bellcrank
- Point of interest
- Actuator
  - A power source link



(a) Cam joint (b) Gear joint

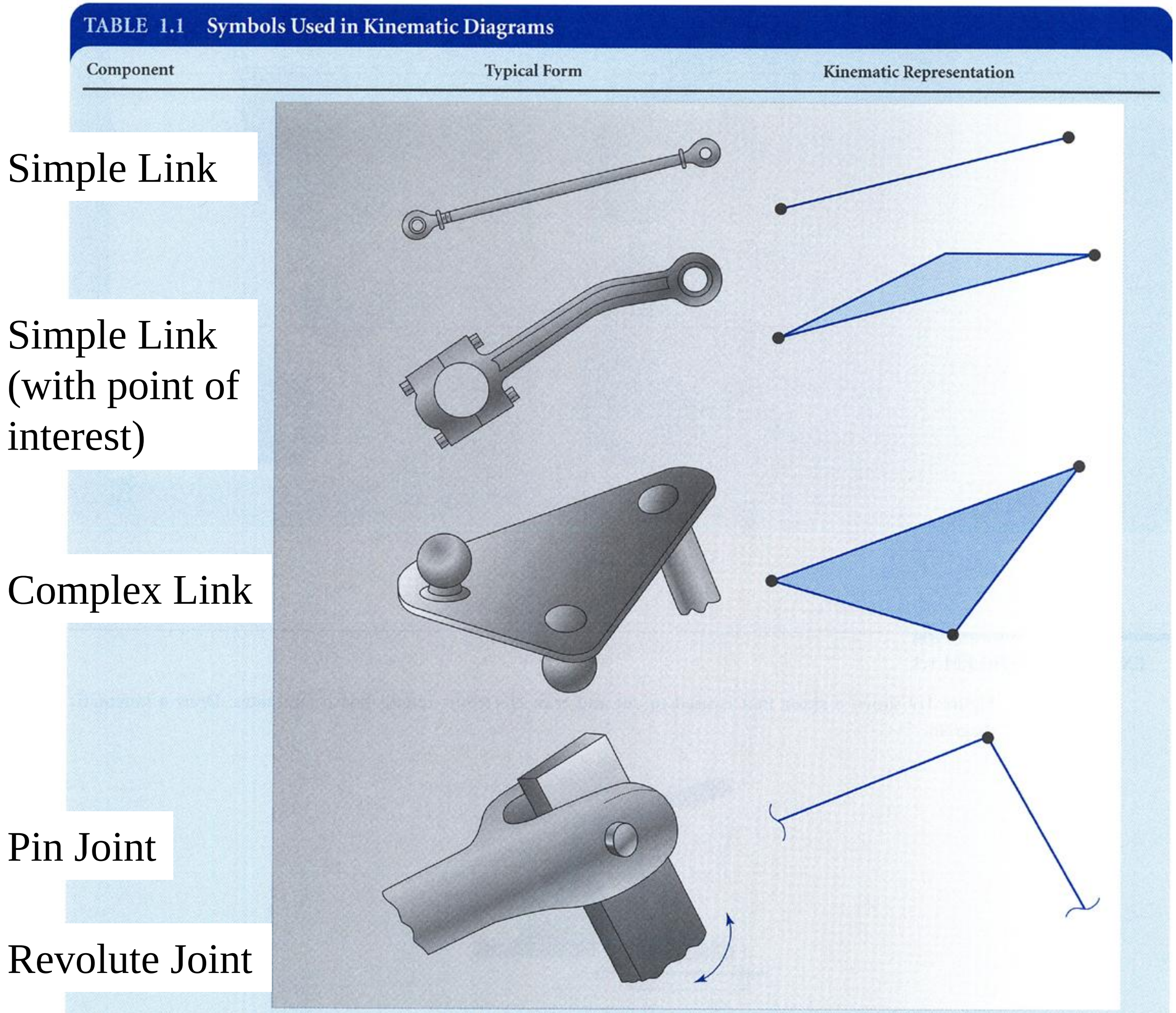
FIGURE 1.5 Higher-order joints: (a) Cam joint and (b) Gear joint.



(a) Simple link (b) Complex link

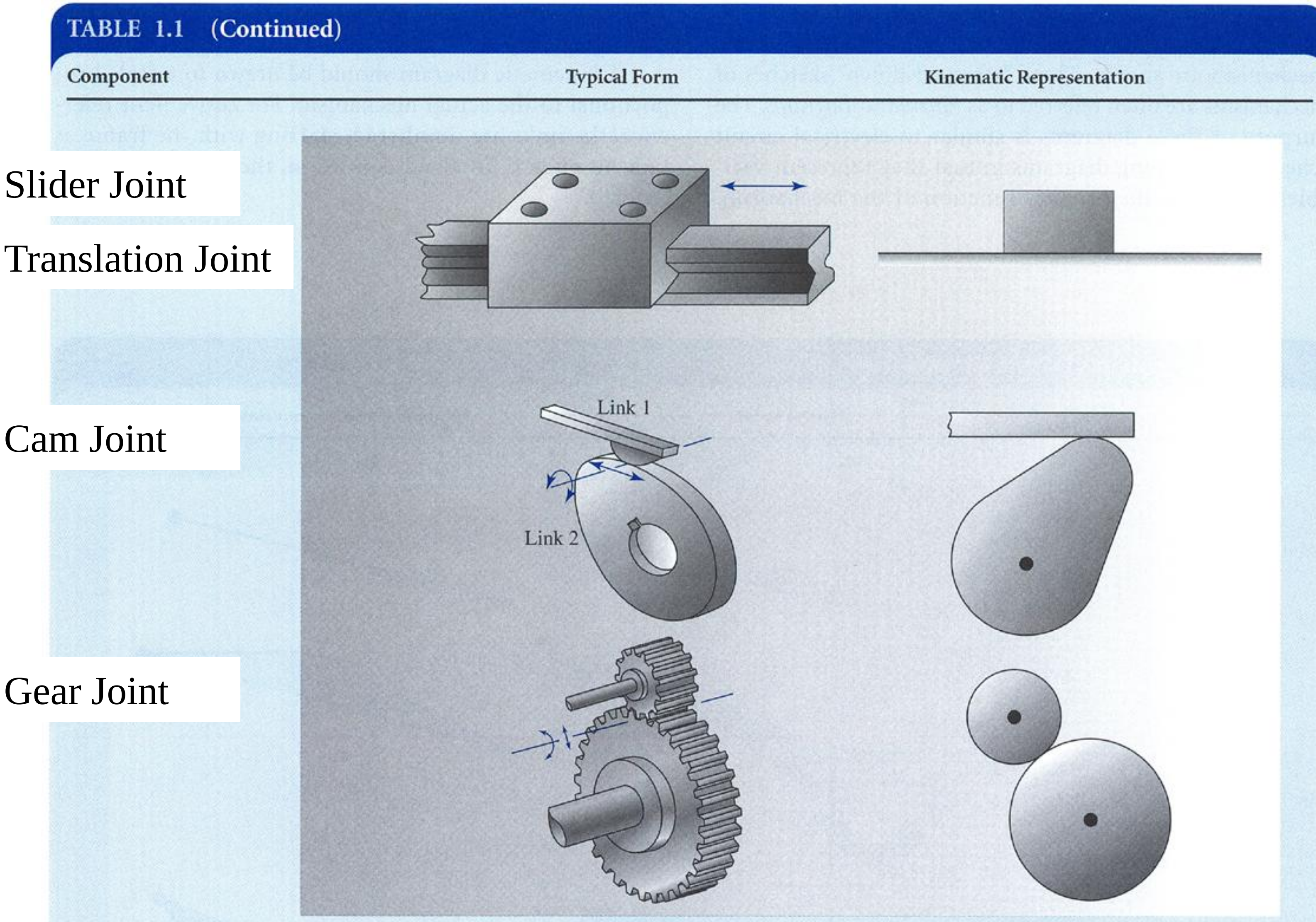
FIGURE 1.6 Links: (a) Simple link and (b) Complex link.

1.5 Kinematic Diagram



Kinematic Diagram

TABLE 1.1 (Continued)



1.7 MOBILITY

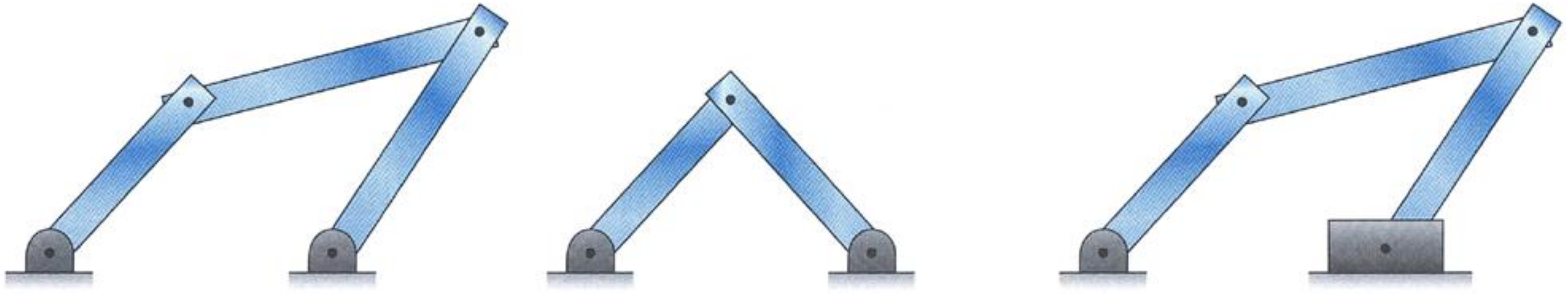
$$M = \text{degrees of freedom} = 3(n - 1) - 2j_p - j_h$$

n = total number of links in the mechanism

j_p = total number of primary joints (pins or sliding joints)
(revolute joint and translation joint)

j_h = total number of higher-order joints (cam or gear joints)

- Constrained mechanism : one degree of freedom
- Locked mechanism : zero degree of freedom
- Unconstrained mechanism : more than one degree of freedom



(a) Single degree-of-freedom ($M = 1$) (b) Locked mechanism ($M = 0$) (c) Multi-degree-of-freedom ($M = 2$)

FIGURE 1.13 Mechanisms and structures with varying mobility.

EXAMPLE PROBLEM 1.3

Figure 1.14 shows a toggle clamp. Draw a kinematic diagram, using the clamping jaw and the handle as points of interest. Also compute the degrees of freedom for the clamp.

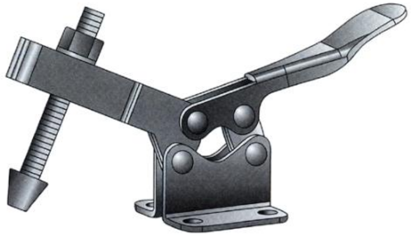
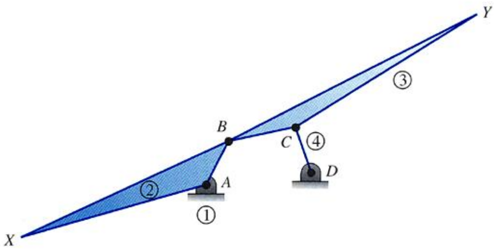


FIGURE 1.14 Toggle clamp for Example Problem 1.3.



SOLUTION:

1. ***Identify the Frame***
2. ***Identify All Other Links***

   Link 2: Handle

   Link 3: Arm that serves as the clamping jaw

   Link 4: Bar that connects the clamping arm and handle
3. ***Identify the Joints***

   Four pin joints are used to connect these different links (link 1 to 2, 2 to 3, 3 to 4, and 4 to 1). These joints are lettered *A* through *D*.
4. ***Identify Any Points of Interest***
5. ***Draw the Kinematic Diagram***
6. ***Calculate Mobility***

   Having four links and four pin joints,

$$n = 4,\ j_p = 4 \text{ pins},\ j_h = 0$$

and

$$M = 3(n - 1) - 2j_p - j_h = 3(4 - 1) - 2(4) - 0 = 1$$

With one degree of freedom, the clamp mechanism is constrained. Moving only one link, the handle, precisely positions all other links in the clamp.

1.7 Actuators and Drivers

- Electric motors (AC/DC)
- Engines
- Servomotors
- Air or hydraulic motors
- Hydraulic or pneumatic cylinders
- Screw actuators
- Manual

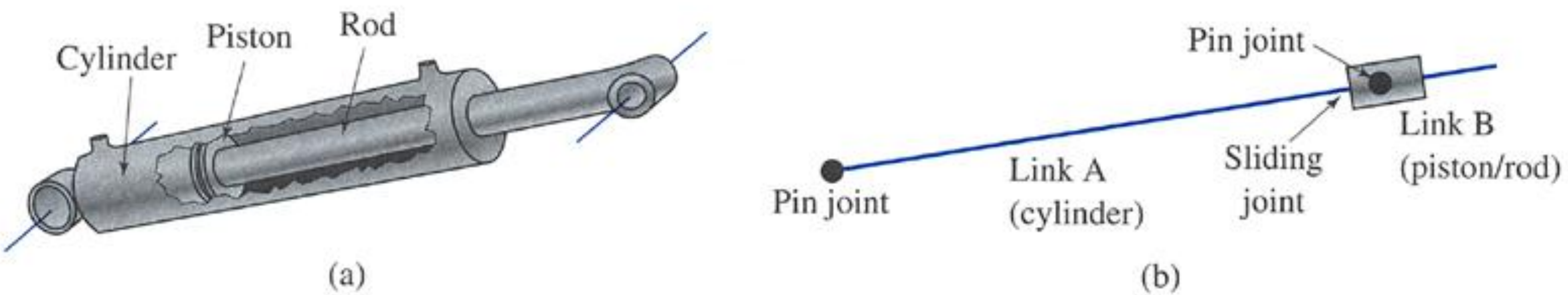


FIGURE 1.20 Hydraulic cylinder.

EXAMPLE PROBLEM 1.7

Figure 1.26 presents a lift table used to adjust the working height of different objects. Draw a kinematic diagram and compute the degrees of freedom.

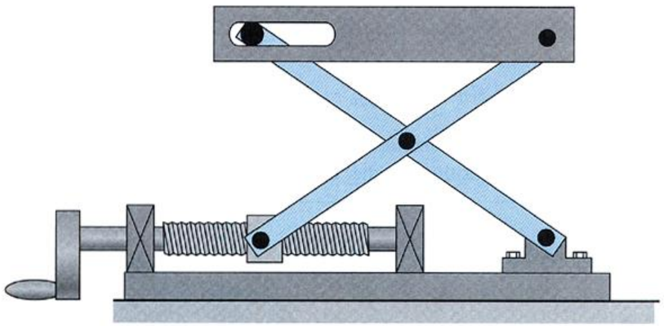


FIGURE 1.26 Lift table for Example Problem 1.7.

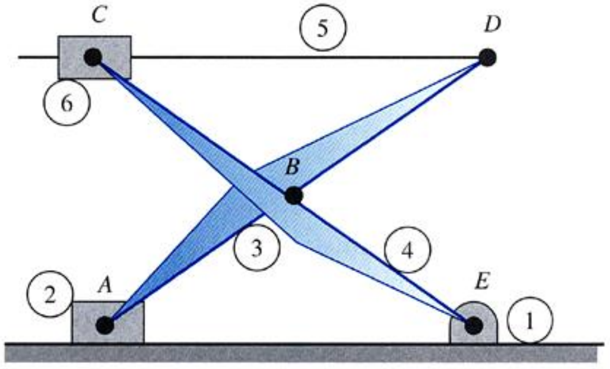


FIGURE 1.27 Kinematic diagram for Example Problem 1.7.

SOLUTION:

1. ***Identify the Frame***
2. ***Identify All Other Links***

   Link 2: Nut

   Link 3: Support arm that ties the nut to the table

   Link 4: Support arm that ties the fixed bearing to the slot in the table

   Link 5: Table

   Link 6: Extra link used to model the pin in slot joint with separate pin and slider joints
3. ***Identify the Joints***
4. ***Draw the Kinematic Diagram***

   The kinematic diagram is given in Figure 1.27.
5. ***Calculate Mobility***

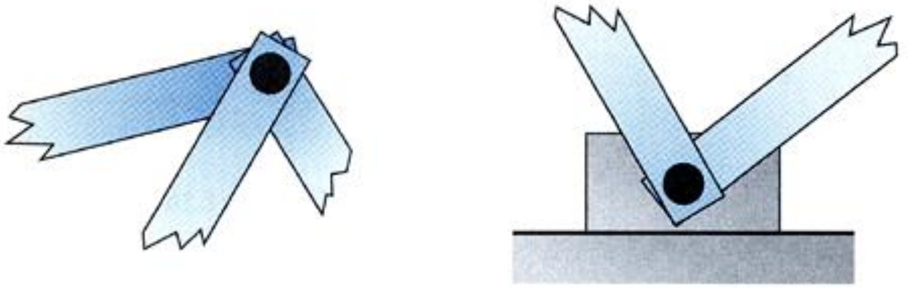
   $n = 6 \quad j_p = (5\text{ pins} + 2\text{ sliders}) = 7 \quad j_h = 0$

   and

   $$M = 3(n-1) - 2j_p - j_h = 3(6-1) - 2(7) - 0 = 15 - 14 = 1$$

1.9 SPECLAL CASES OF THE MOBILITY EQUATION

Coincident Joints



(a) Three rotating links (b) Two rotating and one sliding link

FIGURE 1.28 Three links connected at a common pin joint.

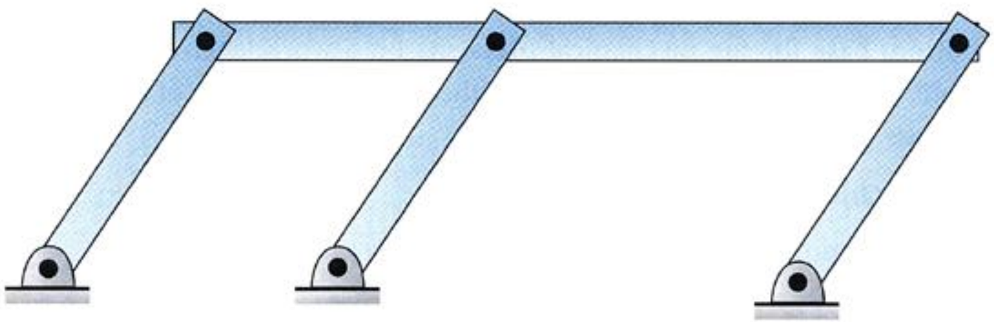


FIGURE 1.31 Mechanism that violates the Gruebler's equation.

- One degree of freedom actually if pivoted links are the same size

EXAMPLE PROBLEM 1.8

Figure 1.29 shows a mechanical press used to exert large forces to insert a small part into a larger one. Draw a kinematic diagram, using the end of the handle as a point of interest. Also compute the degrees of freedom.

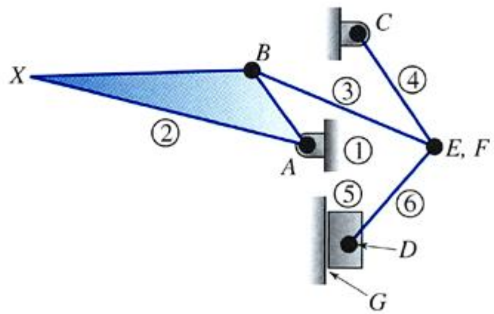


FIGURE 1.30 Kinematic diagram for Example Problem 1.8.

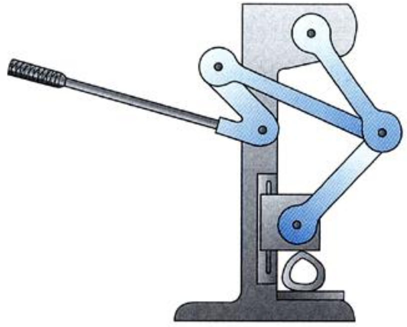


FIGURE 1.29 Mechanical press for Example Problem 1.8.

SOLUTION:

1. ***Identify the Frame***
2. ***Identify All Other Links***

    Link 2: Handle

    Link 3: Arm that connects the handle to the other arms

    Link 4: Arm that connects the base to the other arms

    Link 5: Press head

    Link 6: Arm that connects the head to the other arms

3. ***Identify the Joints***
4. ***Identify Any Points of Interest***
5. ***Draw the Kinematic Diagram***
6. ***Calculate Mobility***

To calculate the mobility, it was determined that there are six links in this mechanism, as well as six pin joints and one slider joint. Therefore,

$$n = 6, j_p = (6 \text{ pins} + 1 \text{ slider}) = 7, j_h = 0$$

and

$$M = 3(n - 1) - 2j_p - j_h = 3(6 - 1) - 2(7) - 0 = 15 - 14 = 1$$

1.10 THE FOUR-BAR MECHANISM

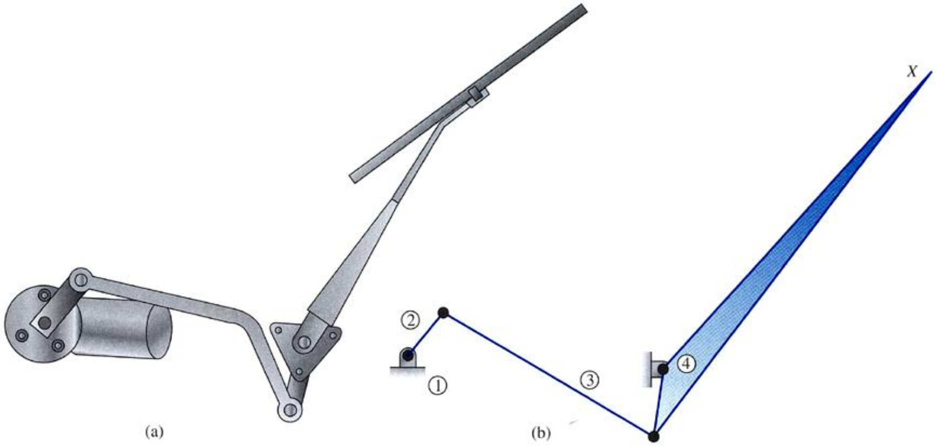


FIGURE 1.33 Rear-window wiper mechanism.

The mobility of a four-bar mechanism consists of the following:

$$n = 4, j_p = 4 \text{ pins}, j_h = 0$$

and

$$M = 3(n - 1) - 2j_p - j_h = 3(4 - 1) - 2(4) - 0 = 1$$

Crank and Rocker

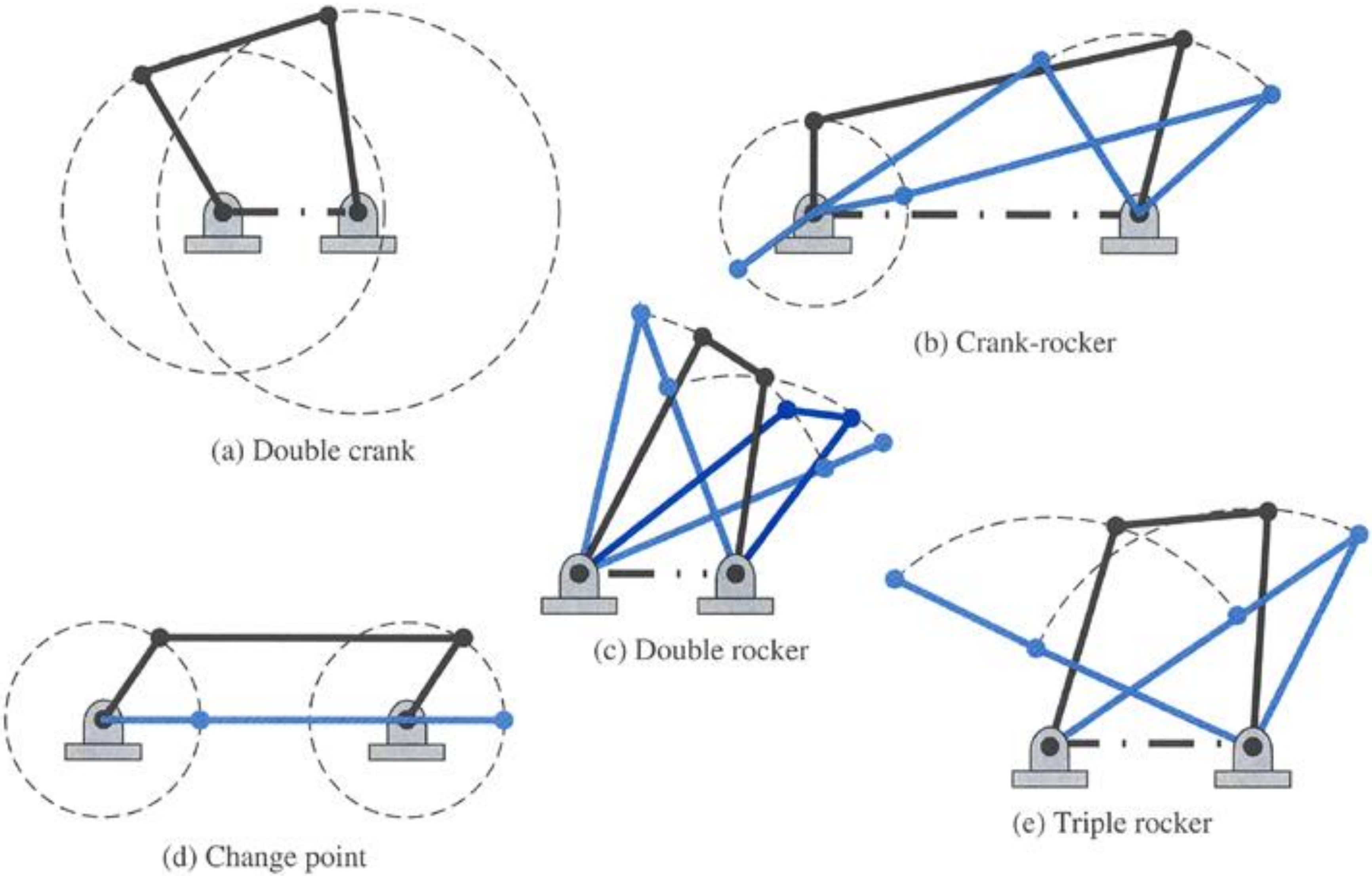


FIGURE 1.34 Categories of four-bar mechanisms.

1.10.1 Design of Crank and Rocker

A four-bar mechanism has at least one revolving link if: $s + l \leq p + q$
Conversely, the three nonfixed links will merely rock if: $s + l > p + q$

s : short link
l : long link
p , q : intermediate link

TABLE 1.2 Categories of Four-Bar Mechanisms

Case	Criteria	Shortest Link	Category
1	$s + l < p + q$	Frame	Double crank
2	$s + l < p + q$	Side	Crank-rocker
3	$s + l < p + q$	Coupler	Double rocker
4	$s + l = p + q$	Any	Change point
5	$s + l > p + q$	Any	Triple rocker

1.11 SLIDER-CRANK MECHANISM

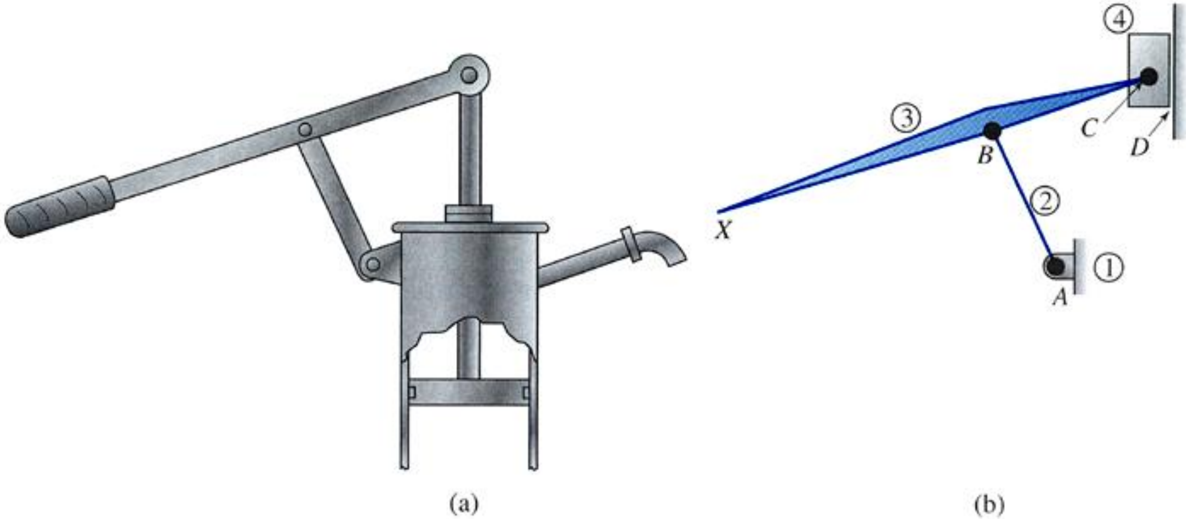


FIGURE 1.37 Pump mechanism for a manual water pump: (a) Mechanism and (b) Kinematic diagram.

The mobility of a slider-crank mechanism is represented by the following:

$$n = 4, j_p = (3 \text{ pins} + 1 \text{ sliding}) = 4, j_h = 0$$

and

$$M = 3(n - 1) - 2j_p - j_h = 3(4 - 1) - 2(4) - 0 = 1.$$

1.12 SPECIAL PURPOSE MECHANISMS

1.12.1 Straight-Line Mechanisms

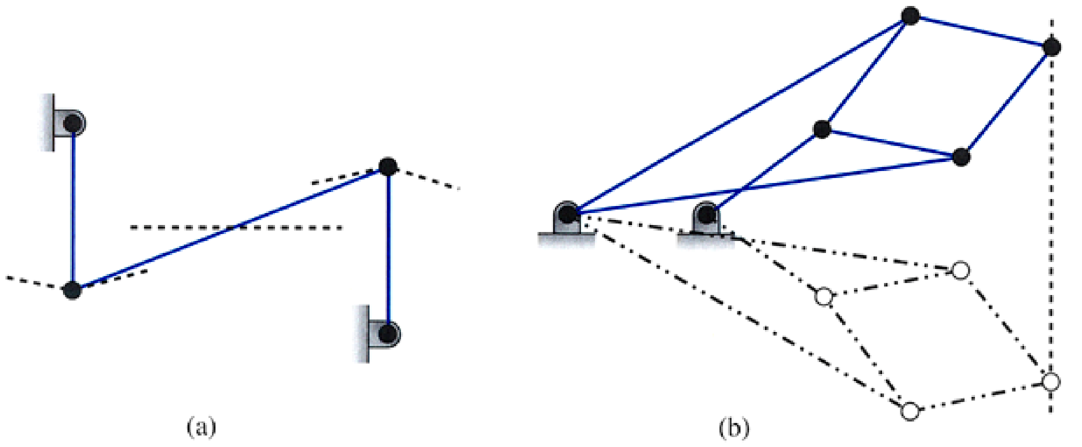


FIGURE 1.38 Straight-line mechanisms

1.12.2 Parallelogram Mechanisms

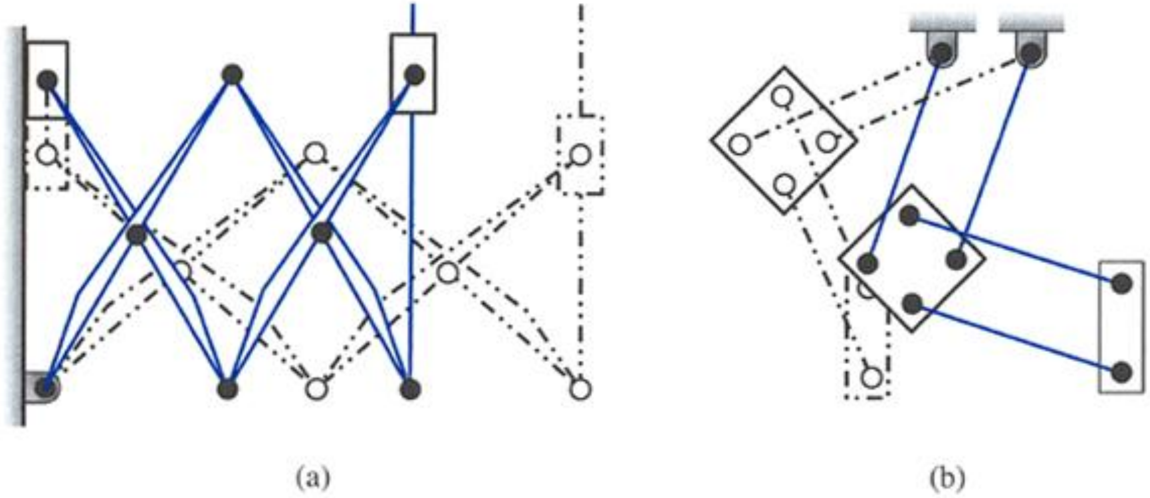
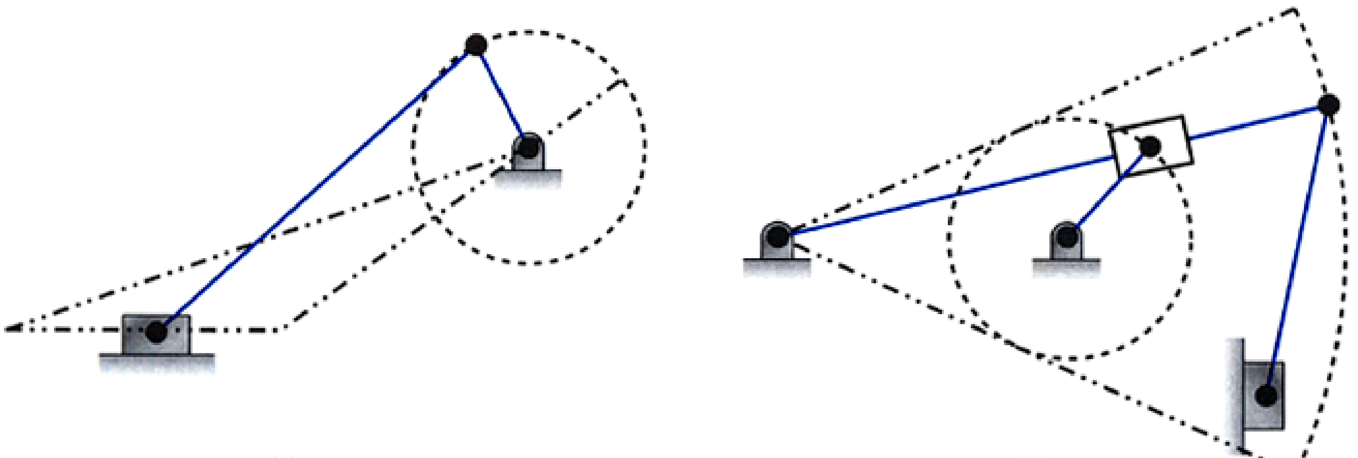


FIGURE 1.39 Parallelogram mechanisms.

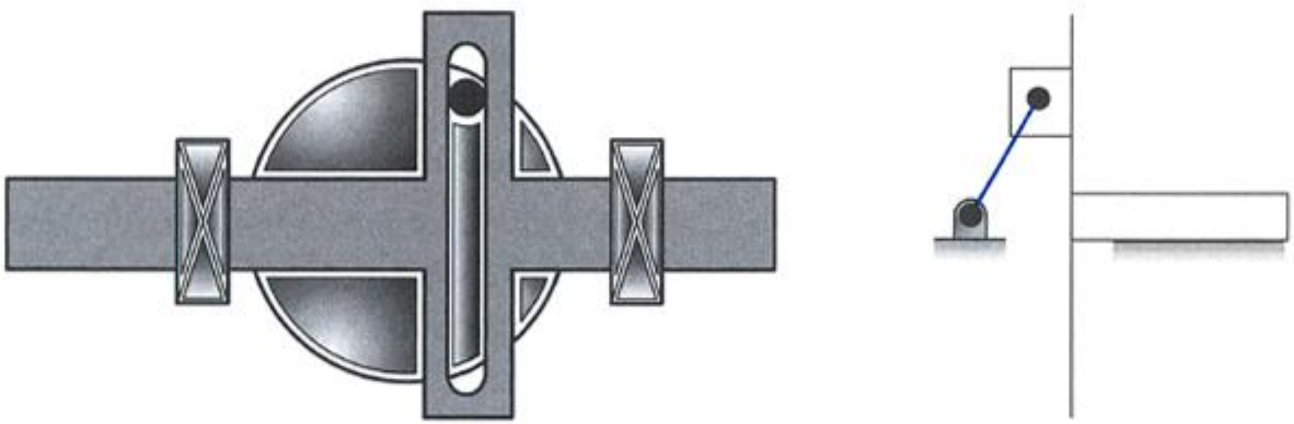
1.12.3 Quick-Return Mechanisms



(a) (b)

FIGURE 1.40 Quick-return mechanisms.

1.12.4 Scotch Yoke Mechanism



(a) Actual mechanism (b) Kinematic diagram

FIGURE 1.41 Scotch yoke mechanism.

4.4 DISPLACEMENT ANALYSIS

- Locate the positions of all links as driver link is displaced
- Configuration
  - Positions of all the links
- One degree of freedom
  - Moving one link will precisely position all other links

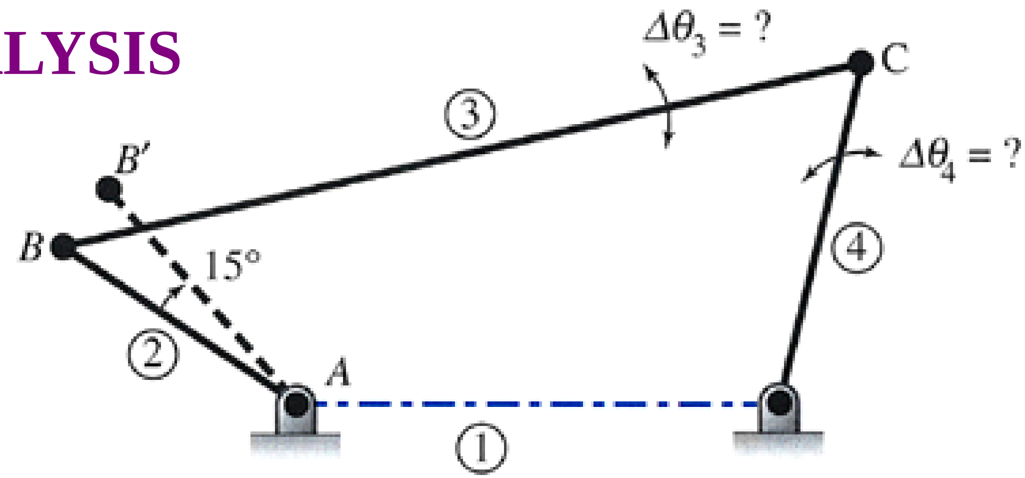


FIGURE 4.5 Typical position analysis.

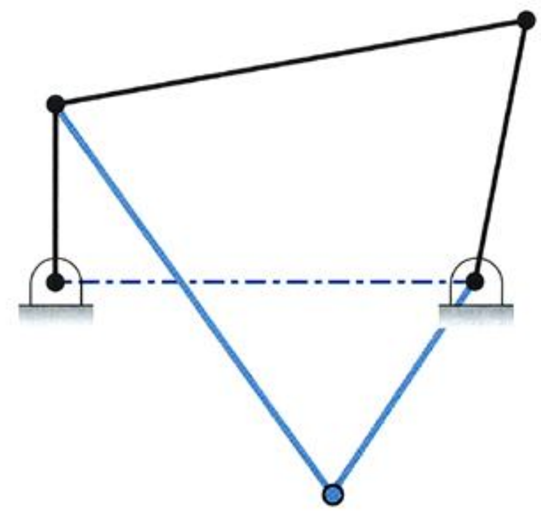


FIGURE 4.6 Two geometric inversions of a four-bar mechanism.

EXAMPLE PROBLEM 4.1

Figure 4.11 shows a kinematic diagram of a mechanism that is driven by moving link 2. Graphically reposition the links of the mechanism as link 2 is displaced 30° counterclockwise. Determine the resulting angular displacement of link 4 and the linear displacement of point E.

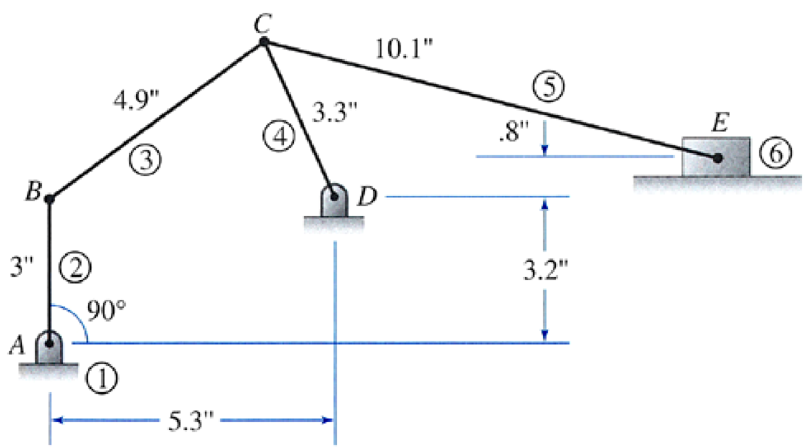


FIGURE 4.11 Kinematic diagram for Example Problem 4.1.

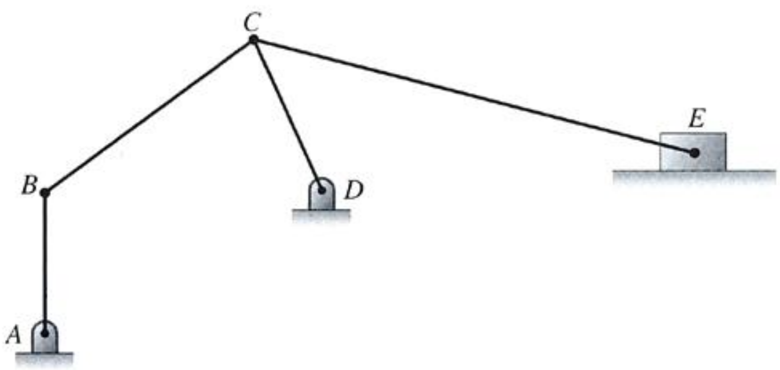
SOLUTION: 1. ***Calculate Mobility***

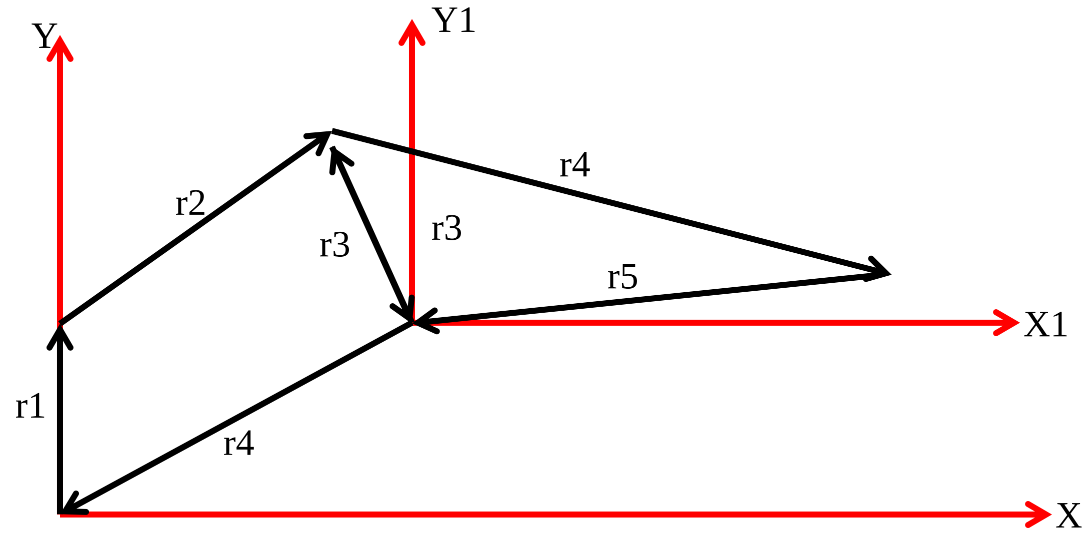
$$n = 6\ j_p = (6 \text{ pins} + 1 \text{ sliding}) = 7\, j_h = 0$$

and

$$M = 3(n-1) - 2j_p - j_h = 3(6-1) - 2(7) - 0 = 15 - 14 = 1$$

4.1 Vector Analysis of Displacement





(1) $\underset{\sim}{r}_1 + \underset{\sim}{r}_2 + \underset{\sim}{r}_3 + \underset{\sim}{r}_4 = 0$

$$\begin{bmatrix} -r_1 s\theta_1 \\ +r_1 c\theta_1 \end{bmatrix} + \begin{bmatrix} r_2 c\theta_2 \\ r_2 s\theta_2 \end{bmatrix} + \begin{bmatrix} r_3 c\theta_3 \\ r_3 s\theta_3 \end{bmatrix} + \begin{bmatrix} -5.3 \\ -3.2 \end{bmatrix} = 0$$

$r_1 = 3,\ \theta_1 = 30,\ r_2 = 4.9,\ r_3 = 3.3$

2 *equations* for 2 *unknows* $\theta_2,\ \theta_3$

(2) $-\underset{\sim}{r}_3 + \underset{\sim}{r}_4 + \underset{\sim}{r}_5 = 0$

$$\begin{bmatrix} -r_3 c\theta_3 \\ -r_3 s\theta_3 \end{bmatrix} + \begin{bmatrix} r_4 c\theta_4 \\ r_4 s\theta_4 \end{bmatrix} + \begin{bmatrix} x_1 \\ 0.8 \end{bmatrix} = 0$$

$r_1 = 10.1$

2 *equations* for 2 *unknows* θ_4 *and* x_1

EXAMPLE PROBLEM 4.6

The mechanism shown in Figure 4.26 is the driving linkage for a reciprocating saber saw. Determine the configurations of the mechanism that places the saw blade in its limiting positions.

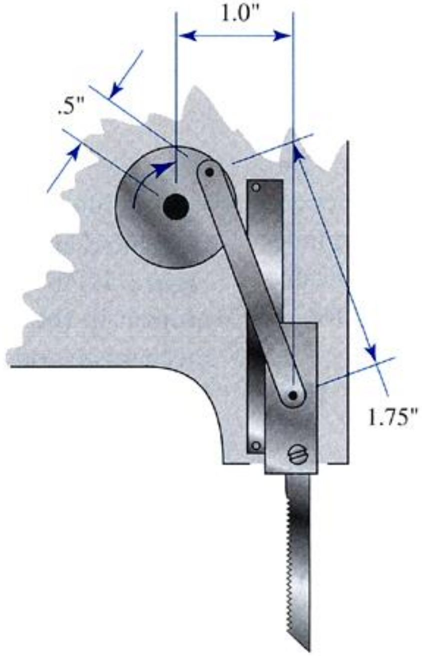
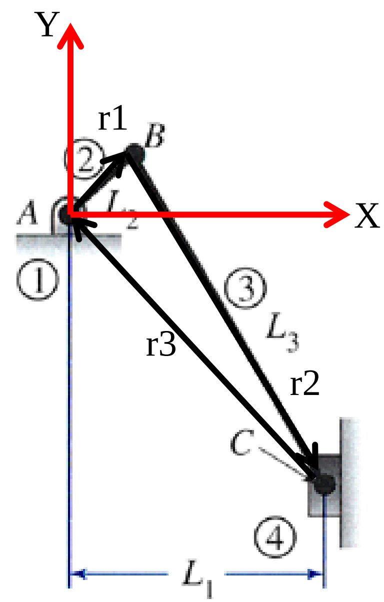


FIGURE 4.26 Saber saw mechanism for Example Problem 4.6.



$$\underset{\sim}{r}_1 + \underset{\sim}{r}_2 + \underset{\sim}{r}_3 = 0$$

$$\begin{bmatrix} 0.5c\theta_1 \\ 0.5s\theta_1 \end{bmatrix} + \begin{bmatrix} 1.75c\theta_2 \\ 1.75s\theta_2 \end{bmatrix} + \begin{bmatrix} 1 \\ y \end{bmatrix} = 0$$

$$for \;\; \theta_1 = \theta_2, \quad solve \;\; for \;\; \theta_1 \;\; and \;\; y_{\max}$$

$$for \;\; \theta_1 = \theta_2 + \pi, \; solve \;\; for \;\; \theta_1 \;\; and \;\; y_{\min}$$

4.7 Limiting Positions and Stroke

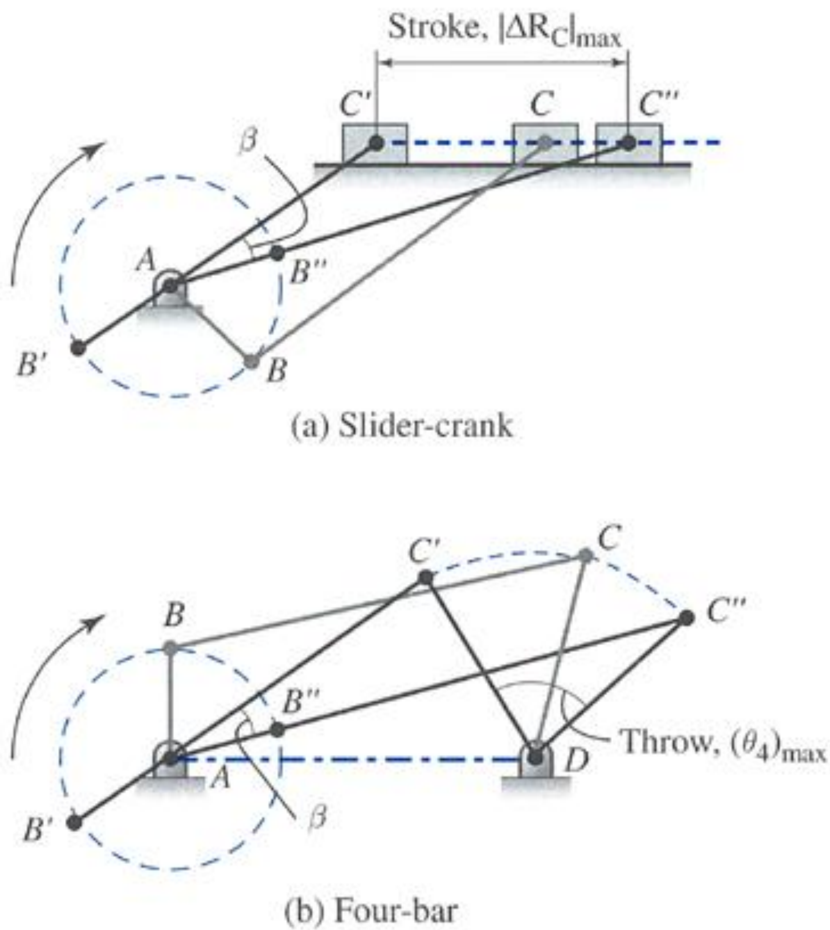


FIGURE 4.25 Limiting positions.

6.2 LINEAR AND ANGULAR VELOCITY

Mathematically, linear velocity of a point is expressed as

$$\mathbf{V} = \lim_{\Delta t \to 0} \frac{d\mathbf{R}}{dt} \quad (6.1)$$

and for short time periods as

$$\mathbf{V} \cong \frac{\Delta \mathbf{R}}{\Delta t} \quad (6.2)$$

$$\omega = \lim_{\Delta t \to o} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt} \quad (6.4)$$

$$\omega \cong \frac{\Delta \theta}{\Delta t} \quad (6.5)$$

6.2.2 Linear Velocity of a General Point

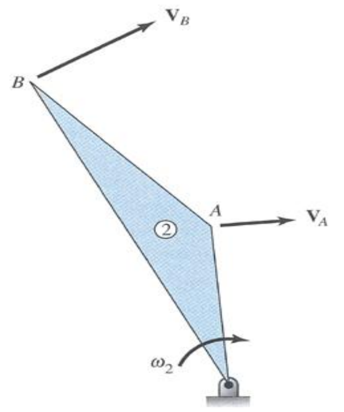


FIGURE 6.2 Linear velocities of points on a link.

EXAMPLE PROBLEM 6.6

Figure 6.11 shows a rock-crushing mechanism. It is used in a machine where large rock is placed in a vertical hopper and falls into this crushing chamber. Properly sized aggregate, which passes through a sieve, is discharged at the bottom. Rock not passing through the sieve is reintroduced into this crushing chamber.

Determine the angular velocity of the crushing ram, in the shown configuration, as the 60-mm crank rotates at 120 rpm, clockwise.

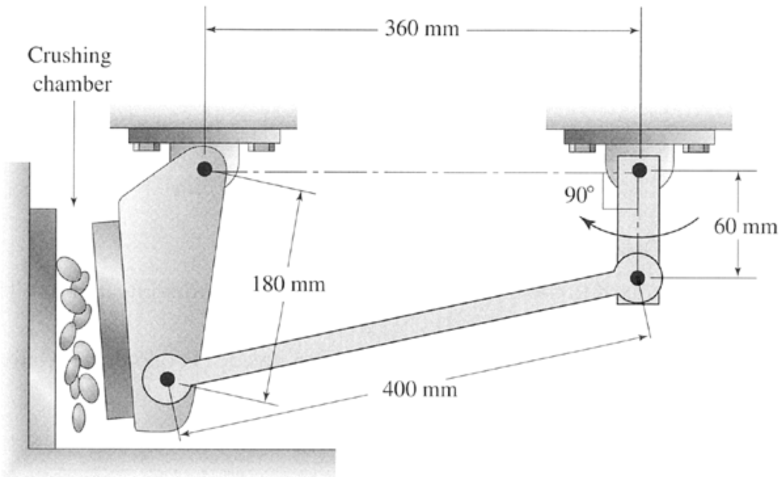
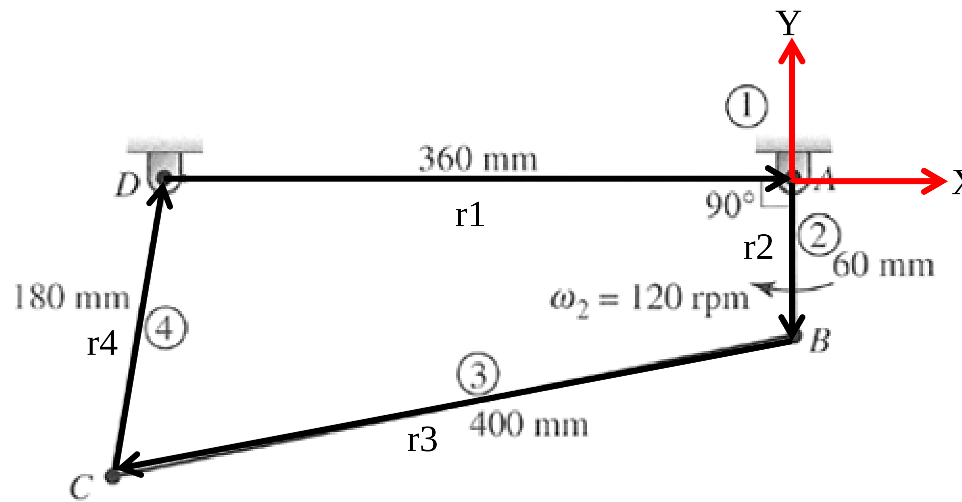


FIGURE 6.11 Mechanism for Example Problem 6.6.



$$\underset{\sim}{r}_1 + \underset{\sim}{r}_2 + \underset{\sim}{r}_3 + \underset{\sim}{r}_4 = 0$$

$$\underset{\sim}{\omega}_2 \times \underset{\sim}{r}_2 + \underset{\sim}{\omega}_3 \times \underset{\sim}{r}_3 + \underset{\sim}{\omega}_4 \times \underset{\sim}{r}_4 = 0$$

2 *eqs for* 2 *unknowns* ω_3 *and* ω_4

6.6 GRAPHICAL VELOCITY ANALYSIS:RELATIVE VELOCITY METHOD

6.6.1 Points on Links Limited to Pure Rotation or Rectilinear Translation

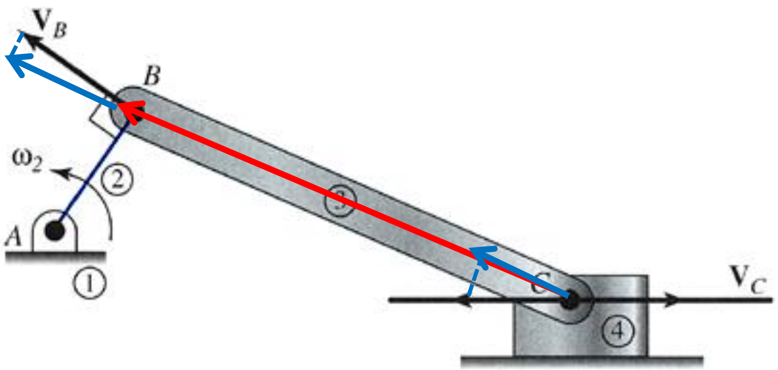
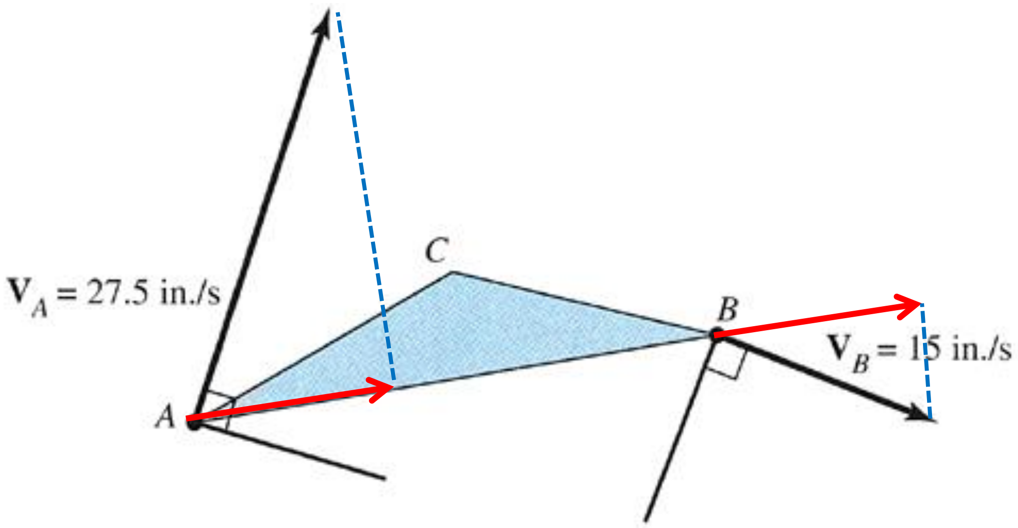


FIGURE 6.10 Links constrained to pure rotation and rectilinear translation.

6.6.2 General Points on a Floating Link



EXAMPLE PROBLEM 6.7

Figure 6.14 illustrates a mechanism that extends reels of cable from a delivery truck. It is operated by a hydraulic cylinder at *A*. At this instant, the cylinder retracts at a rate of 5 mm/s. Determine the velocity of the top joint, point *E*.

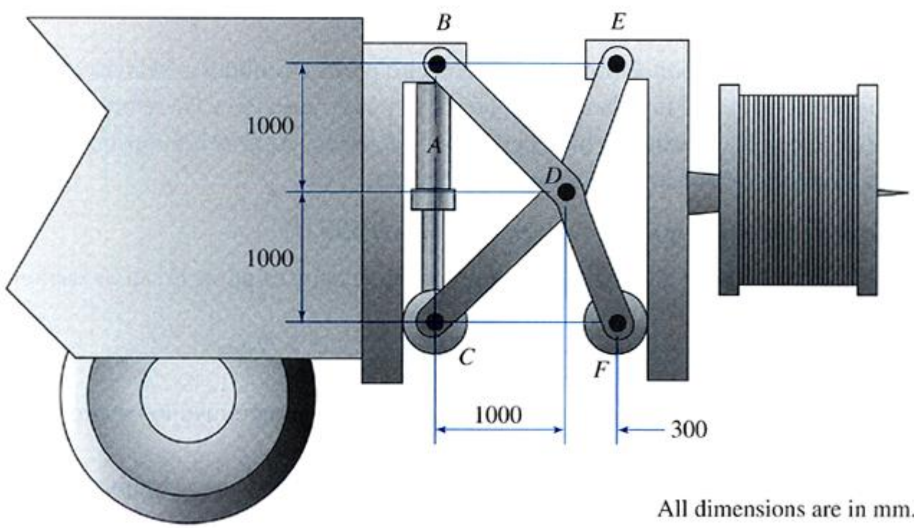
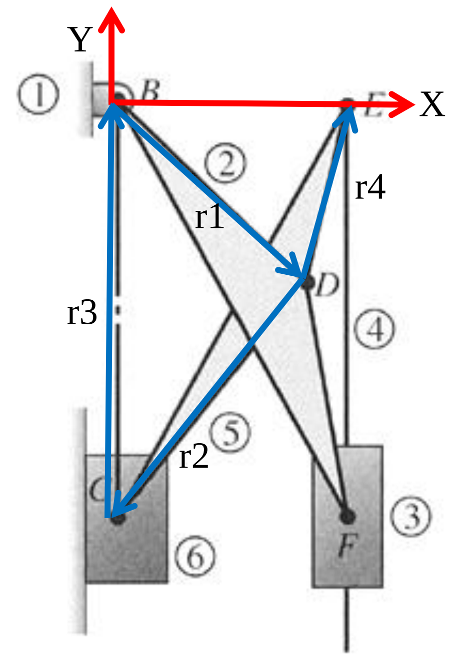


FIGURE 6.14 Mechanism for Example Problem 6.7.



$$\underset{\sim}{r}_1+\underset{\sim}{r}_2+\underset{\sim}{r}_3=0$$

$$\underset{\sim}{\omega}_1\times\underset{\sim}{r}_1+\underset{\sim}{\omega}_2\times\underset{\sim}{r}_2+\begin{bmatrix}0\\5\end{bmatrix}=0$$

$$2\ \ eqs\ \ for\ \ \omega_1\ \ and\ \ \omega_2$$

$$\underset{\sim}{r}=\underset{\sim}{r}_1+\underset{\sim}{r}_4$$

$$\dot{\underset{\sim}{r}}=\underset{\sim}{\omega}_1\times\underset{\sim}{r}_1+\underset{\sim}{\omega}_2\times\underset{\sim}{r}_4$$

EXAMPLE PROBLEM 6.8

Figure 6.16 shows a mechanism that tips the bed of a dump truck. Determine the required speed of the hydraulic cylinder in order to tip the truck at a rate of 5 rad/min.

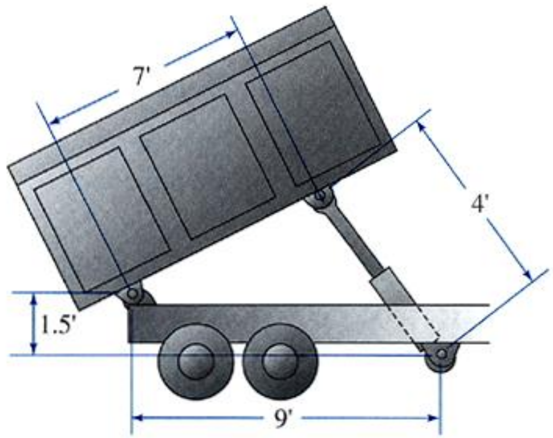
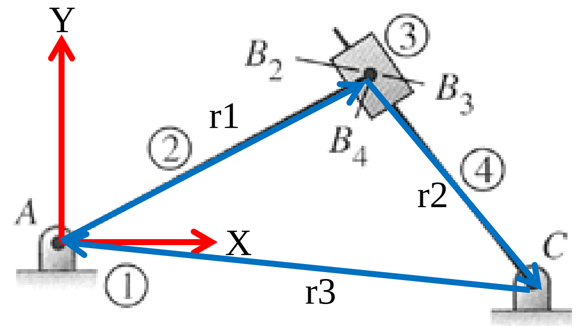


FIGURE 6.16 Dump truck mechanism for Example Problem 6.8.



$$\underset{\sim}{r}_1 + \underset{\sim}{r}_2 + \underset{\sim}{r}_3 = 0$$

$$solve \;\; for \;\; \theta_1 \;\; and \;\; \theta_2$$

$$\underset{\sim}{\omega}_1 \times \underset{\sim}{r}_1 + \underset{\sim}{\omega}_2 \times \underset{\sim}{r}_2 + \dot{r}_2 = 0$$

$$\omega_1 = 5\, rad/\min \,, \;\; \underset{\sim}{\dot{r}}_2 = \begin{bmatrix} vc\theta_2 \\ vs\theta_2 \end{bmatrix}$$

$$2 \;\; eqs \;\; for \;\; 2 \;\; unknowns \;\; \omega_2 \;\; and \;\; v$$

EXAMPLE PROBLEM 6.10

Figure 6.21 illustrates a roofing material delivery truck conveyor. Heavy roofing materials can be transported on the conveyor to the roof. The conveyor is lifted into place by extending the hydraulic cylinder. At this instant, the cylinder is extending at a rate of 8 fpm (ft/min). Determine the rate that the conveyor is being lifted.

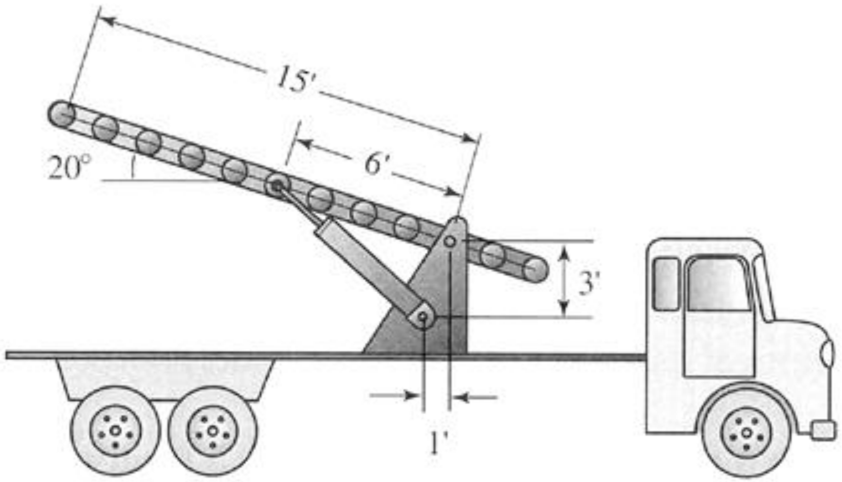
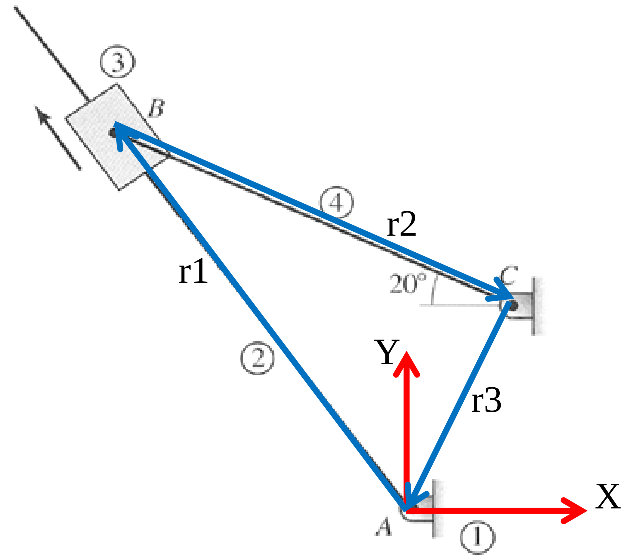


FIGURE 6.21 Conveyor for Example Problem 6.10.



$$\underset{\sim}{r}_1+\underset{\sim}{r}_2+\underset{\sim}{r}_3=0$$

$$r_2=6,\ \theta_2=340^\circ,\ \underset{\sim}{r}_3=\begin{bmatrix}-1\\-3\end{bmatrix}$$

$$2\ eqs\ \ for\ \ 2\ \ unknowns\ \ r_1\ \ and\ \ \theta_1$$

$$\underset{\sim}{\omega}_1\times\underset{\sim}{r}_1+\underset{\sim}{\omega}_2\times\underset{\sim}{r}_2+\underset{\sim}{\dot{r}}_2=0$$

$$\underset{\sim}{\dot{r}}_2=\begin{bmatrix}8c\theta_2\\8s\theta_2\end{bmatrix}$$

$$2\ eqs\ \ for\ \ 2\ \ unknowns\ \ \omega_1\ \ and\ \ \omega_2$$

EXAMPLE PROBLEM 6.16

Figure 6.38 shows a mechanism used in a production line to turn over cartons so that labels can be glued to the bottom of the carton. The driver arm is 15 in. long and, at the instant shown, it is inclined at a 60° angle with a clockwise angular velocity of 5 rad/s. The follower link is 16 in. long. The distance between the pins on the carriage is 7 in., and they are currently in vertical alignment. Determine the angular velocity of the carriage and the slave arm.

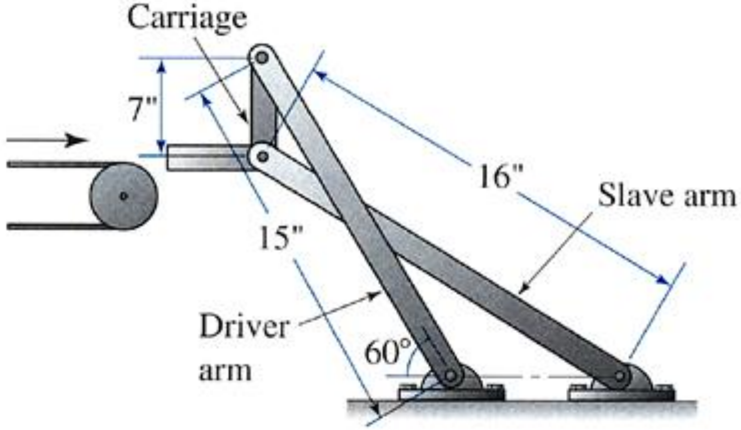
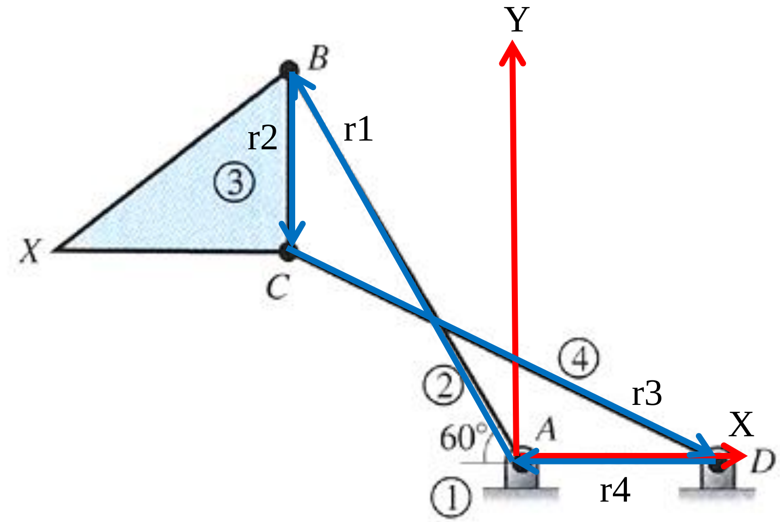


FIGURE 6.38 Turnover mechanism for Example Problem 6.16.



$$\underset{\sim}{r}_1+\underset{\sim}{r}_2+\underset{\sim}{r}_3+\underset{\sim}{r}_4=0$$

$$\underset{\sim}{\omega}_1\times\underset{\sim}{r}_1+\underset{\sim}{\omega}_2\times\underset{\sim}{r}_2+\underset{\sim}{\omega}_3\times\underset{\sim}{r}_3=0$$

$$given\ \ \underset{\sim}{\omega}_1\ \ find\ \ \omega_2\ \ and\ \ \omega_3$$

Chapter 7 Acceleration Analysis

7.2 LINEAR ACCELERATION

7.2.1 Linear Acceleration of Rectilinear Points

$$\mathbf{A} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{V}}{\Delta t} = \frac{\mathbf{dv}}{\mathbf{dt}} \quad (7.1)$$

$$\mathbf{V} = \frac{d\mathbf{R}}{dt}$$

$$\mathbf{A} = \frac{d^2\mathbf{R}}{dt^2} \quad (7.2)$$

$$\mathbf{A} \cong \frac{\Delta \mathbf{V}}{\Delta t} \quad (7.3)$$

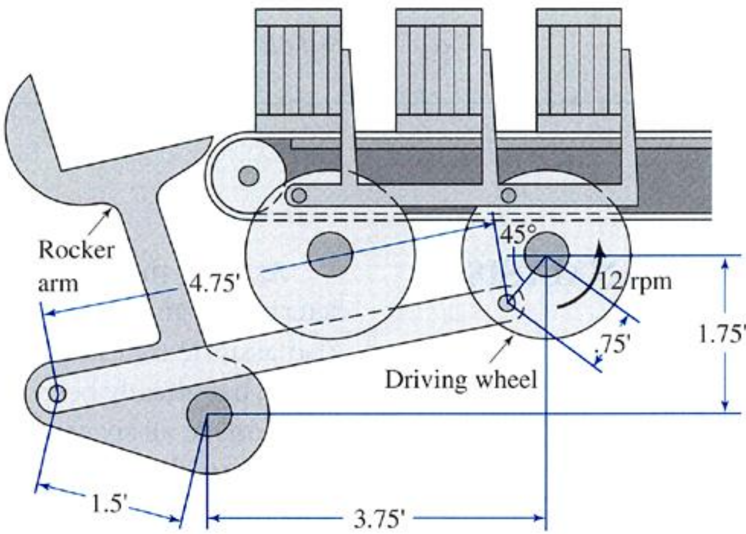
$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt} \quad (7.7)$$

$$\alpha = \frac{d^2\theta}{dt^2} \quad (7.8)$$

$$\alpha \cong \frac{\Delta \omega}{\Delta t} \quad (7.9)$$

EXAMPLE PROBLEM 7.7

The mechanism shown in Figure 7.11 is designed to move parts along a conveyor tray and then rotate and lower those parts to another conveyor. The driving wheel rotates with a constant angular velocity of 12 rpm. Determine the angular acceleration of the rocker arm that rotates and lowers the parts.



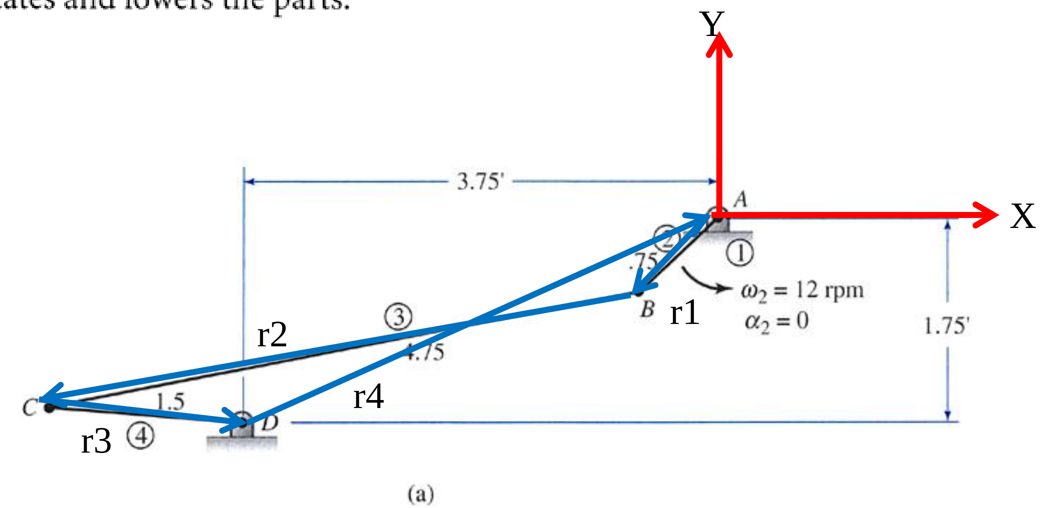


FIGURE 7.11 Mechanism for Example Problem 7.7.

$$\underset{\sim}{r}_1+\underset{\sim}{r}_2+\underset{\sim}{r}_3+\underset{\sim}{r}_4=0$$

$$\underset{\sim}{\omega}_1\times\underset{\sim}{r}_1+\underset{\sim}{\omega}_2\times\underset{\sim}{r}_2+\underset{\sim}{\omega}_3\times\underset{\sim}{r}_3=0$$

$$given\ \ \omega_1\ \ find\ \ \omega_2\ \ and\ \ \omega_3$$

$$\underset{\sim}{\omega}_1\times\left(\underset{\sim}{\omega}_1\times\underset{\sim}{r}_1\right)+\dot{\underset{\sim}{\omega}}_2\times\underset{\sim}{r}_2+\underset{\sim}{\omega}_2\times\left(\underset{\sim}{\omega}_2\times\underset{\sim}{r}_2\right)+\dot{\underset{\sim}{\omega}}_3\times\underset{\sim}{r}_3+\underset{\sim}{\omega}_3\times\left(\underset{\sim}{\omega}_3\times\underset{\sim}{r}_3\right)=0$$

$$solve\ \ for\ \ \dot{\omega}_2\ \ and\ \ \underset{\sim}{\omega}_3$$

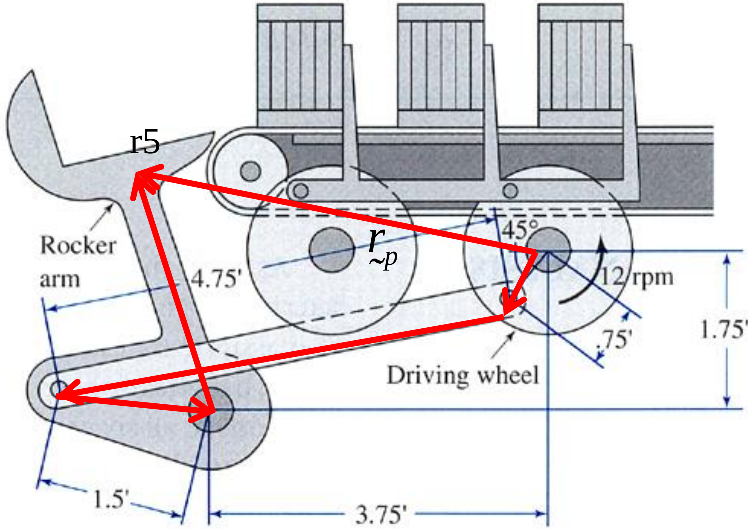


FIGURE 7.11 Mechanism for Example Problem 7.7.

$$r_p = \underset{\sim}{r}_1 + \underset{\sim}{r}_2 + \underset{\sim}{r}_3 + \underset{\sim}{r}_5$$

$$\dot{r}_p = \underset{\sim}{\omega}_1 \times \underset{\sim}{r}_1 + \underset{\sim}{\omega}_2 \times \underset{\sim}{r}_2 + \underset{\sim}{\omega}_3 \times \left(\underset{\sim}{r}_3 + \underset{\sim}{r}_5\right)$$

$$\ddot{r}_p = \underset{\sim}{\omega}_1 \times \left(\underset{\sim}{\omega}_1 \times \underset{\sim}{r}_1\right) + \dot{\underset{\sim}{\omega}}_2 \times \underset{\sim}{r}_2 + \underset{\sim}{\omega}_2 \times \left(\underset{\sim}{\omega}_2 \times \underset{\sim}{r}_2\right) + \dot{\underset{\sim}{\omega}}_3 \times \left(\underset{\sim}{r}_3 + \underset{\sim}{r}_5\right) + \underset{\sim}{\omega}_3 \times \left(\underset{\sim}{\omega}_3 \times \left(\underset{\sim}{r}_3 + \underset{\sim}{r}_5\right)\right)$$

EXAMPLE PROBLEM 7.8

The mechanism shown in Figure 7.13 is a common punch press designed to perform successive stamping operations. The machine has just been powered and at the instant shown is coming up to full speed. The driveshaft rotates clockwise with an angular velocity of 72 rad/s and accelerates at a rate of 250 rad/s^2. At the instant shown, determine the acceleration of the stamping die, which will strike the workpiece.

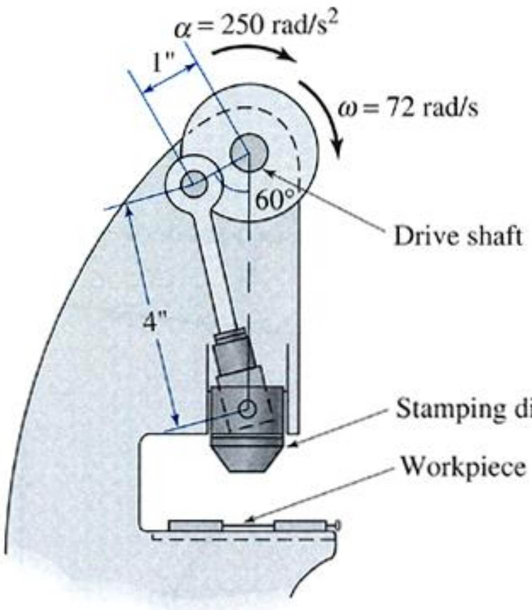


FIGURE 7.13 Mechanism for Example Problem 7.8.

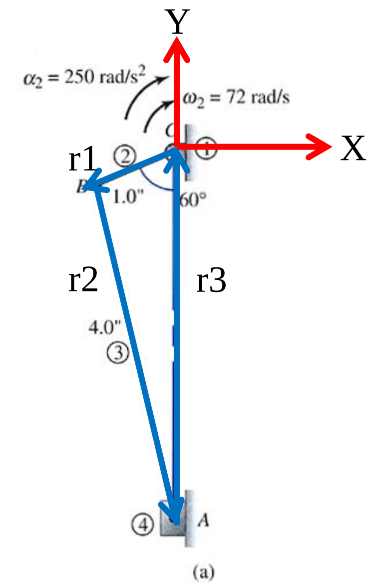


FIGURE 7.14 Diagrams for Example Problem 7.8.

$$\underset{\sim}{r}_1+\underset{\sim}{r}_2+\underset{\sim}{r}_3=0$$

$$\underset{\sim}{\omega}_1\times\underset{\sim}{r}_1+\underset{\sim}{\omega}_2\times\underset{\sim}{r}_2+\begin{bmatrix}0\\v\end{bmatrix}=0$$

solve for ω_2 *and* v

$$\dot{\underset{\sim}{\omega}}_1\times\underset{\sim}{r}_1+\underset{\sim}{\omega}_1\times(\underset{\sim}{\omega}_1\times\underset{\sim}{r}_1)+\dot{\underset{\sim}{\omega}}_2\times\underset{\sim}{r}_2+\underset{\sim}{\omega}_2\times(\underset{\sim}{\omega}_2\times\underset{\sim}{r}_2)+\begin{bmatrix}0\\a\end{bmatrix}=0$$

solve for $\dot{\omega}_2$ *and* a

7.12 EQUIVALENT LINKAGES

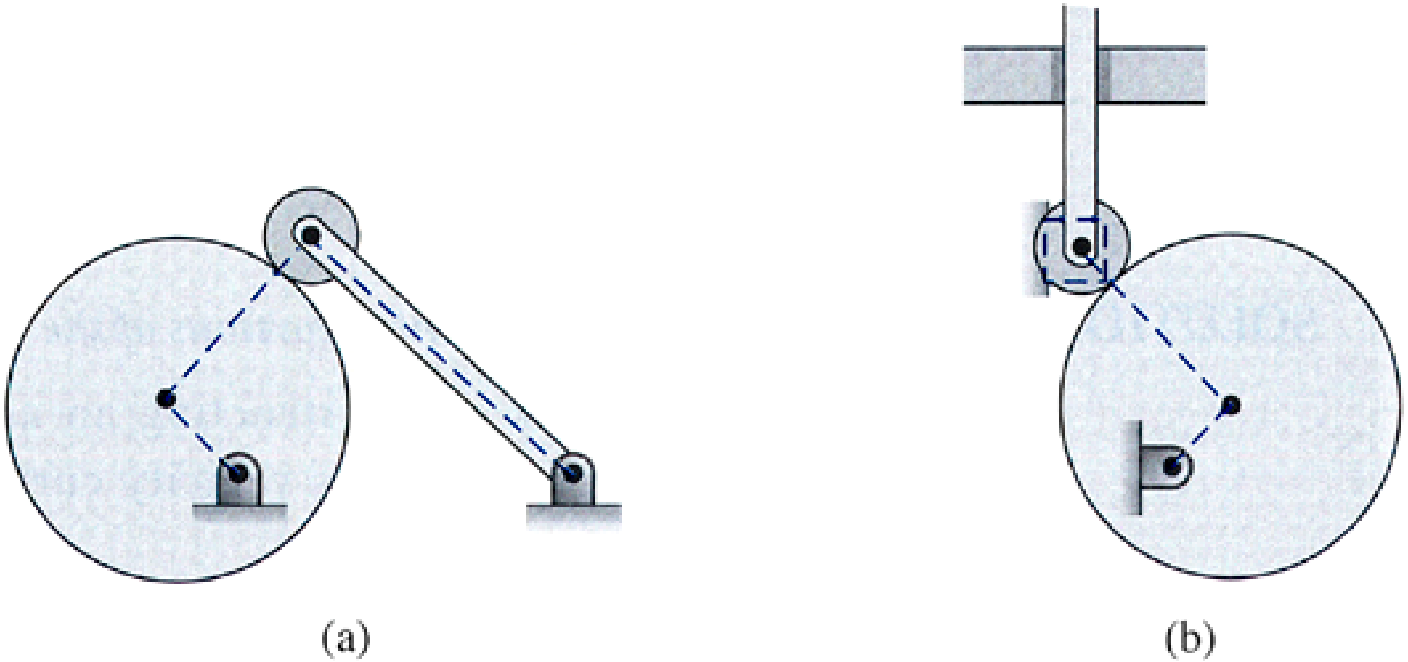
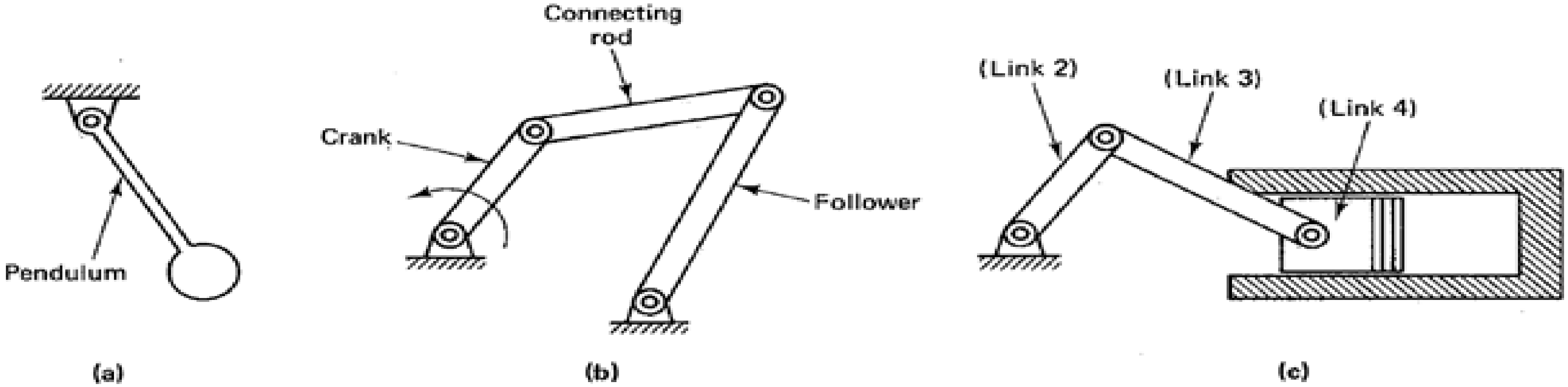


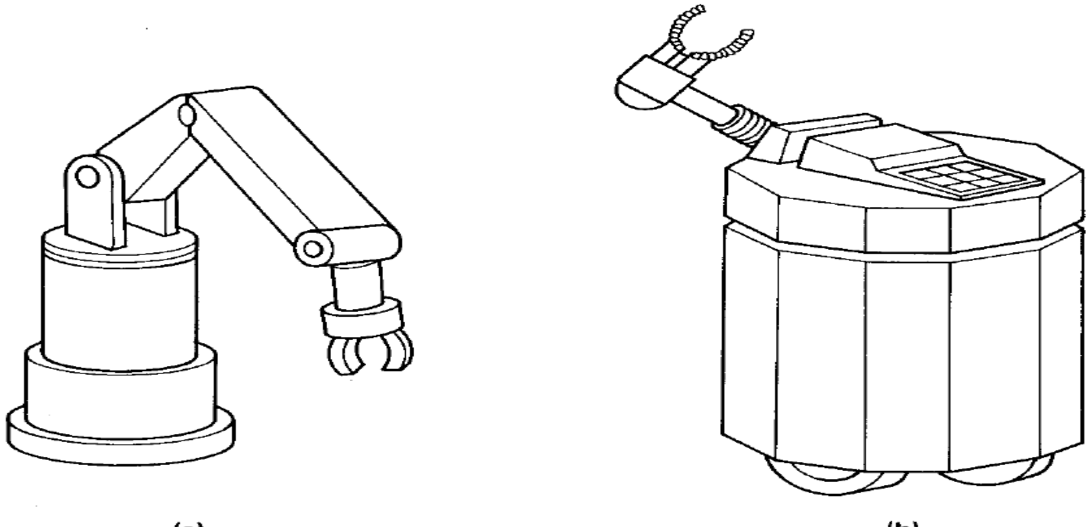
FIGURE 7.25 Equivalent linkages.

Computation of Multibody Kinematics And Dynamics
1.1 Multibody Mechanical Systems

2D Planar Mechanism



3D Spatial Mechanism



Computer-Aided Design (CAD)

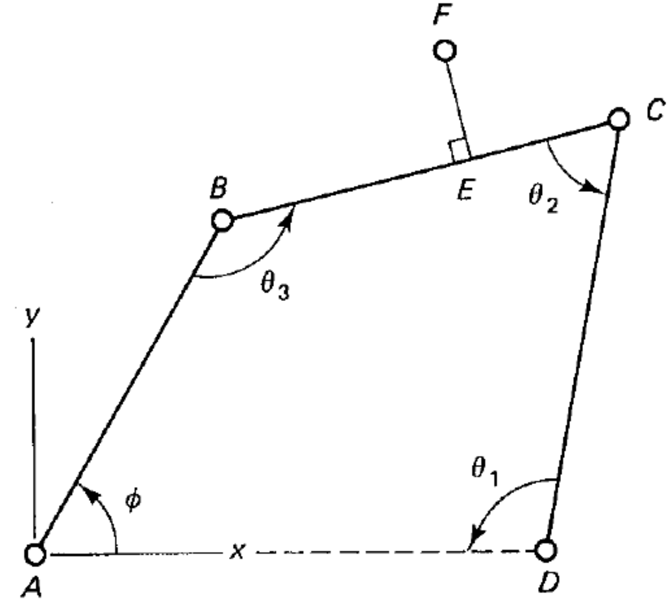
- Mechanics: Statics and dynamics.
- Dynamics : kinematics and kinetics.

- Kinematics is the study of motion, i.e., the study of displacement, velocity, and acceleration, regardless of the forces that produce the motion.

- Kinetics or Dynamics is the study of motion and its relationship with the forces that produce that motion.

1.2 Coordinate Systems

A four-bar mechanism with generalized coordinates.



4 Coordinate

$$\boldsymbol{q} = [\theta_1 \ \theta_2 \ \theta_3 \ \phi]^T$$

3 Constraints

$$(r^2 + l^2 + s^2 - d^2) - 2rl\cos\phi + 2ls\cos\theta_1 - 2rs\cos(\phi - \theta_1) = 0$$

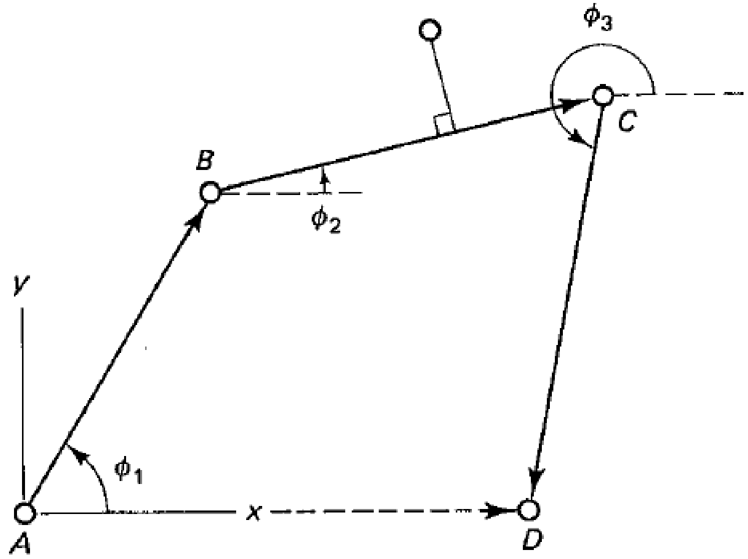
$$(r^2 + l^2 + s^2 - d^2) - 2rl\cos\phi + 2ds\cos\theta_2 = 0$$

$$\phi + \theta_1 + \theta_2 + \theta_3 - 2\pi = 0$$

degrees of freedom 4 – 3 = 1

Generalized Coordinates

$$\boldsymbol{q} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}$$



2 constraints

$$r\cos\phi_1 + d\cos\phi_2 + s\cos\phi_3 - l = 0$$

$$r\sin\phi_1 + d\sin\phi_2 + s\sin\phi_3 = 0$$

$$\text{dof} = 3 - 2 = 1$$

Cartesian Coordinates

12 coordinates and 11 kinematic constraints

$$\boldsymbol{q} = \begin{bmatrix} x_1 & y_1 & \phi_1 & x_2 & y_2 & \phi_2 & x_3 & y_3 & \phi_3 \end{bmatrix}^T$$

$$x_1 - \frac{r}{2}\cos\phi_1 = 0$$

$$y_1 - \frac{r}{2}\sin\phi_1 = 0$$

$$x_1 + \frac{r}{2}\cos\phi_1 - x_2 + \frac{d}{2}\cos\phi_2 = 0$$

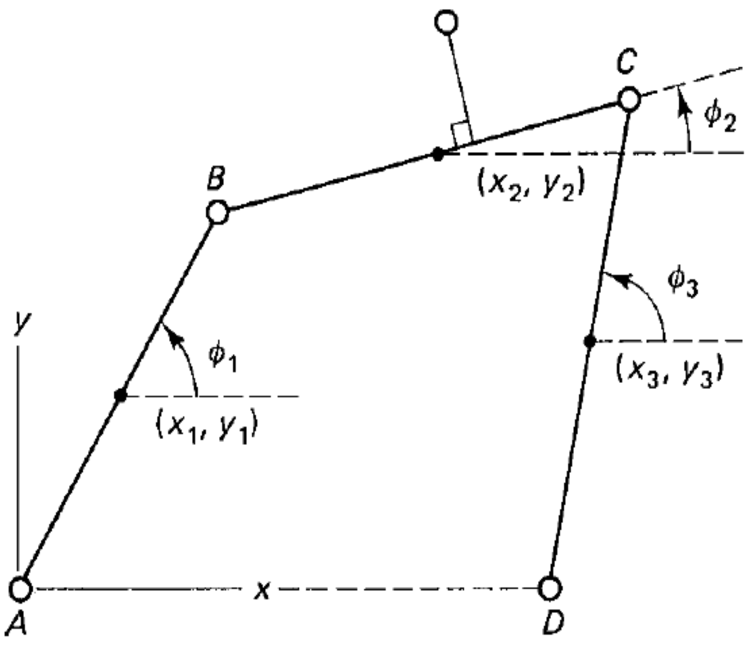
$$y_1 + \frac{r}{2}\sin\phi_1 - y_2 - \frac{d}{2}\sin\phi_2 = 0$$

$$x_2 + \frac{d}{2}\cos\phi_2 - x_3 - \frac{s}{2}\cos\phi_3 = 0$$

$$y_2 + \frac{d}{2}\sin\phi_2 - y_3 - \frac{s}{2}\sin\phi_3 = 0$$

$$x_3 - \frac{s}{2}\cos\phi_3 - l = 0$$

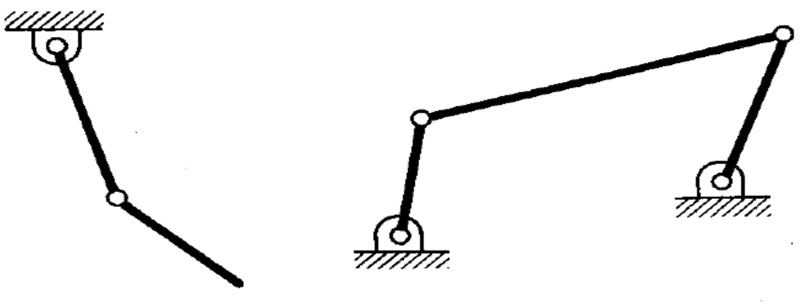
$$y_3 - \frac{s}{2}\sin\phi_3 = 0$$



$$\text{dof} = 9 - 8 = 1$$

1.3 Computation Kinematics

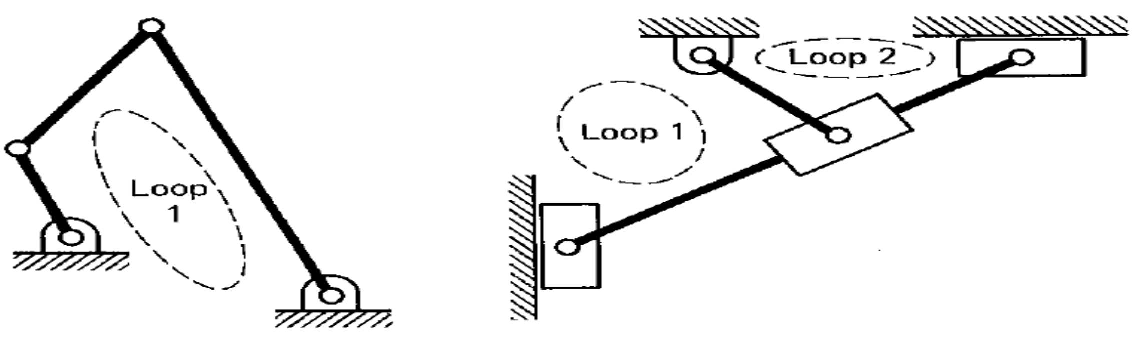
- A mechanism that is formed from a collection of links or bodies kinematically connected to one another.
- An open-loop mechanism may contain links with single joint.
- A closed-loop mechanism is formed from a closed chain, wherein each link is connected to at least two other links of the mechanism.



(a) (b)

Figure (a) Open-loop mechanism—double pendulum and (b) closed-loop mechanism—four-bar linkage.

Single and Multi-Loop Mechanism



(a) (b)

Figure (a) Single-loop mechanism and (b) multi-loop mechanism.

High and Low Pair of Kinematic Joint

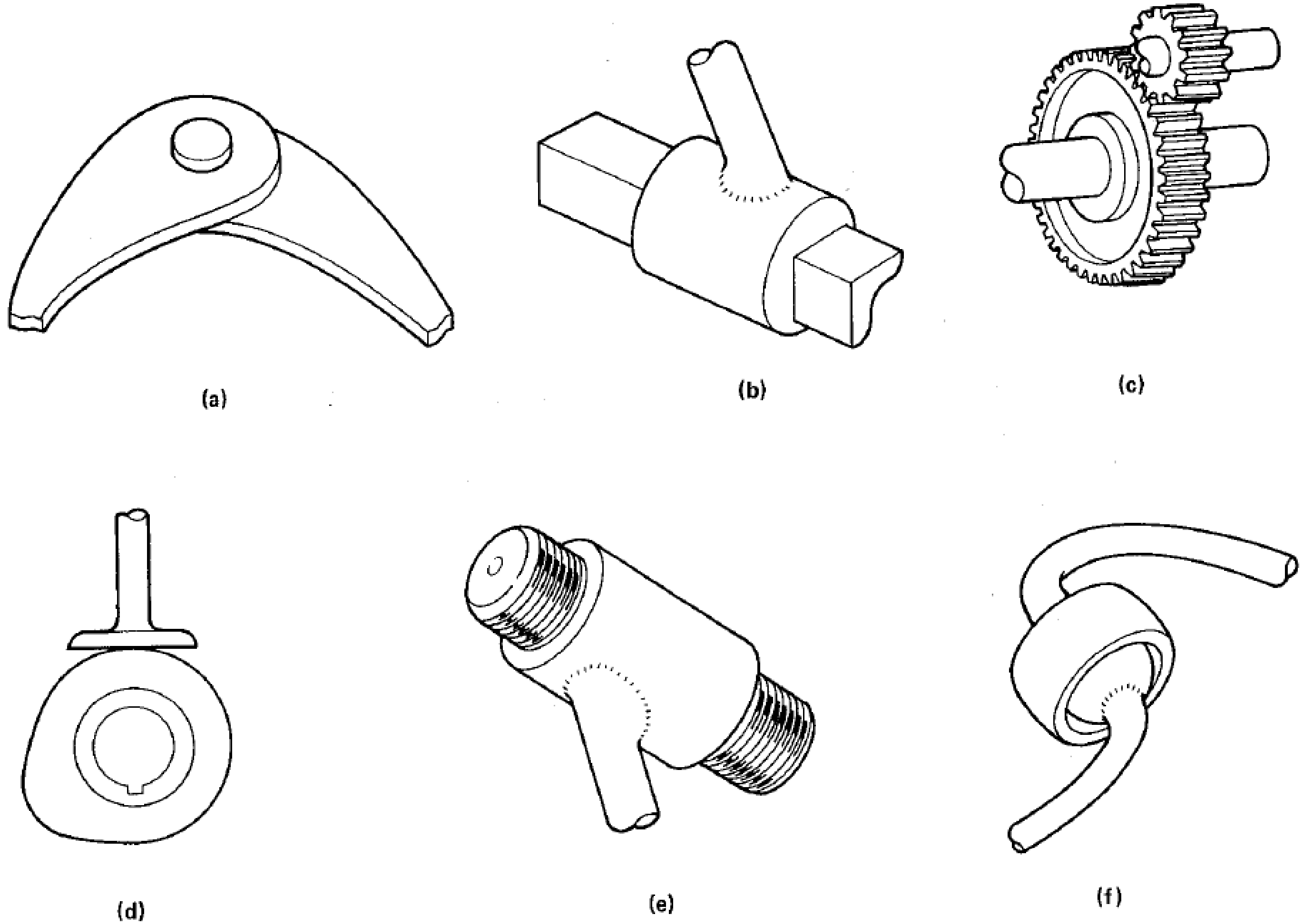


Figure Example of kinematic pairs: (a) revolute joint, (b) translational joint, (c) gear set, (d) cam follower, (e) screw joint, and (f) spherical ball joint.

Generalized Coordinates

3 generalized coordinates,

2 algebraic constraint equations,

$$l_1 \cos\phi_1 + l_2 \cos\phi_2 - l_3 \cos\phi_3 - d_1 = 0$$

$$l_1 \sin\phi_1 + l_2 \sin\phi_2 - l_3 \sin\phi_3 - d_2 = 0$$

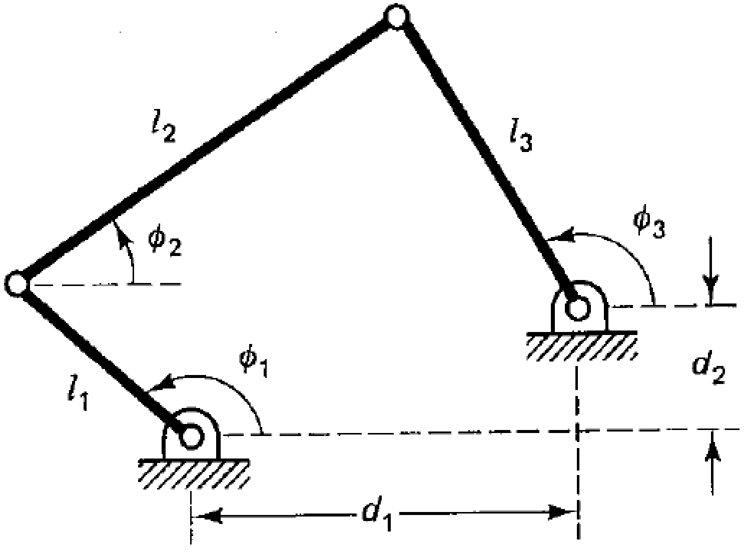
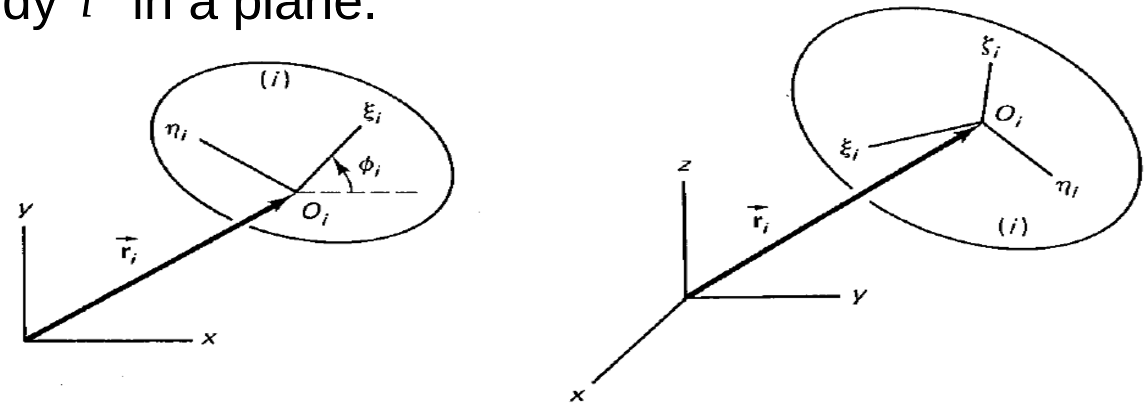


Figure four-bar mechanism.

2.1 Planar Kinematics in Cartesian Coordinates

The column vector $\boldsymbol{q}_i \equiv [x, y, \phi]_i^T$ is the vector of coordinates for body i in a plane.



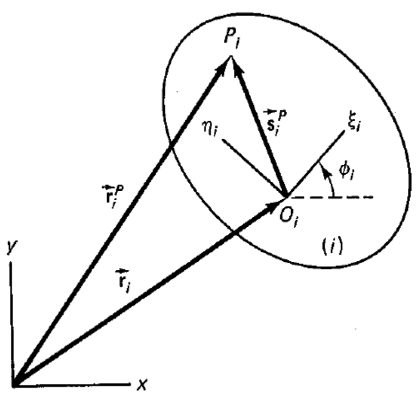
(a) (b)

$$\boldsymbol{q}_i \equiv \left[x, y, z, \phi_1, \phi_2, \phi_3\right]_i^T$$ is the vector of coordinates for body i in three-dimensional space.

Inertial system X − Y

Body−fixed system ξ − η

$$\boldsymbol{r}_i^P = \boldsymbol{r}_i + \boldsymbol{A}_i \boldsymbol{s}_i'^P$$



Coordinate transformation matrix

$$\boldsymbol{A}_i = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}_i$$

Constraint Equation

- A constraint equation describing a condition on the vector of coordinates of a system can be expressed as follows: $\boldsymbol{\Phi} \equiv \boldsymbol{\Phi}(\boldsymbol{q}) = 0$
- In some constraint and driving function, the variable time may appear explicitly: $\boldsymbol{\Phi} \equiv \boldsymbol{\Phi}(\boldsymbol{q}, t) = 0$
- Constraint Jacobian Matrix by differentiating the constraint equations
-

$$\boldsymbol{\Phi}(\boldsymbol{q}) = 0$$

$$\frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} = 0, \text{ often denoted as } \boldsymbol{\Phi}_q \dot{\boldsymbol{q}} = 0$$

$$\text{also denoted as } \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{q}} \ddot{\boldsymbol{q}} + \frac{\partial \left(\dfrac{\partial \phi}{\partial \boldsymbol{q}} \right)}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} = 0$$

$$\boldsymbol{\Phi}_q \ddot{\boldsymbol{q}} + (\boldsymbol{\Phi}_q \dot{\boldsymbol{q}})_q \dot{\boldsymbol{q}} = 0$$

$$\boldsymbol{\Phi}_q \ddot{\boldsymbol{q}} = -(\boldsymbol{\Phi}_q \dot{\boldsymbol{q}})_q \dot{\boldsymbol{q}} \equiv \gamma$$

Redundant Constraint

- Kinematically equivalent.

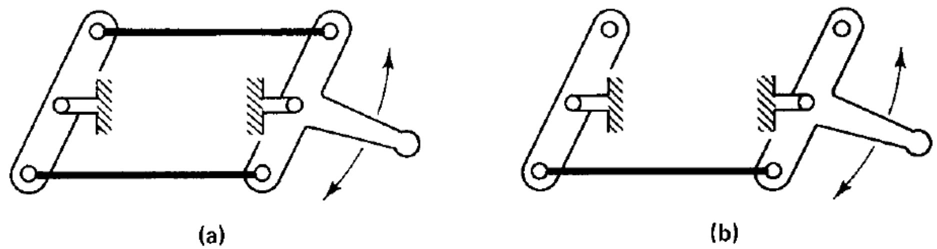


Figure (a) A double parallel-crank mechanism and (b) its kinematically equivalent.

Kinematics of Mechanism

Mainly composed of revolute joint and translation joint

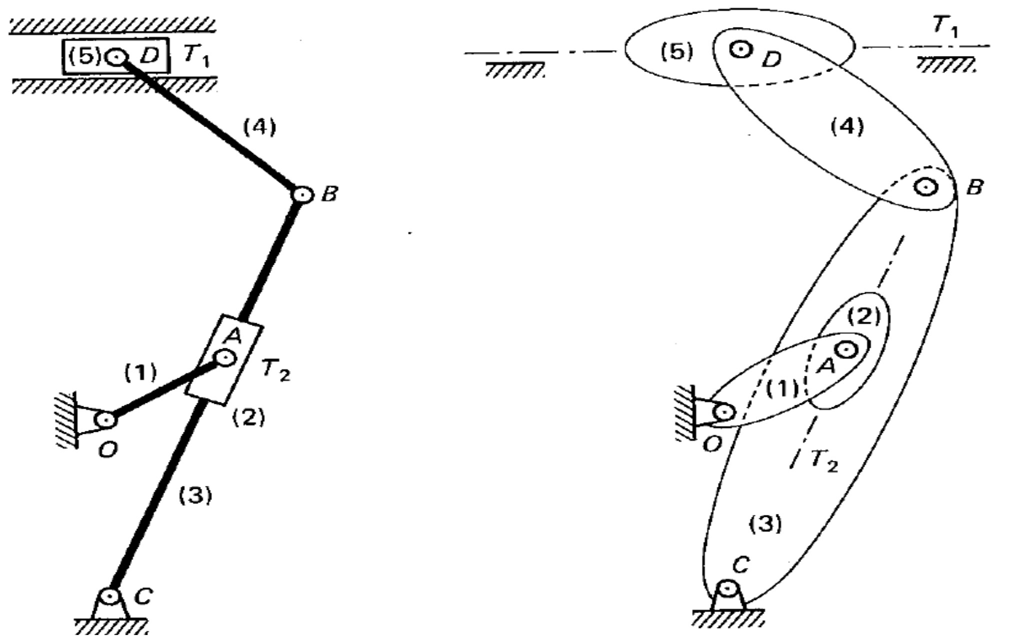


Figure Quick-return mechanism: (a) schematic presentation and
(b) its equivalent representation without showing the actual outlines.

2.2 Revolute Joint

$$\boldsymbol{r}_i + \boldsymbol{s}_i^P - \boldsymbol{r}_j - \boldsymbol{s}_j^P = 0$$

$$\boldsymbol{\Phi}^{(r,2)} \equiv \boldsymbol{r}_i + \boldsymbol{A}_i \boldsymbol{s}_i'^P - \boldsymbol{r}_j - \boldsymbol{A}_j \boldsymbol{s}_j'^P = 0$$

$$\Phi^{(r,2)} \equiv \begin{bmatrix} x_i + \xi_i^P \cos\phi_i - \eta_i^P \sin\phi_i - x_j - \xi_j^P \cos\phi_j + \eta_j^P \sin\phi_j \\ y_i + \xi_i^P \sin\phi_i + \eta_i^P \cos\phi_i - y_j - \xi_j^P \sin\phi_j - \eta_j^P \cos\phi_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

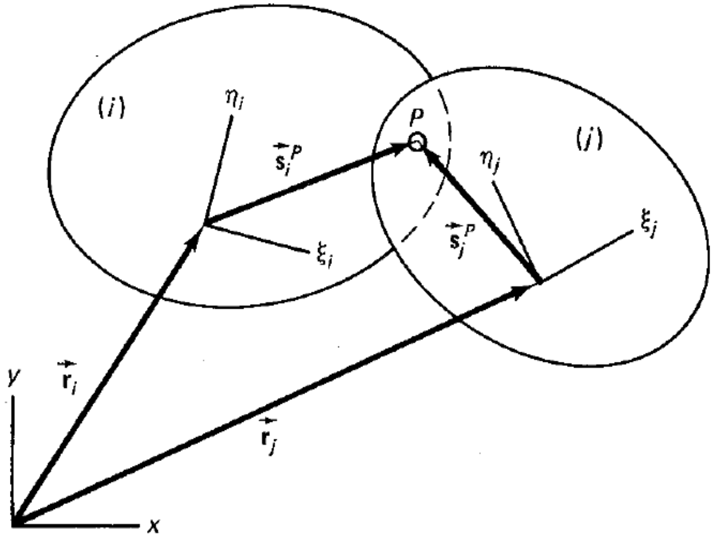


Figure Revolute joint P connecting bodies i and j .

Time Derivative of Revolute Joint Constraint

$$\Phi_1 \equiv x_i + \xi_i^P \cos\phi_i - \eta_i^P \sin\phi_i - x_j - \xi_j^P \cos\phi_j + \eta_j^P \sin\phi_j = 0$$

$$\Phi_2 \equiv y_i + \xi_i^P \sin\phi_i + \eta_i^P \cos\phi_i - y_j - \xi_j^P \sin\phi_j - \eta_j^P \cos\phi_j = 0$$

$$\dot{x}_i - (x_i^P \sin f_i + h_i^P \cos f_i)\dot{f}_i - \dot{x}_j + (x_j^P \sin f_j + h_j^P \cos f_j)\dot{f}_j = 0$$

$$\dot{y}_i + (x_i^P \cos f_i - h_i^P \sin f_i)\dot{f}_i - \dot{y}_j - (x_j^P \cos f_j - h_j^P \sin f_j)\dot{f}_j = 0$$

	$\partial\Phi/\partial x_i$	$\partial\Phi/\partial y_i$	$\partial\Phi/\partial\phi_i$	$\partial\Phi/\partial x_j$	$\partial\Phi/\partial y_j$	$\partial\Phi/\partial\phi_j$
Φ	1	0	$-\left(y_i^p - y_i\right)$	-1	0	$\left(y_i^p - y_i\right)$
	0	1	$\left(x_i^p - x_i\right)$	0	-1	$-\left(x_i^p - x_i\right)$

Time Derivative of Revolute Constraint

$$\Phi_1 \equiv x_i + \xi_i^P \cos\phi_i - \eta_i^P \sin\phi_i - x_j - \xi_j^P \cos\phi_j + \eta_j^P \sin\phi_j = 0$$

$$\Phi_2 \equiv y_i + \xi_i^P \sin\phi_i + \eta_i^P \cos\phi_i - y_j - \xi_j^P \sin\phi_j - \eta_j^P \cos\phi_j = 0$$

$$\dot{x}_i - (\xi_i^P \sin\phi_i + \eta_i^P \cos\phi_i)\dot{\phi}_i - \dot{x}_j + (\xi_j^P \sin\phi_j + \eta_j^P \cos\phi_j)\dot{\phi}_j = 0$$

$$\dot{y}_i + (\xi_i^P \cos\phi_i - \eta_i^P \sin\phi_i)\dot{\phi}_i - \dot{y}_j - (\xi_j^P \cos\phi_j - \eta_j^P \sin\phi_j)\dot{\phi}_j = 0$$

$$\ddot{x}_i - (\xi_i^P \sin\phi_i + \eta_i^P \cos\phi_i)\ddot{\phi}_i - (\xi_i^P \cos\phi_i + \eta_i^P \sin\phi_i)\dot{\phi}_i^2 - \ddot{x}_j$$
$$+ (\xi_j^P \sin\phi_j + \eta_j^P \cos\phi_j)\ddot{\phi}_j + (\xi_j^P \cos\phi_j - \eta_j^P \sin\phi_j)\dot{\phi}_j^2 = 0$$

$$\ddot{y}_i - (\xi_i^P \cos\phi_i + \eta_i^P \sin\phi_i)\ddot{\phi}_i - (\xi_i^P \sin\phi_i + \eta_i^P \cos\phi_i)\dot{\phi}_i^2 - \ddot{y}_j$$
$$- (\xi_j^P \cos\phi_j - \eta_j^P \sin\phi_j)\ddot{\phi}_j - (\xi_j^P \sin\phi_j + \eta_j^P \cos\phi_j)\dot{\phi}_j^2 = 0$$

or

$$\begin{bmatrix} \textcircled{1} & 0 & \textcircled{2} & \textcircled{3} & 0 & \textcircled{4} \\ 0 & \textcircled{5} & \textcircled{6} & 0 & \textcircled{7} & \textcircled{8} \end{bmatrix} \begin{bmatrix} \ddot{x}_i \\ \ddot{y}_i \\ \ddot{\phi}_i \\ \ddot{x}_j \\ \ddot{y}_j \\ \ddot{\phi}_j \end{bmatrix} = \gamma^{(r,2)}$$

$$\gamma^{(r,2)} = \begin{bmatrix} (\xi_i^P \cos\phi_i + \eta_i^P \sin\phi_i)\dot{\phi}_i^2 - (\xi_j^P \cos\phi_j + \eta_j^P \sin\phi_j)\dot{\phi}_j^2 \\ (\xi_i^P \sin\phi_i + \eta_i^P \cos\phi_i)\dot{\phi}_i^2 - (\xi_j^P \sin\phi_j + \eta_j^P \cos\phi_j)\dot{\phi}_j^2 \end{bmatrix}$$

$$= \begin{bmatrix} (x_i^P - x_i)\dot{\phi}_i^2 - (x_j^P - x_j)\dot{\phi}_j^2 \\ (y_i^P - y_i)\dot{\phi}_i^2 - (y_j^P - y_j)\dot{\phi}_j^2 \end{bmatrix} = s_i^P \dot{\phi}_i^2 - s_j^P \dot{\phi}_j^2$$

2.3 Translational Joint

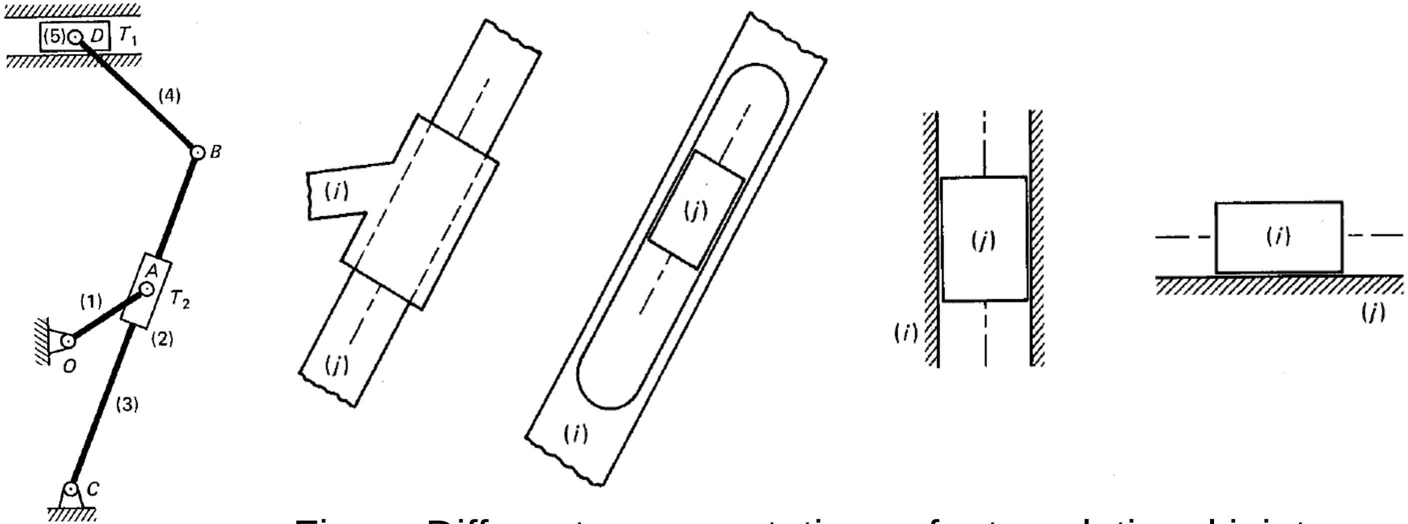
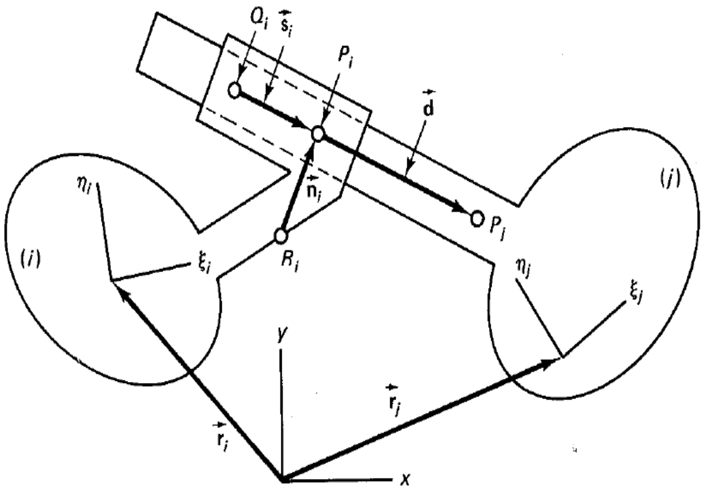


Figure Different representations of a translational joint.

Translational Joint



$$\boldsymbol{n}_i^T \boldsymbol{d} = 0$$

$$\begin{bmatrix} x_i^P - x_i^R & y_i^P - y_i^R \end{bmatrix} \begin{bmatrix} x_i^P - x_j^P \\ y_i^P - y_j^P \end{bmatrix} = 0$$

$$\Phi = \begin{bmatrix} +(x_i^P - x_j^P)(y_i^P - y_i^Q) + (y_i^P - y_j^P)(x_i^P - x_j^Q) \\ \phi_i - \phi_j - (\phi_i^0 - \phi_j^0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Figure A translational joint between bodies i and j.

Time Derivative of Translational Joint Constraint

$$\Phi = \begin{bmatrix} (x_i^P - x_i^Q)(y_j^P - y_i^P) - (y_i^P - y_i^Q)(x_j^P - x_i^P) \\ \phi_i - \phi_j - (\phi_i^0 - \phi_j^0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Φ	$(y_i^P - y_i^Q)$	$-(x_i^P - x_i^Q)$	$-(x_j^P - x_i)(x_i^P - x_i^Q)$ $-(y_j^P - y_i)(y_i^P - y_i^Q)$	$-(y_i^P - y_i^Q)$	$(x_i^P - x_i^Q)$	$(x_j^P - x_i)(x_i^P - x_i^Q)$ $+(y_j^P - y_i)(y_i^P - y_i^Q)$
	0	0	1	0	0	-1

$$\gamma = -2[(x_i^P - x_i^Q)(\dot{x}_i - \dot{x}_j) + (y_i^P - y_i^Q)(\dot{y}_i - \dot{y}_j)]\dot{\phi}_i - [(x_i^P - x_i^Q)(y_i - y_j) - (y_i^P - y_i^Q)(x_i - x_j)]\dot{\phi}_i^2$$

Driving Link

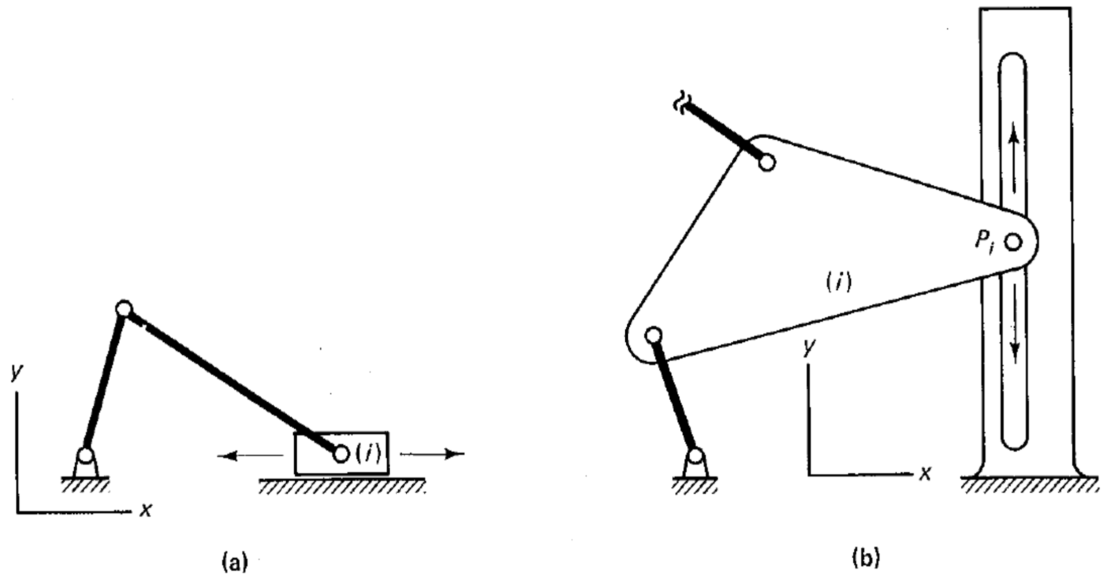


Figure (a) The motion of the slider is controlled in the $x-$direction and (b) the motion of point P is controlled in the $y-$ direction.

$$\Phi \equiv x_i - d(t) = 0$$

$$\Phi \equiv y_i^P - d(t) = 0$$

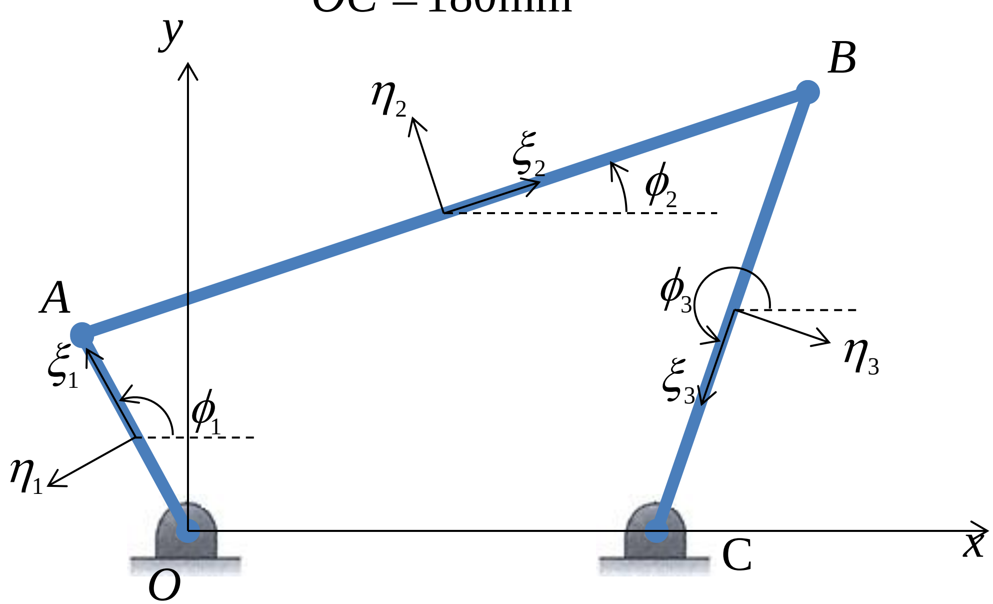
A Matlab Program for Kinematics of a four-bar linkage

$\overline{OA} = 80\text{mm}$

$\overline{AB} = 260\text{mm}$

$\overline{BC} = 180\text{mm}$

$\overline{OC} = 180\text{mm}$



8 Constraint equations:

$$-x_1 + 40\,\mathbf{cos}\,\phi_1 = 0$$

$$-y_1 + 40\,\mathbf{sin}\,\phi_1 = 0$$

$$x_1 + 40\,\mathbf{cos}\,\phi_1 - x_2 + 130\,\mathbf{cos}\,\phi_2 = 0$$

$$y_1 + 40\,\mathbf{sin}\,\phi_1 - y_2 + 130\,\mathbf{sin}\,\phi_2 = 0$$

$$x_2 + 130\,\mathbf{cos}\,\phi_2 - x_3 + 90\,\mathbf{cos}\,\phi_3 = 0$$

$$y_2 + 130\,\mathbf{sin}\,\phi_2 - y_3 + 90\,\mathbf{sin}\,\phi_3 = 0$$

$$x_3 + 90\,\mathbf{cos}\,\phi_3 - 180 = 0$$

$$y_3 + 90\,\mathbf{sin}\,\phi_3 = 0$$

driving link

$$\phi_1 - 2\pi t - \pi/2 = 0$$

To solve the 9 equations for 9 unknown $\boldsymbol{q}^T = [x_1, y_1, \phi_1, x_2, y_2, \phi_2, x_3, y_3, \phi_3]$

Jacobian matrix and velocity equations

$$\mathbf{J} = \begin{matrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \\ \Phi_6 \\ \Phi_7 \\ \Phi_8 \\ \Phi_9 \end{matrix} \begin{bmatrix} -1 & 0 & -40\sin\varphi_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 40\cos\varphi_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -40\sin\varphi_1 & -1 & 0 & -130\sin\varphi_2 & 0 & 0 & 0 \\ 0 & 1 & 40\cos\varphi_1 & 0 & -1 & 130\cos\varphi_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -130\sin\varphi_2 & -1 & 0 & -90\sin\varphi_3 \\ 0 & 0 & 0 & 0 & 1 & 130\cos\varphi_2 & 0 & -1 & 90\cos\varphi_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -90\sin\varphi_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 90\cos\varphi_3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \beta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2\pi \end{bmatrix}$$

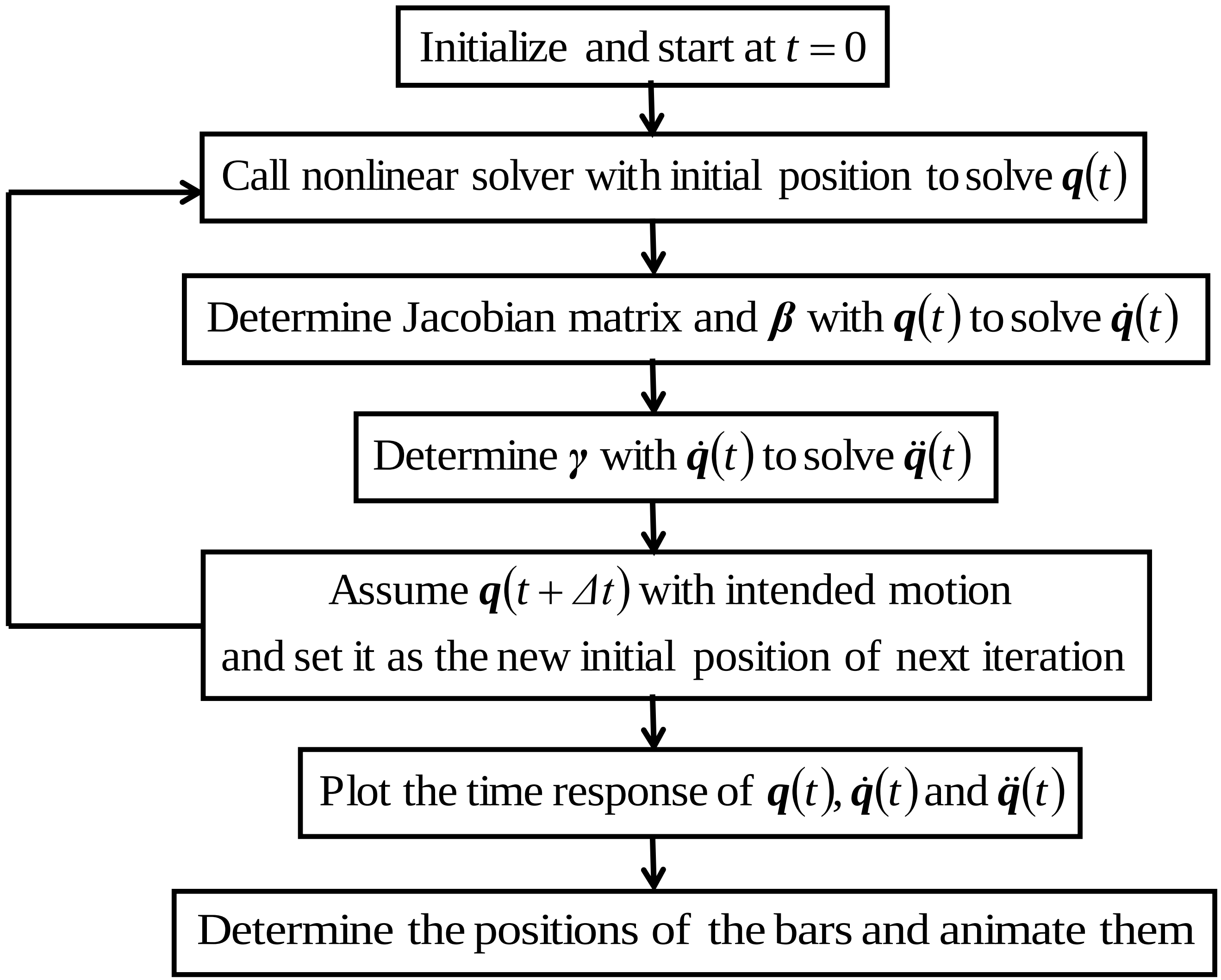
To solve $\mathbf{J}\dot{\boldsymbol{q}} = \boldsymbol{\beta}$ for the velocity $\dot{\boldsymbol{q}}$

Acceleration equations

$$\boldsymbol{\gamma} = \begin{bmatrix} 40\cos\phi_1 \cdot \dot{\phi}_1^2 \\ 40\sin\phi_1 \cdot \dot{\phi}_1^2 \\ 50\cos\phi_1 \cdot \dot{\phi}_1^2 + 130\cos\phi_2 \cdot \dot{\phi}_2^2 \\ 50\sin\phi_1 \cdot \dot{\phi}_1^2 + 130\sin\phi_2 \cdot \dot{\phi}_2^2 \\ 130\cos\phi_2 \cdot \dot{\phi}_2^2 + 90\cos\phi_3 \cdot \dot{\phi}_3^2 \\ 130\sin\phi_2 \cdot \dot{\phi}_2^2 + 90\sin\phi_3 \cdot \dot{\phi}_3^2 \\ 90\cos\phi_3 \cdot \dot{\phi}_3^2 \\ 90\sin\phi_3 \cdot \dot{\phi}_3^2 \\ 0 \end{bmatrix}$$

To solve $\mathbf{J}\ddot{\boldsymbol{q}} = \boldsymbol{\gamma}$ for the acceleration $\ddot{\boldsymbol{q}}$

The procedure of m-file



The code of m.file (1)

```
% Set up the time interval and the initial positions of the nine coordinates
T_Int=0:0.01:2;
X0=[0 50 pi/2   125.86 132.55 0.2531   215.86 82.55 4.3026];
global T
Xinit=X0;

% Do the loop for each time interval
for Iter=1:length(T_Int);
    T=T_Int(Iter);
    % Determine the displacement at the current time
    [Xtemp,fval] = fsolve(@constrEq4bar,Xinit);

    % Determine the velocity at the current time
    phi1=Xtemp(3); phi2=Xtemp(6); phi3=Xtemp(9);
    JacoMatrix=Jaco4bar(phi1,phi2,phi3);
    Beta=[0 0 0    0 0 0    0 0 2*pi]';
    Vtemp=JacoMatrix\Beta;

    % Determine the acceleration at the current time
    dphi1=Vtemp(3); dphi2=Vtemp(6); dphi3=Vtemp(9);
    Gamma=Gamma4bar(phi1,phi2,phi3,dphi1,dphi2,dphi3);
    Atemp=JacoMatrix\Gamma;

    % Record the results of each iteration
    X(:,Iter)=Xtemp; V(:,Iter)=Vtemp; A(:,Iter)=Atemp;

    % Determine the new initial position to solve the equation of the next
    % iteration and assume that the kinematic motion is with inertia
    if Iter==1
        Xinit=X(:,Iter);
    else
        Xinit=X(:,Iter)+(X(:,Iter)-X(:,Iter-1));
    end

end
```

The code of m.file (2)

```
% T vs displacement plot for the nine coordinates
figure
for i=1:9;
    subplot(9,1,i)
    plot (T_Int,X(i,:))
    set(gca,'xtick',[], 'FontSize', 5)
end
% Reset the bottom subplot to have xticks
set(gca,'xtickMode', 'auto')

% T vs velocity plot for the nine coordinates
figure
for i=1:9;
    subplot(9,1,i)
    plot (T_Int,V(i,:))
    set(gca,'xtick',[], 'FontSize', 5)
end
set(gca,'xtickMode', 'auto')

% T vs acceleration plot for the nine coordinates
figure
for i=1:9;
    subplot(9,1,i)
    plot (T_Int,A(i,:))
    AxeSup=max(A(i,:));
    AxeInf=min(A(i,:));
    if AxeSup-AxeInf<0.01
        axis([-inf,inf,(AxeSup+AxeSup)/2-0.1 (AxeSup+AxeSup)/2+0.1]);
    end
    set(gca,'xtick',[], 'FontSize', 5)
end
set(gca,'xtickMode', 'auto')
```

The code of m.file (3)

```
% Determine the positions of the four revolute joints at each iteration
Ox=zeros(1,length(T_Int));
Oy=zeros(1,length(T_Int));
Ax=80*cos(X(3,:));
Ay=80*sin(X(3,:));
Bx=Ax+260*cos(X(6,:));
By=Ay+260*sin(X(6,:));
Cx=180*ones(1,length(T_Int));
Cy=zeros(1,length(T_Int));

% Animation
figure
for t=1:length(T_Int);
    bar1x=[Ox(t) Ax(t)];
    bar1y=[Oy(t) Ay(t)];
    bar2x=[Ax(t) Bx(t)];
    bar2y=[Ay(t) By(t)];
    bar3x=[Bx(t) Cx(t)];
    bar3y=[By(t) Cy(t)];

    plot (bar1x,bar1y,bar2x,bar2y,bar3x,bar3y);
    axis([-120,400,-120,200]);
    axis normal

    M(:,t)=getframe;
end
```

Initialization

```
1. % Set up the time interval and the initial positions of the nine coordinates
2. T_Int=0:0.01:2;
3. X0=[0 50 pi/2   125.86 132.55 0.2531   215.86 82.55 4.3026];
4. global T
5. Xinit=X0;
```

1. The sentence is notation that is behind symbol “%”.
2. Simulation time is set from 0 to 2 with Δt = 0.01.
3. Set the appropriate initial positions of the 9 coordinates which are used to solve nonlinear solver.
4. Declare a global variable T which is used to represent the current time *t* and determine the driving constraint for angular velocity.

Determine the displacement

```
10. [Xtemp,fval] = fsolve(@constrEq4bar,Xinit);
```

10. Call the nonlinear solver fsolve in which the constraint equations and initial values are necessary. The initial values is mentioned in above script. The constraint equations is written as a function (which can be treated a kind of subroutine in Matlab) as following and named as constrEq4bar. The fsolve finds a root of a system of nonlinear equations and adopts the trust-region dogleg algorithm by default.

```
function F=constrEq4bar(X)

global T

x1=X(1); y1=X(2); phi1=X(3);
x2=X(4); y2=X(5); phi2=X(6);
x3=X(7); y3=X(8); phi3=X(9);

F=[ -x1+40*cos(phi1);
    -y1+40*sin(phi1);
    x1+40*cos(phi1)-x2+130*cos(phi2);
    y1+40*sin(phi1)-y2+130*sin(phi2);
    x2+130*cos(phi2)-x3+90*cos(phi3);
    y2+130*sin(phi2)-y3+90*sin(phi3);
    x3+90*cos(phi3)-180;
    y3+90*sin(phi3);
    phi1-2*pi*T-pi/2];
```

The equation of driving constraint is depended on current time T

Determine the velocity

```
14.    phi1=Xtemp(3); phi2=Xtemp(6); phi3=Xtemp(9);
15.    JacoMatrix=Jaco4bar(phi1,phi2,phi3);
16.    Beta=[0 0 0    0 0 0    0 0 2*pi]';
17.    Vtemp=JacoMatrix\Beta;
```

15. Call the function Jaco4bar to obtain the Jacobian Matrix depended on current values of displacement.
16. Declare the right-side of the velocity equations.
17. Solve linear equation by left matrix division "\" roughly the same as $\mathbf{J}^{-1}\boldsymbol{\beta}$. The algorithm adopts several methods such as LAPACK, CHOLMOD, and LU. Please find the detail in Matlab Help.

```
function JacoMatrix=Jaco4bar(phi1,phi2,phi3)

JacoMatrix=[   -1 0 -40*sin(phi1)     0 0 0                 0 0 0;
               0 -1 40*cos(phi1)      0 0 0                 0 0 0;
               1 0 -40*sin(phi1)      -1 0 -130*sin(phi2)   0 0 0;
               0 1 40*cos(phi1)       0 -1 130*cos(phi2)    0 0 0;
               0 0 0                  1 0 -130*sin(phi2)    -1 0 -90*sin(phi3);
               0 0 0                  0 1 130*cos(phi2)     0 -1 90*cos(phi3);
               0 0 0                  0 0 0                 1 0 -90*sin(phi3);
               0 0 0                  0 0 0                 0 1 90*cos(phi3);
               0 0 1                  0 0 0                 0 0 0];
```

Determine the acceleration

```
20.    dphi1=Vtemp(3); dphi2=Vtemp(6); dphi3=Vtemp(9);
21.    Gamma=Gamma4bar(phi1,phi2,phi3,dphi1,dphi2,dphi3);
22.    Atemp=JacoMatrix\Gamma;
```

21. Call the function Gamma4bar to obtain the right-side of the velocity equations depended on current values of velocity.
22. Solve linear equation to obtain the current acceleration.

```
function Gamma=Gamma4bar(phi1,phi2,phi3,dphi1,dphi2,dphi3)

Gamma=[ 40*cos(phi1)*dphi1^2;
        40*sin(phi1)*dphi1^2;
        40*cos(phi1)*dphi1^2+130*cos(phi2)*dphi2^2;
        40*sin(phi1)*dphi1^2+130*sin(phi2)*dphi2^2;
        130*cos(phi2)*dphi2^2+90*cos(phi3)*dphi3^2;
        130*sin(phi2)*dphi2^2+90*sin(phi3)*dphi3^2;
        90*cos(phi3)*dphi3^2;
        90*sin(phi3)*dphi3^2;
        0];
```

Determine next initial positions

```
29.    if Iter==1
30.        Xinit=X(:,Iter);
31.    else
32.        Xinit=X(:,Iter)+(X(:,Iter)-X(:,Iter-1));
33.    end
```

29.~33. Predict the next initial positions with assumption of inertia except the first time of the loop.

Plot time response

```
37.figure
38.for i=1:9;
39.    subplot(9,1,i)
40.    plot (T_Int,X(i,:))
41.    set(gca,'xtick',[], 'FontSize', 5)
42.end
43.% Reset the bottom subplot to have xticks
44.set(gca,'xtickMode', 'auto')
45.
46.% T vs velocity plot for the nine coordinates
47.figure
48.for i=1:9;
37.…
```

37. Create a blank figure .

39. Locate the position of subplot in the figure.

40. Plot the nine subplots for the time responses of nine coordinates.

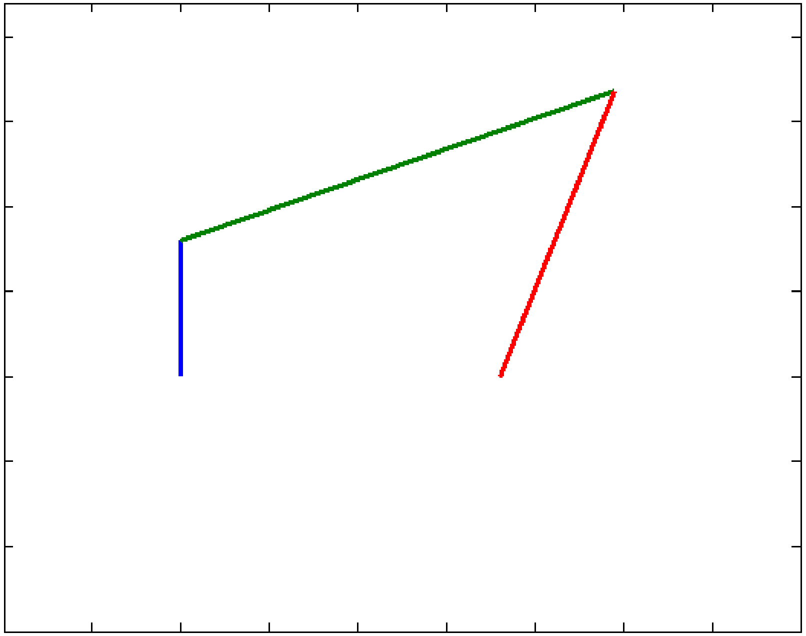
41. Eliminate x-label for time-axis and set the font size of y-label.

44. Resume x-label at bottom because the nine subplots share the same time-axis.

47.~ It is similar to above.

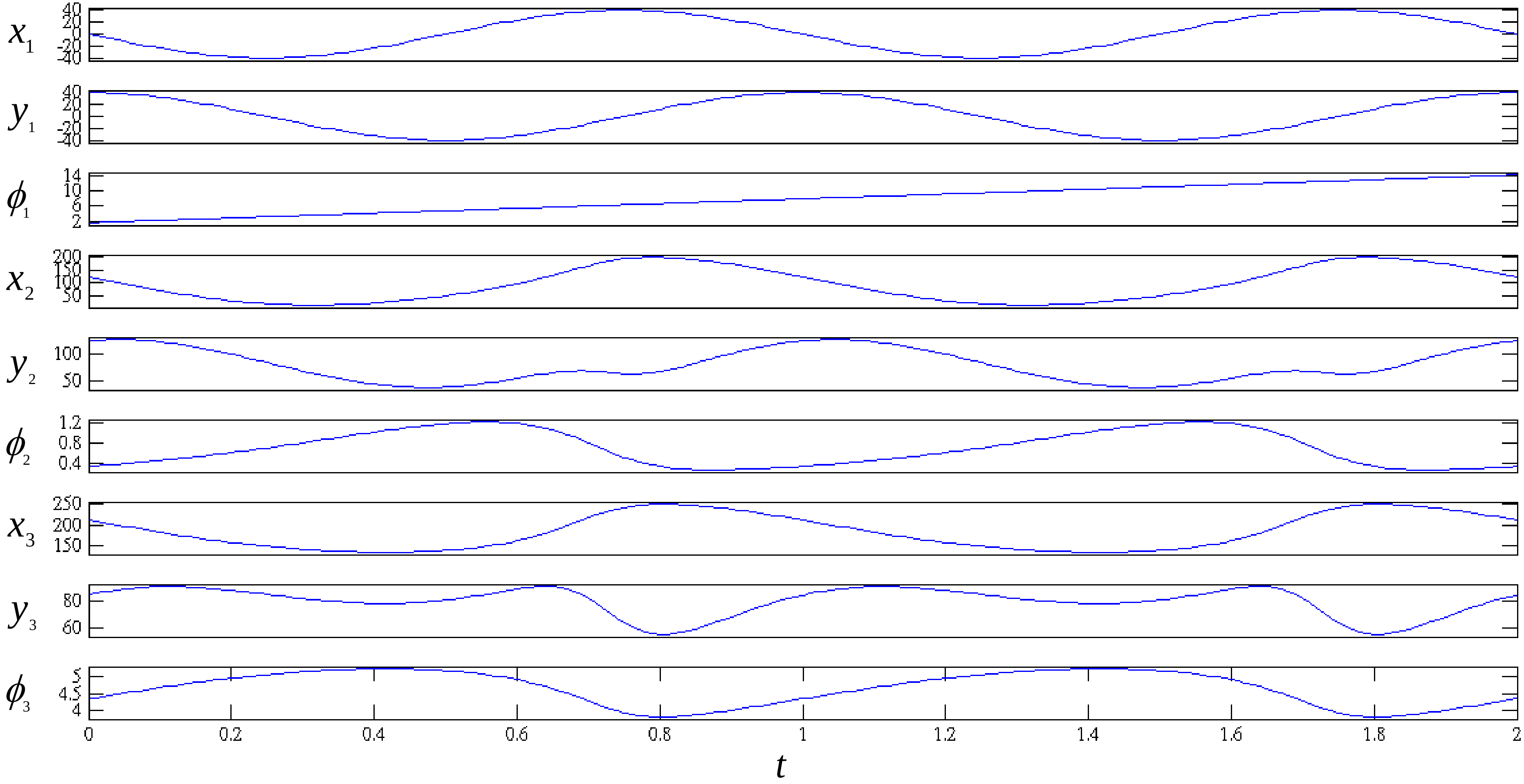
Animation

```
69.Ox=zeros(1,length(T_Int));
70.Oy=zeros(1,length(T_Int));
71.Ax=80*cos(X(3,:));
72.Ay=80*sin(X(3,:));
73.Bx=Ax+260*cos(X(6,:));
74.…

80.for t=1:length(T_Int);
81.    bar1x=[Ox(t) Ax(t)];
82.    bar1y=[Oy(t) Ay(t)];
83.    bar2x=[Ax(t) Bx(t)];
84.    bar2y=[Ay(t) By(t)];
85.    bar3x=[Bx(t) Cx(t)];
86.    bar3y=[By(t) Cy(t)];
87.
88.    plot (bar1x,bar1y,bar2x,bar2y,bar3x,bar3y)
89.    axis([-120,400,-120,200]);
90.    axis normal
91.
92.    M(:,t)=getframe;
93.end
```

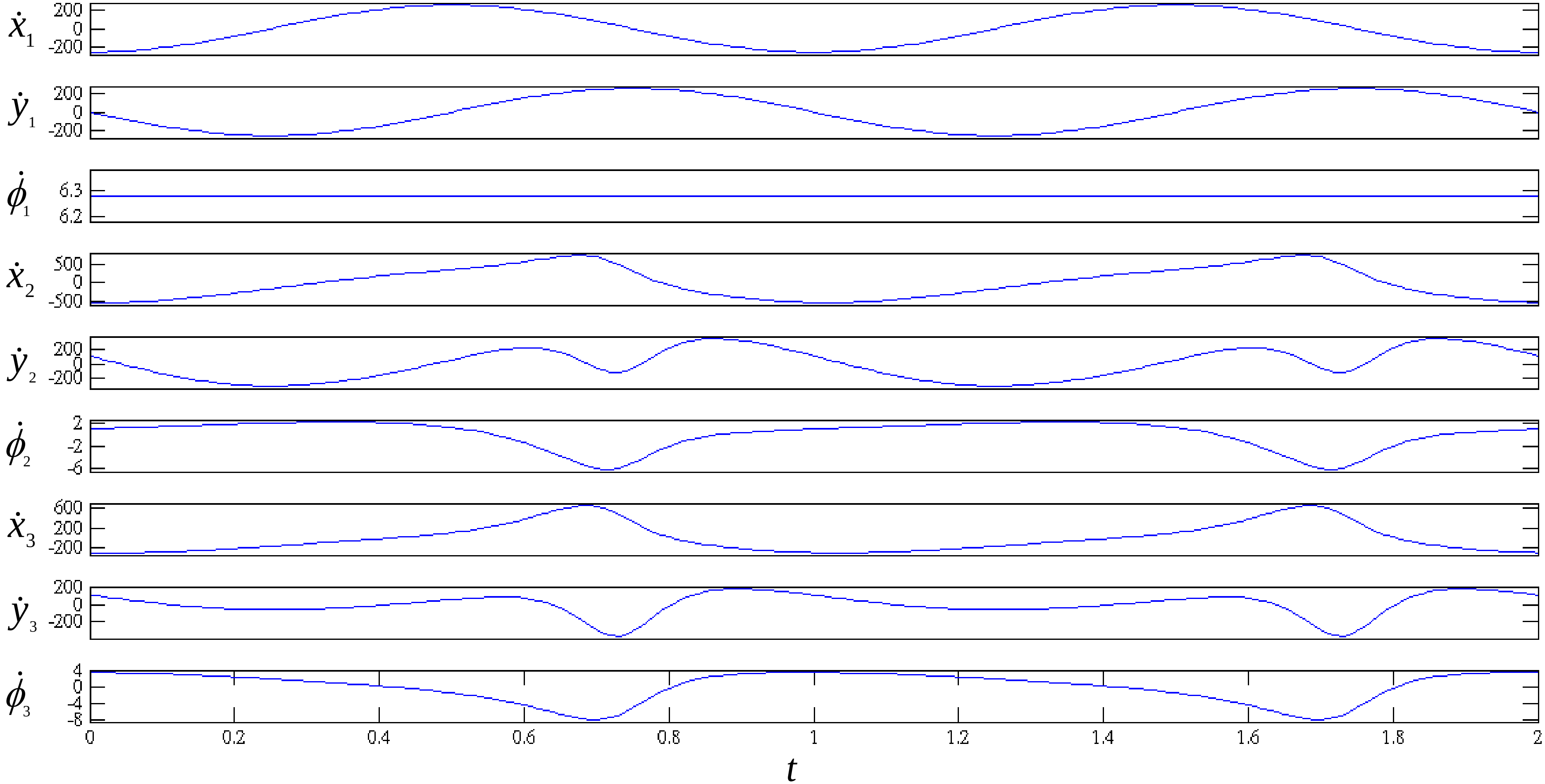


69. Determine the displacement of revolute joint.
80. Repeat to plot the locations by continue time elapsing.
81. Determine the horizontal location of $\overline{OA}$.
88. Plot $\overline{OA}$, $\overline{AB}$, $\overline{BC}$, and $\overline{OC}$.
89. Set an appropriate range of axis.

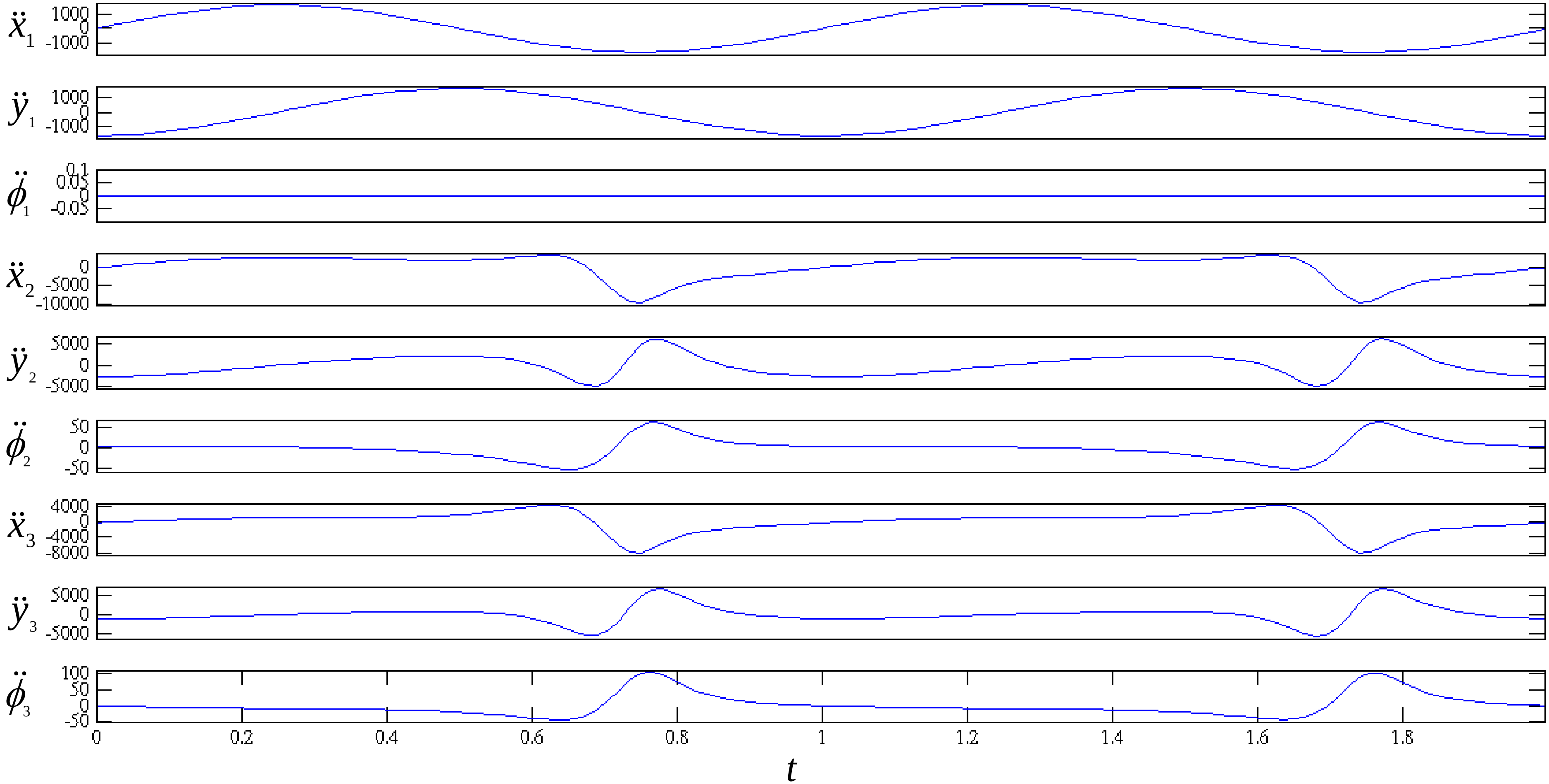
Time response of displacement



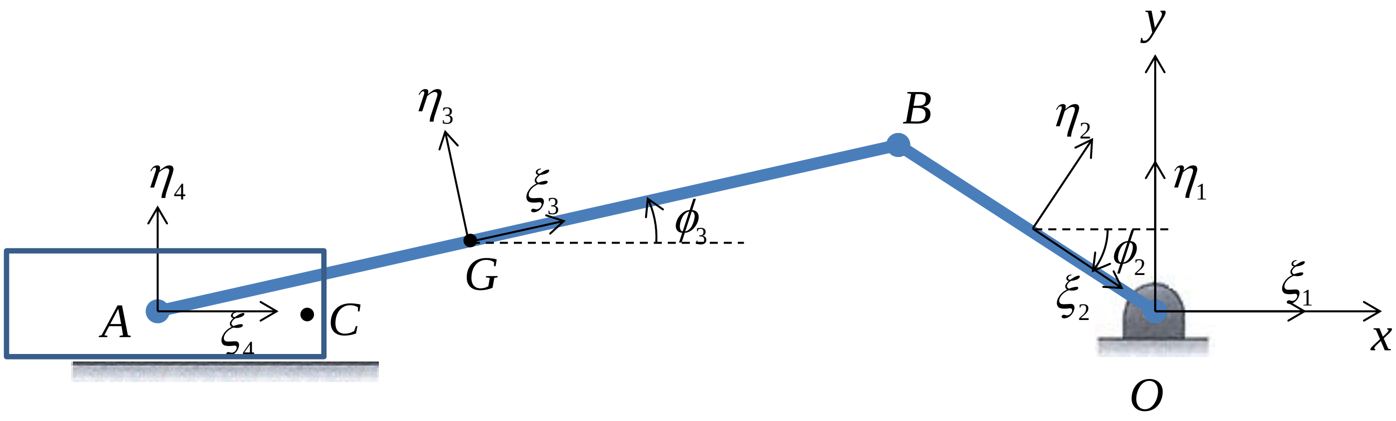
Time response of velocity



Time response of acceleration



Example of a slider-crank mechanism



11 Constraint equations:

$x_1 = 0$

$y_1 = 0$

$\phi_1 = 0$

$x_4 - x_3 + 200\cos\phi_3 = 0$

$y_4 - y_3 + 200\sin\phi_3 = 0$

$x_3 + 300\cos\phi_3 - x_2 + 100\cos\phi_2 = 0$

$y_3 + 300\sin\phi_3 - y_2 + 100\sin\phi_2 = 0$

$x_2 + 100\cos\phi_2 - x_1 = 0$

$y_2 + 100\cos\phi_2 - y_1 = 0$

$100\cos\phi_4\left(y_1 - y_4 - 100\sin\phi_4\right) - 100\sin\phi_4\left(x_1 - x_4 - 100\cos\phi_4\right) = 0$

$\phi_4 - \phi_1 = 0$

driving link

$\phi_2 - 5.76 + 1.2t = 0$

$\overline{AG} = 200\text{mm}$

$\overline{GB} = 300\text{mm}$

$\overline{BO} = 200\text{mm}$

To solve the 12 equations for 12 unknown $\mathbf{q}^T = \left[x_1, y_1, \phi_1, x_2, y_2, \phi_2, x_3, y_3, \phi_3, x_4, y_4, \phi_4\right]$

Jacobian matrix

$\langle 1\rangle,\langle 2\rangle,\langle 3\rangle,\langle 6\rangle,\langle 10\rangle,\langle 14\rangle,\langle 18\rangle,\langle 22\rangle,\langle 26\rangle,\langle 34\rangle,\langle 36\rangle = 1$

$\langle 4\rangle,\langle 8\rangle,\langle 12\rangle,\langle 16\rangle,\langle 20\rangle,\langle 24\rangle,\langle 35\rangle = -1$

$\langle 5\rangle = -200\sin\phi_3$

$\langle 9\rangle = 200\cos\phi_3$

$\langle 13\rangle,\langle 23\rangle = -100\sin\phi_2$

$\langle 15\rangle = -300\sin\phi_3$

$\langle 17\rangle,\langle 27\rangle = 100\cos\phi_2$

$\langle 19\rangle = 300\cos\phi_3$

$\langle 28\rangle = -100\sin\phi_4$

$\langle 29\rangle = 100\cos\phi_4$

$\langle 31\rangle = -\langle 28\rangle$

$\langle 32\rangle = -\langle 29\rangle$

$\langle 33\rangle = 100\left[\cos\phi_4\left(x_4 - x_1\right) + \sin\phi_4\left(y_4 - y_1\right)\right]$

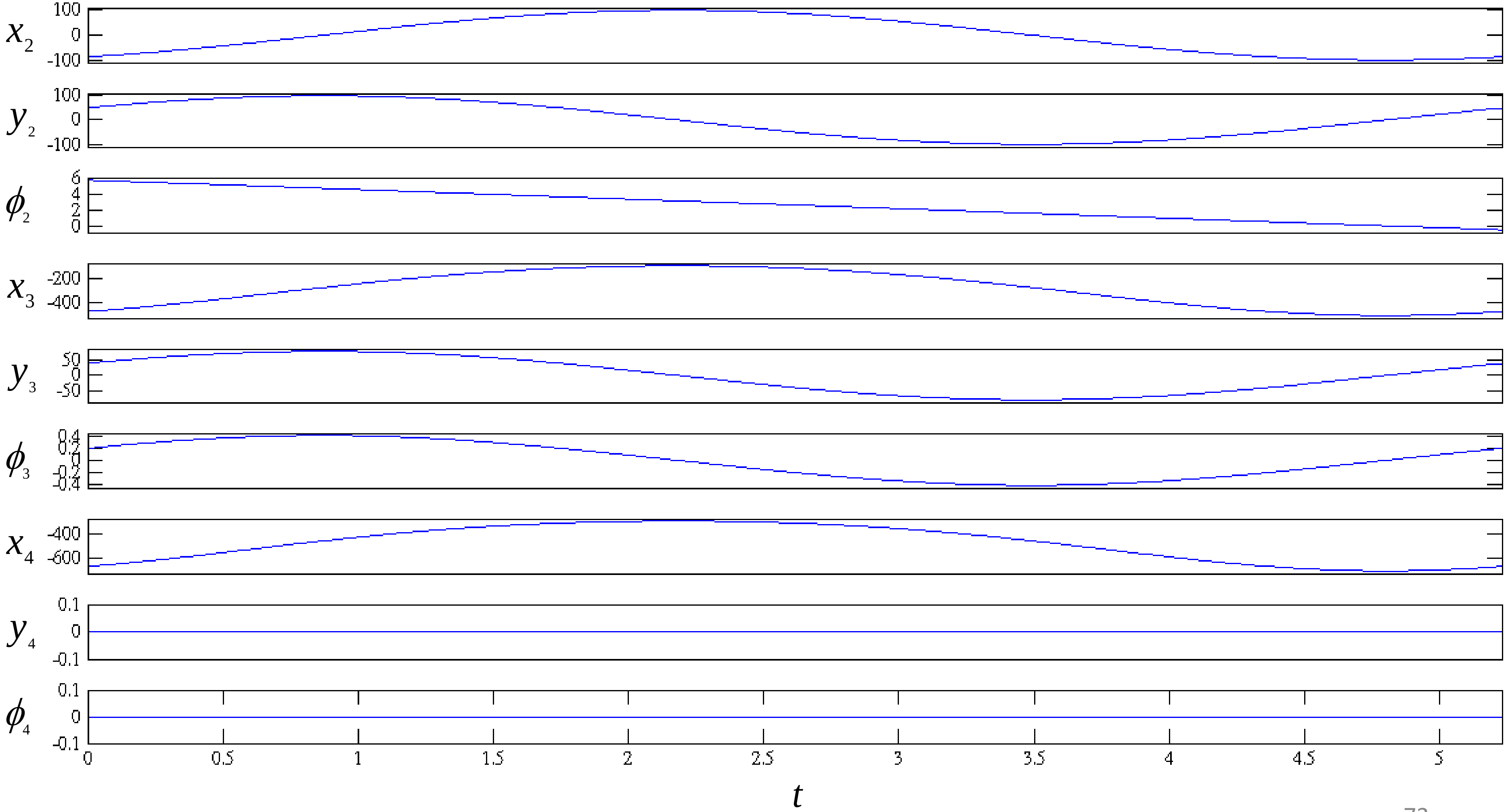
$$\mathbf{J} = \begin{matrix} \partial\Phi_1/\partial \\ \partial\Phi_2/\partial \\ \partial\Phi_3/\partial \\ \partial\Phi_4/\partial \\ \partial\Phi_5/\partial \\ \partial\Phi_6/\partial \\ \partial\Phi_7/\partial \\ \partial\Phi_8/\partial \\ \partial\Phi_9/\partial \\ \partial\Phi_{10}/\partial \\ \partial\Phi_{11}/\partial \\ \partial\Phi_{12}/\partial \end{matrix} \begin{bmatrix} \langle 1\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \langle 2\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \langle 3\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \langle 4\rangle & 0 & \langle 5\rangle & \langle 6\rangle & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \langle 8\rangle & \langle 9\rangle & 0 & \langle 10\rangle & 0 \\ 0 & 0 & 0 & \langle 12\rangle & 0 & \langle 13\rangle & \langle 14\rangle & 0 & \langle 15\rangle & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \langle 16\rangle & \langle 17\rangle & 0 & \langle 18\rangle & \langle 19\rangle & 0 & 0 & 0 \\ \langle 20\rangle & 0 & 0 & \langle 22\rangle & 0 & \langle 23\rangle & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \langle 24\rangle & 0 & 0 & \langle 26\rangle & \langle 27\rangle & 0 & 0 & 0 & 0 & 0 & 0 \\ \langle 28\rangle & \langle 29\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \langle 31\rangle & \langle 32\rangle & \langle 33\rangle \\ 0 & 0 & \langle 34\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \langle 35\rangle \\ 0 & 0 & 0 & 0 & 0 & \langle 36\rangle & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \beta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1.2 \end{bmatrix}$$

The right-side of the acceleration equations

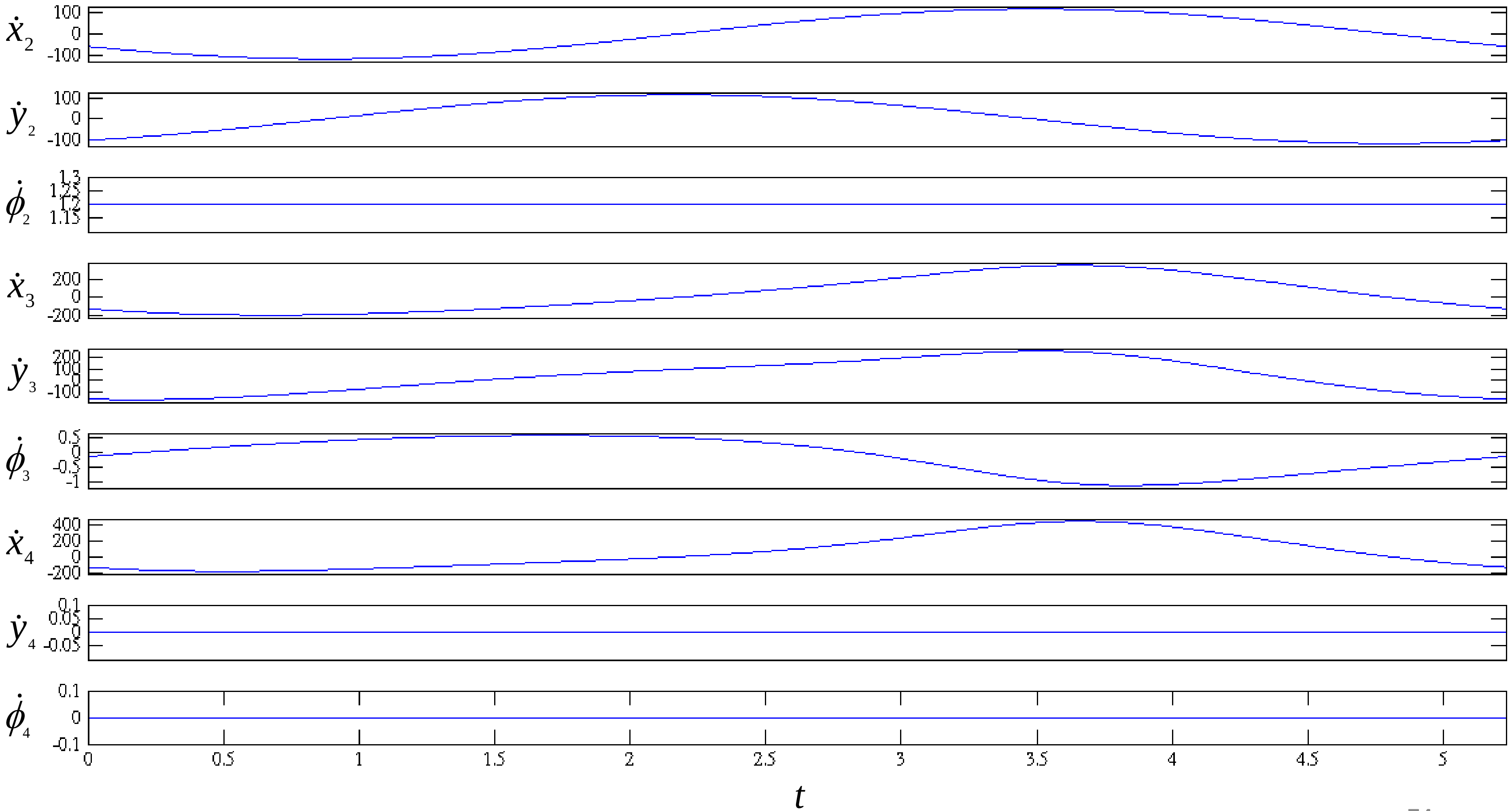
$$\boldsymbol{\gamma} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 200\cos\phi_3\dot{\phi}_3^{\,2} \\ 200\sin\phi_3\dot{\phi}_3^{\,2} \\ 300\cos\phi_3\dot{\phi}_3^{\,2} + 100\cos\phi_2\dot{\phi}_2^{\,2} \\ 300\sin\phi_3\dot{\phi}_3^{\,2} + 100\sin\phi_2\dot{\phi}_2^{\,2} \\ 100\cos\phi_2\dot{\phi}_2^{\,2} \\ 100\sin\phi_2\dot{\phi}_2^{\,2} \\ \gamma(10) \\ 0 \\ 0 \end{bmatrix}$$

where $\gamma(10) = 200\cos\phi_4(\dot{x}_1 - \dot{x}_4)\dot{\phi}_4 + 200\sin\phi_4(\dot{y}_1 - \dot{y}_4)\dot{\phi}_4 - \dot{\phi}_4^{\,2}[100\sin\phi_4(x_1 - x_4) - 100\cos\phi_4(y_1 - y_4)]$

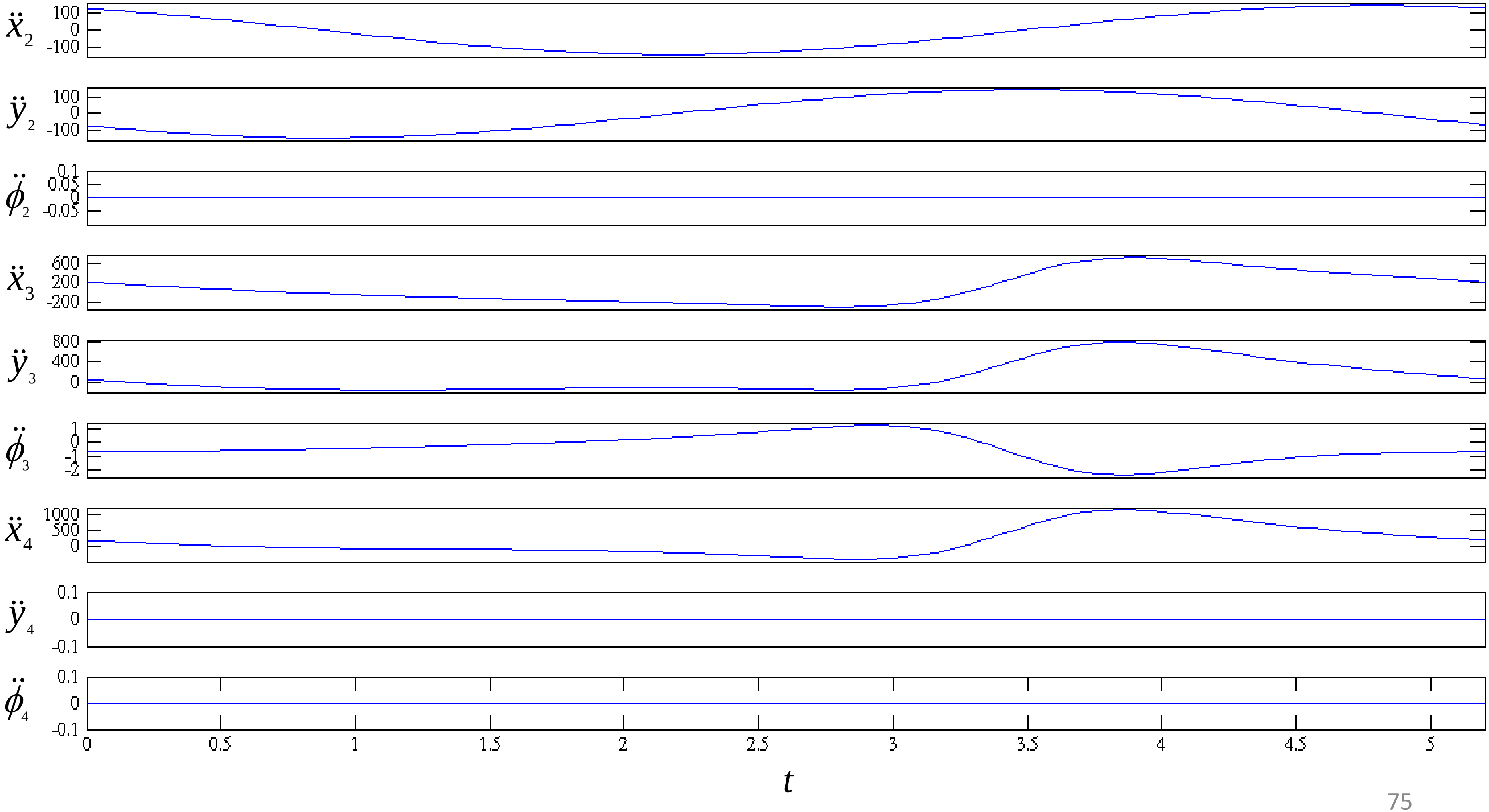
Time response of displacement



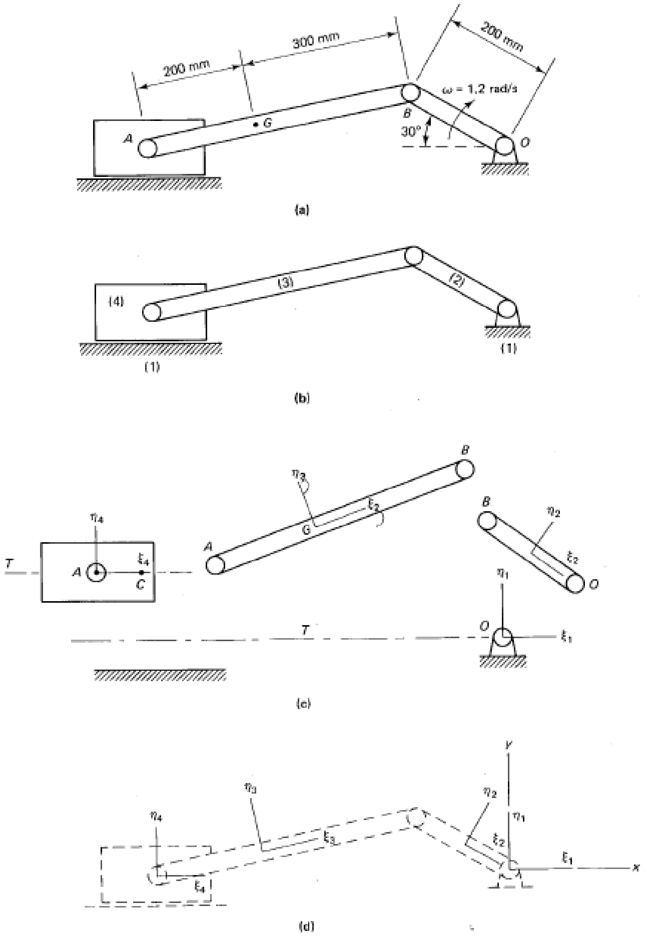
Time response of velocity



Time response of acceleration



Kinematic Modeling



Ground

$$x_1 = 0.0, \quad y_1 = 0.0, \quad \phi_1 = 0.0$$

Revolute joint

$$\xi_4^A = 0.0, \quad \eta_4^A = 0.0, \quad \xi_3^A = -200.0, \quad \eta_3^A = 0.0$$

$$\xi_3^B = 300.0, \quad \eta_3^B = 0.0, \quad \xi_2^B = -100.0, \quad \eta_2^B = 0.0$$

$$\xi_2^0 = 100.0, \quad \eta_2^0 = 0.0, \quad \xi_1^0 = 0.0, \quad \eta_1^0 = 0.0$$

translational joint

$$\xi_4^A = 0.0, \quad \eta_4^A = 0.0, \quad \xi_1^B = -100.0, \quad \eta_1^B = 0.0,$$

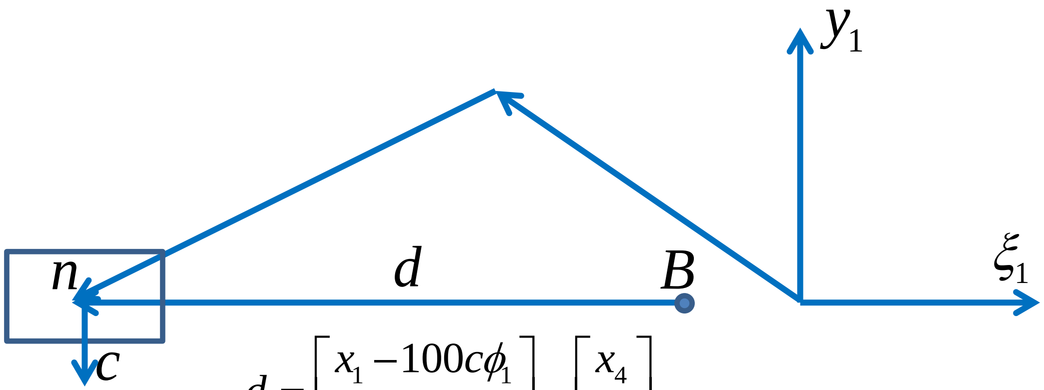
$$\xi_4^C = 100.0, \quad \eta_4^C = 0.0, \quad \xi_1^0 = 0.0, \quad \eta_1^0 = 0.0$$

driving link

$$\phi_2 - 5.76 + 1.2t = 0.0$$

Figure Kinematic modeling of a slider-crank mechanism.

Translation Joint



$$\underset{\sim}{d} = \begin{bmatrix} x_1 - 100c\phi_1 \\ y_1 - 100s\phi_1 \end{bmatrix} - \begin{bmatrix} x_4 \\ y_4 \end{bmatrix}$$

$$\mathbf{\Phi} = (-100c\phi_4)(y_1 - 100s\phi_1 - y_4) - (x_1 - 100c\phi_1 - x_4)(-100s\phi_4)$$

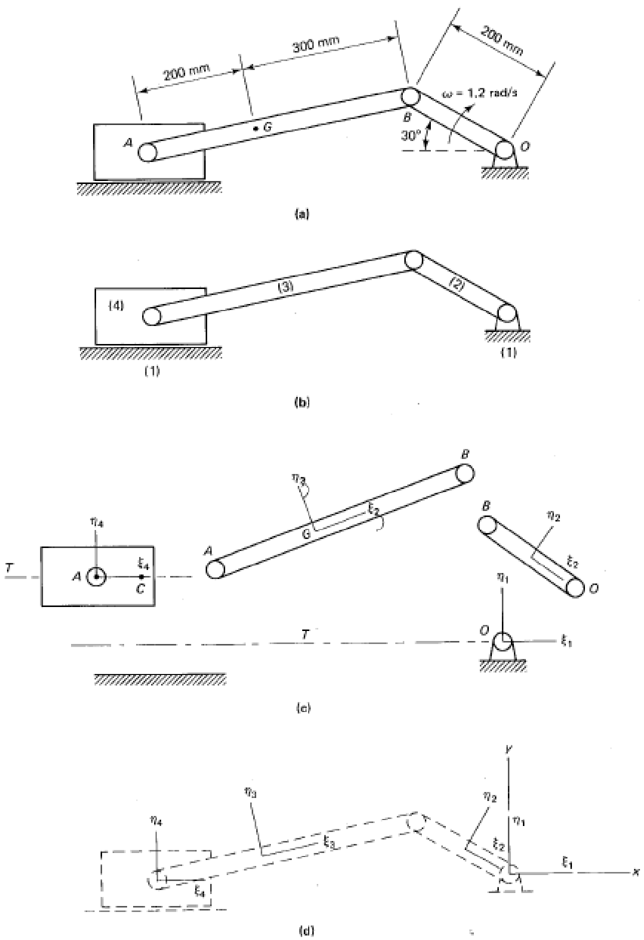
$$\begin{bmatrix} \partial\mathbf{\Phi}/\partial x_1 = +100s\phi_4 \\ \partial\mathbf{\Phi}/\partial y_1 = -100c\phi_4 \\ \partial\mathbf{\Phi}/\partial \phi_1 = -100(-100c\phi_4)c\phi_1 + 100s\phi_4 s\phi_1 \\ \partial\mathbf{\Phi}/\partial \phi_4 = (y_1 - 100s\phi_1 - y_4)100s\phi_4 + (x_1 - 100c\phi_1 - x_4)100c\phi_4 = 0 \end{bmatrix}$$

$$\underset{\sim}{B} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix} \begin{bmatrix} -100 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 - 100c\phi_1 \\ y_1 - 100s\phi_1 \end{bmatrix}$$

$$\underset{\sim}{n} = \begin{bmatrix} c\phi_4 & -s\phi_4 \\ s\phi_4 & c\phi_4 \end{bmatrix} \begin{bmatrix} 0 \\ -100 \end{bmatrix} = \begin{bmatrix} +100s\phi_4 \\ -100c\phi_4 \end{bmatrix}$$

$$\underset{\sim}{n}^T \underset{\sim}{d} = \begin{bmatrix} 100s\phi_4 & -100c\phi_4 \end{bmatrix} \begin{bmatrix} x_1 - 100c\phi_1 - x_4 \\ y_1 - 100s\phi_1 - y_4 \end{bmatrix}$$

Kinematic Modeling



Ground

$$\Phi_1 \equiv x_1 = 0.0$$

$$\Phi_2 \equiv y_1 = 0.0$$

$$\Phi_3 \equiv \phi_1 = 0.0$$

revolute joints

$$\Phi_4 \equiv x_4 - x_3 + 200\cos\phi_3 = 0$$

$$\Phi_5 \equiv y_4 - y_3 + 200\sin\phi_3 = 0$$

$$\Phi_6 \equiv x_3 + 300\cos\phi_3 - x_2 + 100\cos\phi_2 = 0$$

$$\Phi_7 \equiv y_3 + 300\sin\phi_3 - y_2 + 100\sin\phi_2 = 0$$

$$\Phi_8 \equiv x_2 + 100\cos\phi_2 - x_1 = 0$$

$$\Phi_9 \equiv y_2 + 100\sin\phi_2 - y_1 = 0$$

translational joint

$$\Phi_{10} \equiv (-100\cos\phi_4)(y_1 - 100\sin\phi_1 - y_4) - (x_1 - 100\cos\phi_1 - x_4)(-100\sin\phi_4) = 0$$

$$\Phi_{11} \equiv \phi_4 - \phi_1 = 0$$

driving constraint

$$\Phi_{12} \equiv \phi_2 - 5.76 + 1.2t = 0$$

Figure Kinematic modeling of a slider-crank mechanism.

Jacobian Matrix

	x_1	y_1	ϕ_1	x_2	y_2	ϕ_2	x_3	y_3	ϕ_3	x_4	y_4	ϕ_4
$\partial\Phi_1/\partial\ldots$	(1)	0	0	0	0	0	0	0	0	0	0	0
$\partial\Phi_2/\partial\ldots$	0	(2)	0	0	0	0	0	0	0	0	0	0
$\partial\Phi_3/\partial\ldots$	0	0	(3)	0	0	0	0	0	0	0	0	0
$\partial\Phi_4/\partial\ldots$	0	0	0	0	0	0	(4)	0	(5)	(6)	0	(7)
$\partial\Phi_5/\partial\ldots$	0	0	0	0	0	0	0	(8)	(9)	0	(10)	(11)
$\partial\Phi_6/\partial\ldots$	0	0	0	(12)	0	(13)	(14)	0	(15)	0	0	0
$\partial\Phi_7/\partial\ldots$	0	0	0	0	(16)	(17)	0	(18)	(19)	0	0	0
$\partial\Phi_8/\partial\ldots$	(20)	0	(21)	(22)	0	(23)	0	0	0	0	0	0
$\partial\Phi_9/\partial\ldots$	0	(24)	(25)	0	(26)	(27)	0	0	0	0	0	0
$\partial\Phi_{10}/\partial\ldots$	(28)	(29)	(30)	0	0	0	0	0	0	(31)	(32)	(33)
$\partial\Phi_{11}/\partial\ldots$	0	0	(34)	0	0	0	0	0	0	0	0	(35)
$\partial\Phi_{12}/\partial\ldots$	0	0	0	0	0	(36)	0	0	0	0	0	0

Figure The Jacobian matrix.

Cont'd

	x_1	y_1	ϕ_1	x_2	y_2	ϕ_2	x_3	y_3	ϕ_3	x_4	y_4	ϕ_4
$\partial\Phi_1/\partial\ldots$	(1)	0	0	0	0	0	0	0	0	0	0	0
$\partial\Phi_2/\partial\ldots$	0	(2)	0	0	0	0	0	0	0	0	0	0
$\partial\Phi_3/\partial\ldots$	0	0	(3)	0	0	0	0	0	0	0	0	0
$\partial\Phi_4/\partial\ldots$	0	0	0	0	0	0	(4)	0	(5)	(6)	0	(7)
$\partial\Phi_5/\partial\ldots$	0	0	0	0	0	0	0	(8)	(9)	0	(10)	(11)
$\partial\Phi_6/\partial\ldots$	0	0	0	(12)	0	(13)	(14)	0	(15)	0	0	0
$\partial\Phi_7/\partial\ldots$	0	0	0	0	(16)	(17)	0	(18)	(19)	0	0	0
$\partial\Phi_8/\partial\ldots$	(20)	0	(21)	(22)	0	(23)	0	0	0	0	0	0
$\partial\Phi_9/\partial\ldots$	0	(24)	(25)	0	(26)	(27)	0	0	0	0	0	0
$\partial\Phi_{10}/\partial\ldots$	(28)	(29)	(30)	0	0	0	0	0	0	(31)	(32)	(33)
$\partial\Phi_{11}/\partial\ldots$	0	0	(34)	0	0	0	0	0	0	0	0	(35)
$\partial\Phi_{12}/\partial\ldots$	0	0	0	0	0	(36)	0	0	0	0	0	0

(1), (2), (3), (6), (10), (14), (18), (22), (26), (34), (36) = 1

(4), (8), (12), (16), (20), (24), (35) = −1

(7), (11), (21), (25), = 0

(5) $= -200 \sin \phi_3$

(9) $= 200 \cos \phi_3$

(13), (23) $= 100 \sin \phi_2$

(15) $= -300 \sin \phi_3$

(17), (27) $= 100 \cos \phi_2$

(19) $= 300 \cos \phi_3$

(28) $= 100\, s\phi_4$

(29) $= -100\, c\phi_4$

(30) $= +10{,}000(\cos \phi_1 \cos \phi_4 + \sin \phi_1 \sin \phi_4)$

(31) = − (28)

(32) = − (29)

(33) $= 100[\cos \phi_4 (y_1 + 100 \sin \phi_1 - y_4) + c\phi(x_1 - 100 c\phi_1 - x_4)]$

3.2 Solution Technique

kinematic constraints $\boldsymbol{\Phi} \equiv \boldsymbol{\Phi}(\boldsymbol{q}) = 0$

driving link $\boldsymbol{\Phi}^{(d)} \equiv \boldsymbol{\Phi}(\boldsymbol{q}, t) = 0$

velocity equations
$$\frac{\partial \phi}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} = 0 \text{ or } \boldsymbol{\Phi}_q \dot{\boldsymbol{q}} = 0$$

$$\frac{\partial \phi^{(d)}}{\partial \boldsymbol{q}} \underset{\sim}{\dot{\boldsymbol{q}}} + \frac{\partial \phi}{\partial t} = 0 \text{ or } \boldsymbol{\Phi}_q{}^{(d)} \dot{\boldsymbol{q}} + \boldsymbol{\Phi}_t{}^{(d)} = 0$$

$$\begin{bmatrix} \boldsymbol{\Phi}_{\boldsymbol{q}} \\ \boldsymbol{\Phi}_{\boldsymbol{q}}{}^{(d)} \end{bmatrix} \dot{\boldsymbol{q}} = \begin{bmatrix} 0 \\ -\boldsymbol{\Phi}_{\boldsymbol{t}}{}^{(d)} \end{bmatrix}$$

acceleration
$$\frac{\partial \phi}{\partial \boldsymbol{q}} \ddot{\boldsymbol{q}} + \frac{\partial \left(\dfrac{\partial \phi}{\partial \boldsymbol{q}} \right)}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} = 0 \text{ or } \boldsymbol{\Phi}_q \ddot{\boldsymbol{q}} + (\boldsymbol{\Phi}_q \dot{\boldsymbol{q}})_q \dot{\boldsymbol{q}} = 0$$

$$\boldsymbol{\Phi}_q{}^{(d)} \ddot{\boldsymbol{q}} + (\boldsymbol{\Phi}_q{}^{(d)} \dot{\boldsymbol{q}})_q \dot{\boldsymbol{q}} + 2\boldsymbol{\Phi}_{qt}{}^{(d)} \dot{\boldsymbol{q}} + \boldsymbol{\Phi}_{tt}{}^{(d)} = 0$$

$$\begin{bmatrix} \boldsymbol{\Phi}_q \\ \boldsymbol{\Phi}_q^{(d)} \end{bmatrix} \ddot{\boldsymbol{q}} = \begin{bmatrix} -(\boldsymbol{\Phi}_q \dot{\boldsymbol{q}})_q \dot{\boldsymbol{q}} \\ -(\boldsymbol{\Phi}_q^{(d)} \dot{\boldsymbol{q}})_q \dot{\boldsymbol{q}} - 2\boldsymbol{\Phi}_{qt}^{(d)} \dot{\boldsymbol{q}} - \boldsymbol{\Phi}_{tt}^{(d)} \end{bmatrix} = \begin{bmatrix} \gamma \\ -(\boldsymbol{\Phi}_q^{(d)} \dot{\boldsymbol{q}})_q \dot{\boldsymbol{q}} - 2\boldsymbol{\Phi}_{qt}^{(d)} \dot{\boldsymbol{q}} - \boldsymbol{\Phi}_{tt}^{(d)} \end{bmatrix}$$

Solution Technique

At any given instant t

(1) Solve $$\begin{cases} \Phi_q(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{0} \\ \Phi_q^{(d)}(\boldsymbol{q},t)\dot{\boldsymbol{q}} = \text{(the right hand side)} \end{cases}$$

n equations for n unknowns $\boldsymbol{q}(t)$

(2) Solve $$\begin{cases} \Phi_q(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{0} \\ \Phi_q^{(d)}(\boldsymbol{q},t)\dot{\boldsymbol{q}} = \text{(the right hand side)} \end{cases}$$

n equations for n unknowns $\dot{\boldsymbol{q}}(t)$

(3) Solve $$\begin{cases} \Phi_q(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{\gamma} \\ \Phi_q^{(d)}(\boldsymbol{q})\boldsymbol{q} = \text{(the right hand side)} \end{cases}$$

n equations for n unknowns $\ddot{\boldsymbol{q}}(t)$

4.1 Planar Rigid Body Dynamics

$$m_i \ddot{x}_i = f_{xi}$$
$$m_i \ddot{y}_i = f_{yi}$$
$$\mu_i \ddot{\phi}_i = n_i$$

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \\ n \end{bmatrix}_i$$

$$\boldsymbol{M}_i \ddot{\boldsymbol{q}}_i = \boldsymbol{g}_i$$

Constraint Force

$$\boldsymbol{\Phi}(\boldsymbol{q},t)=0,\ \boldsymbol{\Phi}\in\square^{n\times 1}$$

$$\frac{\partial\phi}{\partial\boldsymbol{q}}\delta\boldsymbol{q}=0\ \text{ or }\ \boldsymbol{\Phi}_q\delta\boldsymbol{q}=0,\quad \boldsymbol{\Phi}_q\in\square^{m\times n}$$

There exists Lagrange Multiplier $\boldsymbol{\lambda}$, $\boldsymbol{\lambda}\in\square^{m\times 1}$

such that $\boldsymbol{\lambda}^T\boldsymbol{\Phi}_q$ is that constraint force.

The equations of motion can be written as

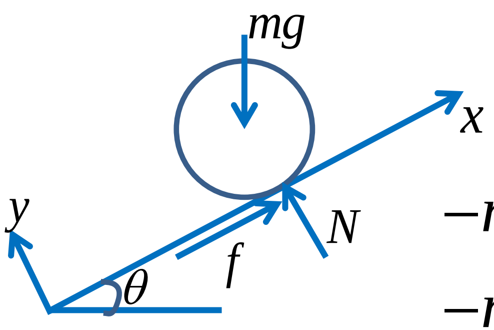
$$\boldsymbol{M}\ddot{\boldsymbol{q}}=\boldsymbol{g}+\boldsymbol{g}^{(c)}$$

$$\boldsymbol{g}^{(c)}=-\boldsymbol{\Phi}_q{}^T\boldsymbol{\lambda}$$

$$\boldsymbol{M}\ddot{\boldsymbol{q}}+\boldsymbol{\Phi}_q^T\boldsymbol{\lambda}=\boldsymbol{g}$$

Illustration of Constraint Force

Pure rolling of a disk down a slope



$$-mgs\theta + f = m\ddot{x}$$

$$-mgc\theta + N = m\ddot{y}$$

$$-mgc\theta + N = 0$$

$$f \cdot R = I_c \ddot{\phi}$$

4 eqs. for 5 unknowns $\ddot{x}, \ddot{y}, \ddot{\phi}, f, N$

constraint equation $\begin{cases} \ddot{x} + R\ddot{\phi} = 0 \\ y - R = 0 \end{cases}$

$$\ddot{x} = -\frac{2}{3} gs\theta,\ \ \ddot{y} = 0,\ \ \ddot{\phi} = \frac{2g}{3R} s\theta,\ \ N = mgc\theta,\ \ f = \frac{1}{3} mgs\theta$$

Generalized coordinate $x,\ \phi$

Constraint eq. $\ddot{x}+R\ddot{\phi}=0 \quad y-R=0$

$$\Rightarrow -x-R\phi=0$$

$$\begin{bmatrix} \underline{M} & \dfrac{\partial\phi}{\partial \underset{\sim}{q}}^T \\ \dfrac{\partial\phi}{\partial \underset{\sim}{q}} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\underset{\sim}{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} -mgs\theta \\ -mgc\theta \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} m & 0 & 0 & -1 & 0 \\ 0 & m & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2}mR^2 & -R & 0 \\ -1 & 0 & -R & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} -mgs\theta \\ -mgc\theta \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

3 eqs. for 3 unknowns $\ddot{x}=-\dfrac{2}{3}gs\theta,\ \ddot{y}=0,\ \ddot{\phi}=\dfrac{2g}{3R}s\theta,\ \lambda=\dfrac{1}{3}mgs\theta$

The constraint force $\quad \dfrac{\partial\phi}{\partial \underset{\sim}{q}}^T \underset{\sim}{\lambda} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ -R & 0 \end{bmatrix} \lambda = \begin{bmatrix} -\frac{1}{3}mgs\theta \\ mgc\theta \\ -\frac{1}{3}mgRs\theta \end{bmatrix}_{3\times 1}$

is the friction force for pure rolling.

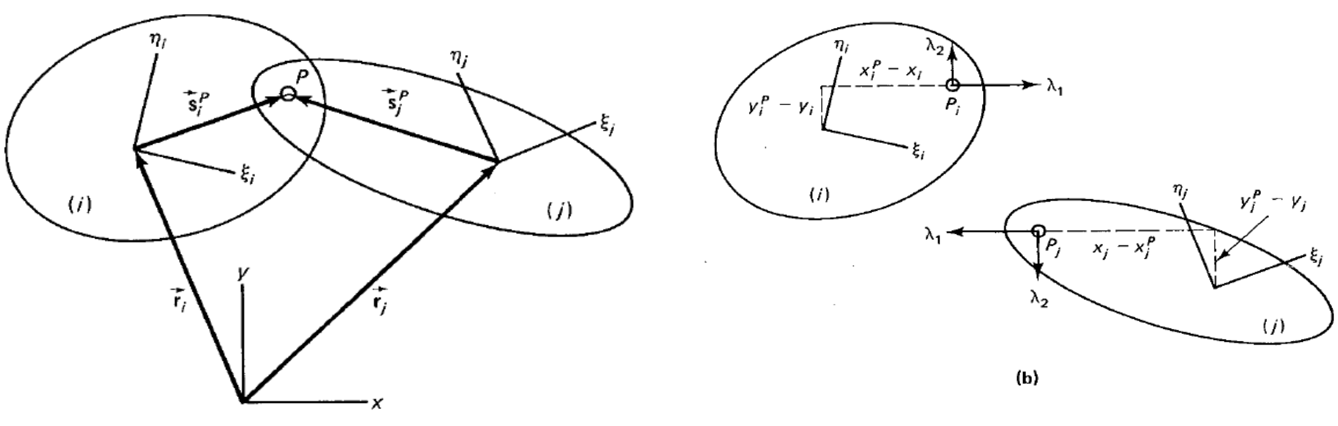
4.2 Constraint Force in Revolute Joint

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{bmatrix}_i + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -(y_i^P - y_i) & (x_i^P - x_i) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \\ n \end{bmatrix}_i$$

$$m_i \ddot{x}_i = f_{(x)i} - \lambda_1$$

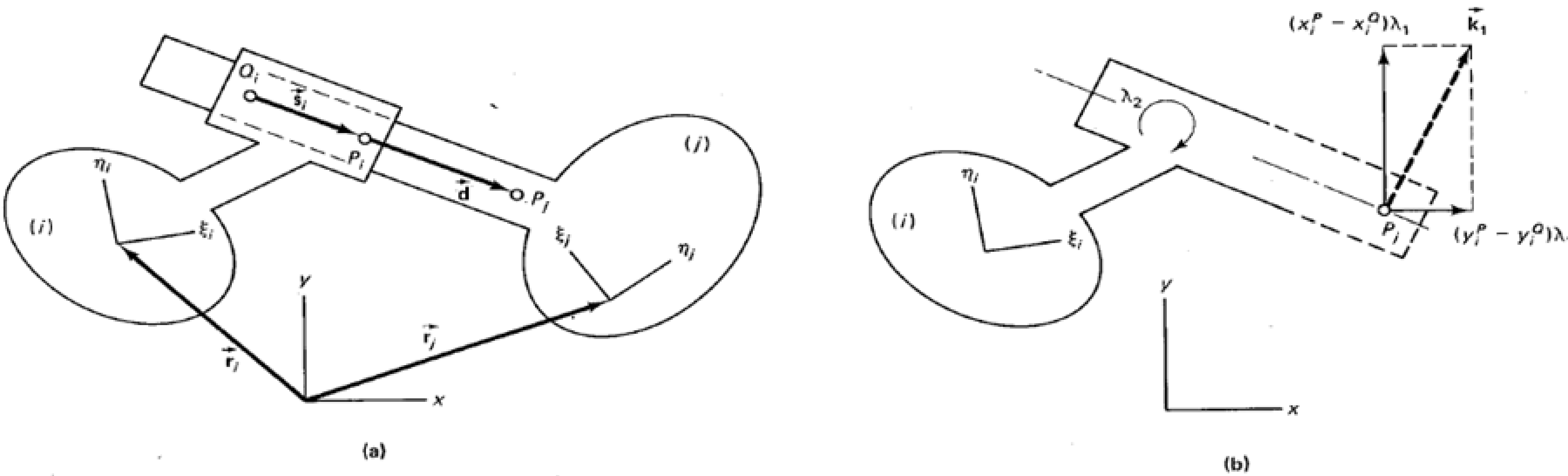
$$m_i \ddot{y}_i = f_{(y)i} - \lambda_2$$

$$\mu_i \ddot{\phi}_i = n_i - (y_i^P - y_i)\lambda_1 + (x_i^P - x_i)\lambda_2$$



(a) (b)

Constraint Force in Translational Joint



For a translational joint between *i* and *j*,
the equation of motion for body *i* can be written as

$$m_i \ddot{x}_i = f_{xi} + (y_i^P - y_i^Q)\lambda_1$$

$$m_i \ddot{y}_i = f_{yi} + (x_i^P - x_i^Q)\lambda_1$$

$$\mu_i \ddot{\phi}_i = n_i - [(x_j^P - x_i)(x_i^P - x_i^Q) + (y_j^P - y_i)(y_i^P - y_i^Q)]\lambda_1 + \lambda_2$$

4.3 Formulation of Multi-body Dynamic Systems

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \Phi_q^T \boldsymbol{\lambda} = \boldsymbol{g}$$

$$\Phi_q \ddot{\boldsymbol{q}} - \boldsymbol{\gamma} = 0$$

$$\begin{bmatrix} \boldsymbol{M} & \Phi_q^T \\ \Phi_q & 0 \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{g} \\ \boldsymbol{\gamma} \end{bmatrix}$$

$n+m$ linear algebraic equations in $n+m$ unknowns for $\ddot{q}$ and λ.

A Matlab Program for

Dynamics of a four-bar linkage

$\overline{OA} = 0.08\text{m}$

$\overline{AB} = 0.26\text{m}$

$\overline{BC} = 0.18\text{m}$

$\overline{OC} = 0.18\text{m}$

$M_{\overline{OA}} = 0.08\text{kg}$

$M_{\overline{AB}} = 0.26\text{kg}$

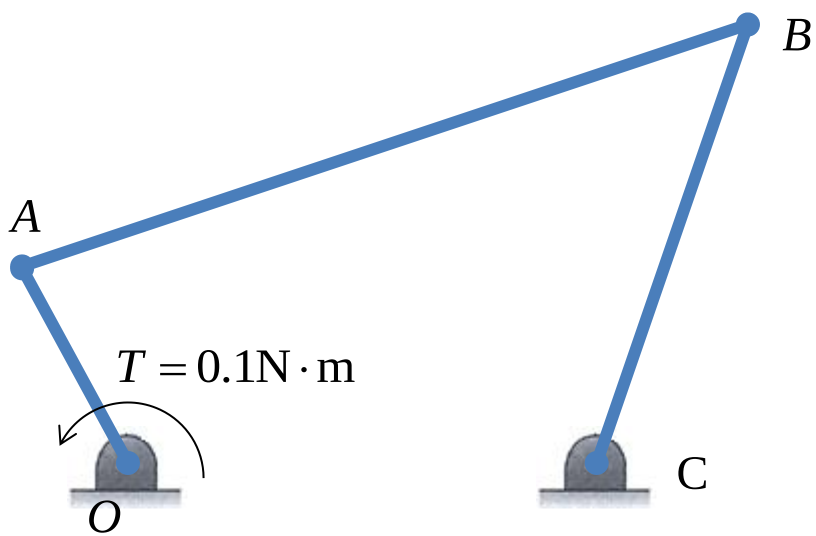
$M_{\overline{BC}} = 0.18\text{kg}$

$I_{\overline{OA}} = 4.27 \times 10^{-5}\,\text{kg}\cdot\text{m}^2$

$I_{\overline{AB}} = 1.46 \times 10^{-3}\,\text{kg}\cdot\text{m}^2$

$I_{\overline{BO}} = 4.86 \times 10^{-4}\,\text{kg}\cdot\text{m}^2$

$g = 9.8\text{m/s}^2 \downarrow$



Mass matrix and external force vector

$$M_{\overline{OA}} = 0.08\text{kg} \qquad I_{\overline{OA}} = 4.27\times10^{-5}\,\text{kg}\cdot\text{m}^2$$

$$M_{\overline{AB}} = 0.26\text{kg} \qquad I_{\overline{AB}} = 1.46\times10^{-3}\,\text{kg}\cdot\text{m}^2$$

$$M_{\overline{BC}} = 0.18\text{kg} \qquad I_{\overline{BO}} = 4.86\times10^{-4}\,\text{kg}\cdot\text{m}^2$$

$$g = 9.8\text{m/s}^2 \downarrow$$

$$T = 0.1\text{N}\cdot\text{m}$$

$$\mathbf{M} = \begin{bmatrix} 0.08 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.08 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4.27\times10^{-5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.26 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.26 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.46\times10^{-3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.18 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.18 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4.86\times10^{-4} \end{bmatrix}$$

$$\boldsymbol{g} = \begin{bmatrix} 0 \\ 0.08\times9.8 \\ 0.1 \\ 0 \\ 0.26\times9.8 \\ 0 \\ 0 \\ 0.18\times9.8 \\ 0 \end{bmatrix}$$

Jacobian matrix and $\boldsymbol{\gamma}$

$$\overline{OA} = 0.08\text{m}$$

$$\overline{AB} = 0.26\text{m}$$

$$\overline{BC} = 0.18\text{m}$$

$$\overline{OC} = 0.18\text{m}$$

$$\mathbf{J}_{9\times 9} = \begin{bmatrix} -1 & 0 & -0.04\sin\phi_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0.04\cos\phi_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -0.04\sin\phi_1 & -1 & 0 & -0.13\sin\phi_2 & 0 & 0 & 0 \\ 0 & 1 & 0.04\cos\phi_1 & 0 & -1 & 0.13\cos\phi_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -0.13\sin\phi_2 & -1 & 0 & -0.09\sin\phi_3 \\ 0 & 0 & 0 & 0 & 1 & 0.13\cos\phi_2 & 0 & -1 & 0.09\cos\phi_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -0.09\sin\phi_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.09\cos\phi_3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{\gamma} = \begin{bmatrix} 0.04\cos\phi_1 \cdot \dot{\phi}_1^2 \\ 0.04\sin\phi_1 \cdot \dot{\phi}_1^2 \\ 0.05\cos\phi_1 \cdot \dot{\phi}_1^2 + 0.13\cos\phi_2 \cdot \dot{\phi}_2^2 \\ 0.05\sin\phi_1 \cdot \dot{\phi}_1^2 + 0.13\sin\phi_2 \cdot \dot{\phi}_2^2 \\ 0.13\cos\phi_2 \cdot \dot{\phi}_2^2 + 0.09\cos\phi_3 \cdot \dot{\phi}_3^2 \\ 0.13\sin\phi_2 \cdot \dot{\phi}_2^2 + 0.09\sin\phi_3 \cdot \dot{\phi}_3^2 \\ 0.09\cos\phi_3 \cdot \dot{\phi}_3^2 \\ 0.09\sin\phi_3 \cdot \dot{\phi}_3^2 \\ 0 \end{bmatrix}$$

Computation

The objective is to solve the differential equation $\begin{bmatrix} \mathbf{M}_{9\times9} & \mathbf{J}_{9\times8}^{T} \\ \mathbf{J}_{9\times8} & \mathbf{O}_{8\times8} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{g} \\ \boldsymbol{\gamma} \end{bmatrix}$ for $\boldsymbol{q}(t)$.

In numerial computing, the first step is to solve $\begin{bmatrix} \mathbf{M} & \mathbf{J}_{9\times8}^{T} \\ \mathbf{J}_{9\times8} & \mathbf{O}_{8\times8} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{g} \\ \boldsymbol{\gamma} \end{bmatrix}$

for $\ddot{\boldsymbol{q}}(0)_{9\times1}$ and $\boldsymbol{\lambda}(0)_{8\times1}$ with initial position $\boldsymbol{q}(0)$ and velocity $\dot{\boldsymbol{q}}(0)$.

Integrate $\begin{bmatrix} \ddot{\boldsymbol{q}}(0) \\ \dot{\boldsymbol{q}}(0) \end{bmatrix} \xrightarrow{\Delta t} \begin{bmatrix} \dot{\boldsymbol{q}}(\Delta t) \\ \boldsymbol{q}(\Delta t) \end{bmatrix}$, and use the $\dot{\boldsymbol{q}}(\Delta t)$ and $\boldsymbol{q}(\Delta t)$

to repeat the above step for $\ddot{\boldsymbol{q}}(\Delta t)$ and $\boldsymbol{\lambda}(\Delta t)$.

Repeat $t + \Delta t$ until terminal condition.

A convenient way with Matlab solver

Solve initial value problems for ordinary differential equations with ode45(commended), ode23, ode113…

The equations are described in the form of $\boldsymbol{z}'=f(t,\boldsymbol{z})$

$$\text{Let } \boldsymbol{z} = \begin{bmatrix} x_1 \\ y_1 \\ \vdots \\ \phi_3 \\ \dot{x}_1 \\ \dot{y}_1 \\ \vdots \\ \dot{\phi}_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{bmatrix}, \boldsymbol{z}' = \begin{bmatrix} \dot{\boldsymbol{q}} \\ \ddot{\boldsymbol{q}} \end{bmatrix}$$

The syntax for calling solver in Matlab

Solution array

A vector of initial conditions

```
[T,Z] = ode45(@Func4Bar,[0:0.005:2],Z0);
```

column vector of time points

A vector specifying the interval of integration

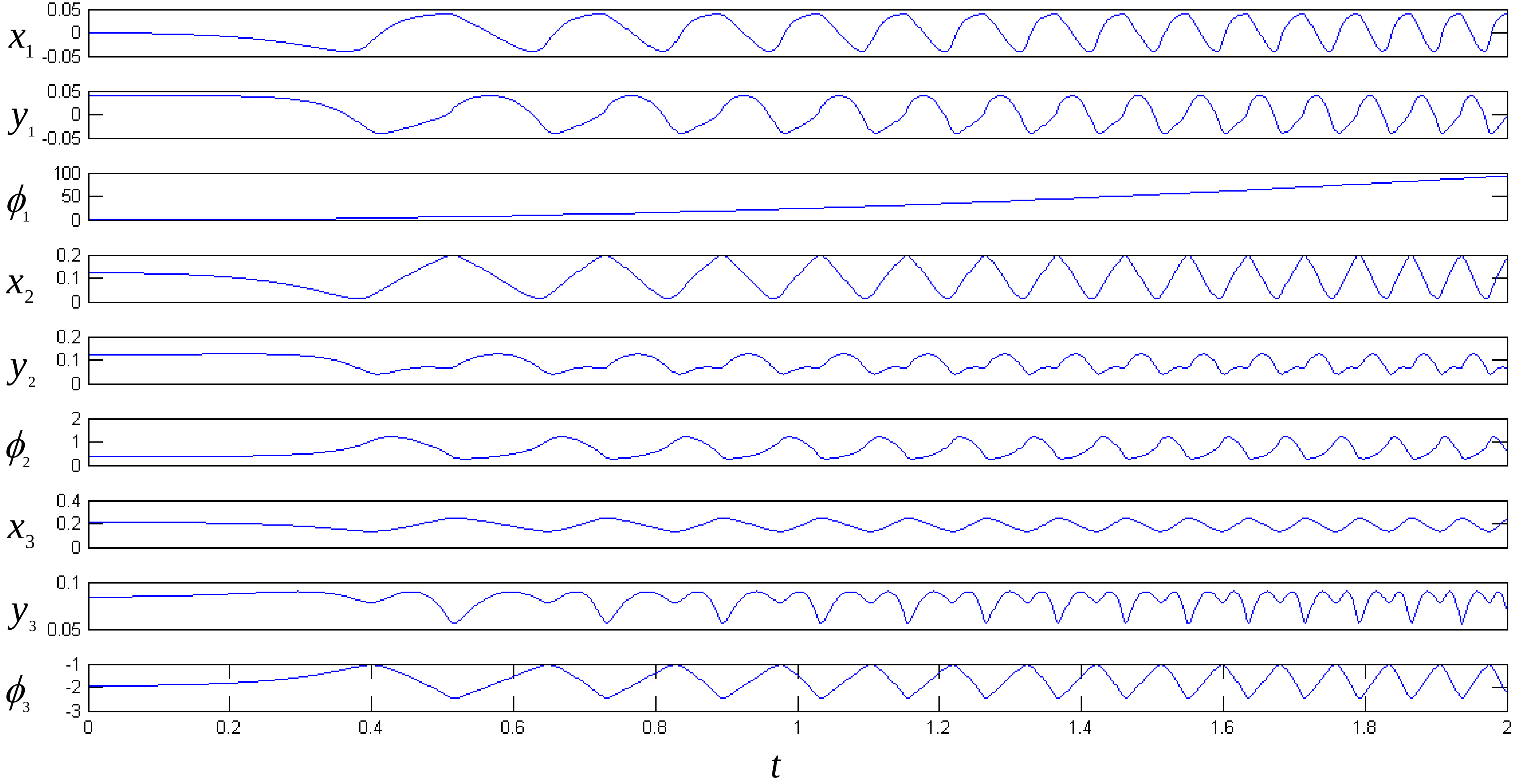
A function that evaluates the right side of the differential equations

```
function dz=Func4Bar(t,z)

global L1 L2 L3 L4 torque gravity

phi1=z(3); phi2=z(6); phi3=z(9);
dphi1=z(12); dphi2=z(15); dphi3=z(18);

M=diag([L1 L1 L1^3/12 L2 L2 L2^3/12 L3 L3 L3^3/12]);

J=[ -1 0 -0.5*L1*sin(phi1)  0 0 0                    0 0 0;
    0 -1 0.5*L1*cos(phi1)   0 0 0                    0 0 0;
    1 0  -0.5*L1*sin(phi1)  -1 0 -0.5*L2*sin(phi2)   0 0 0;
    0 1  0.5*L1*cos(phi1)   0 -1 0.5*L2*cos(phi2)    0 0 0;
    0 0 0                   1 0 -0.5*L2*sin(phi2)    -1 0 -0.5*L3*sin(phi3);
    0 0 0                   0 1  0.5*L2*cos(phi2)    0 -1 0.5*L3*cos(phi3);
    0 0 0                   0 0 0                    1 0 -0.5*L3*sin(phi3);
    0 0 0                   0 0 0                    0 1 0.5*L3*cos(phi3)];
```

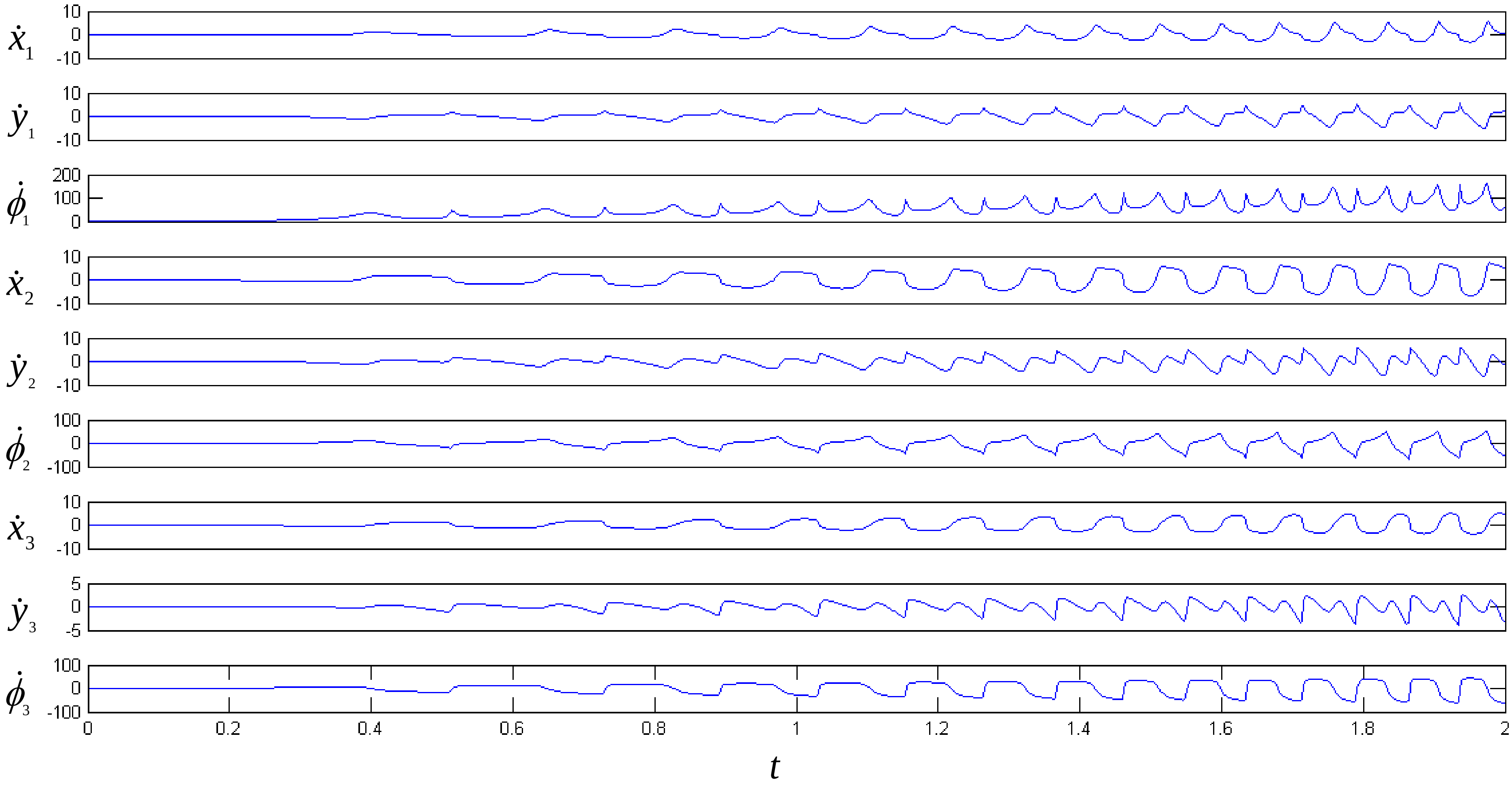
The syntax for calling solver in Matlab

```
J=[ -1 0 -0.5*L1*sin(phi1)  0 0 0                    0 0 0;
    0 -1 0.5*L1*cos(phi1)   0 0 0                    0 0 0;
    1 0  -0.5*L1*sin(phi1)  -1 0 -0.5*L2*sin(phi2)  0 0 0;
    0 1  0.5*L1*cos(phi1)   0 -1 0.5*L2*cos(phi2)   0 0 0;
    0 0 0                   1 0 -0.5*L2*sin(phi2)   -1 0 -0.5*L3*sin(phi3);
    0 0 0                   0 1  0.5*L2*cos(phi2)   0 -1 0.5*L3*cos(phi3);
    0 0 0                   0 0 0                    1 0 -0.5*L3*sin(phi3);
    0 0 0                   0 0 0                    0 1 0.5*L3*cos(phi3)];


gamma=[ 0.5*L1*cos(phi1)*dphi1^2;
        0.5*L1*sin(phi1)*dphi1^2;
        0.5*L1*cos(phi1)*dphi1^2+0.5*L2*cos(phi2)*dphi2^2;
        0.5*L1*sin(phi1)*dphi1^2+0.5*L2*sin(phi2)*dphi2^2;
        0.5*L2*cos(phi2)*dphi2^2+0.5*L3*cos(phi3)*dphi3^2;
        0.5*L2*sin(phi2)*dphi2^2+0.5*L3*sin(phi3)*dphi3^2;
        0.5*L3*cos(phi3)*dphi3^2;
        0.5*L3*sin(phi3)*dphi3^2];

g=[0 gravity*L1 torque  0 gravity*L2 0  0 gravity*L3 0]';

Matrix=[M J';
        J zeros(size(J,1),size(J,1))];

d2q=Matrix\[g;gamma];

dz=[z(10:18,:); d2q(1:9,:)];
```

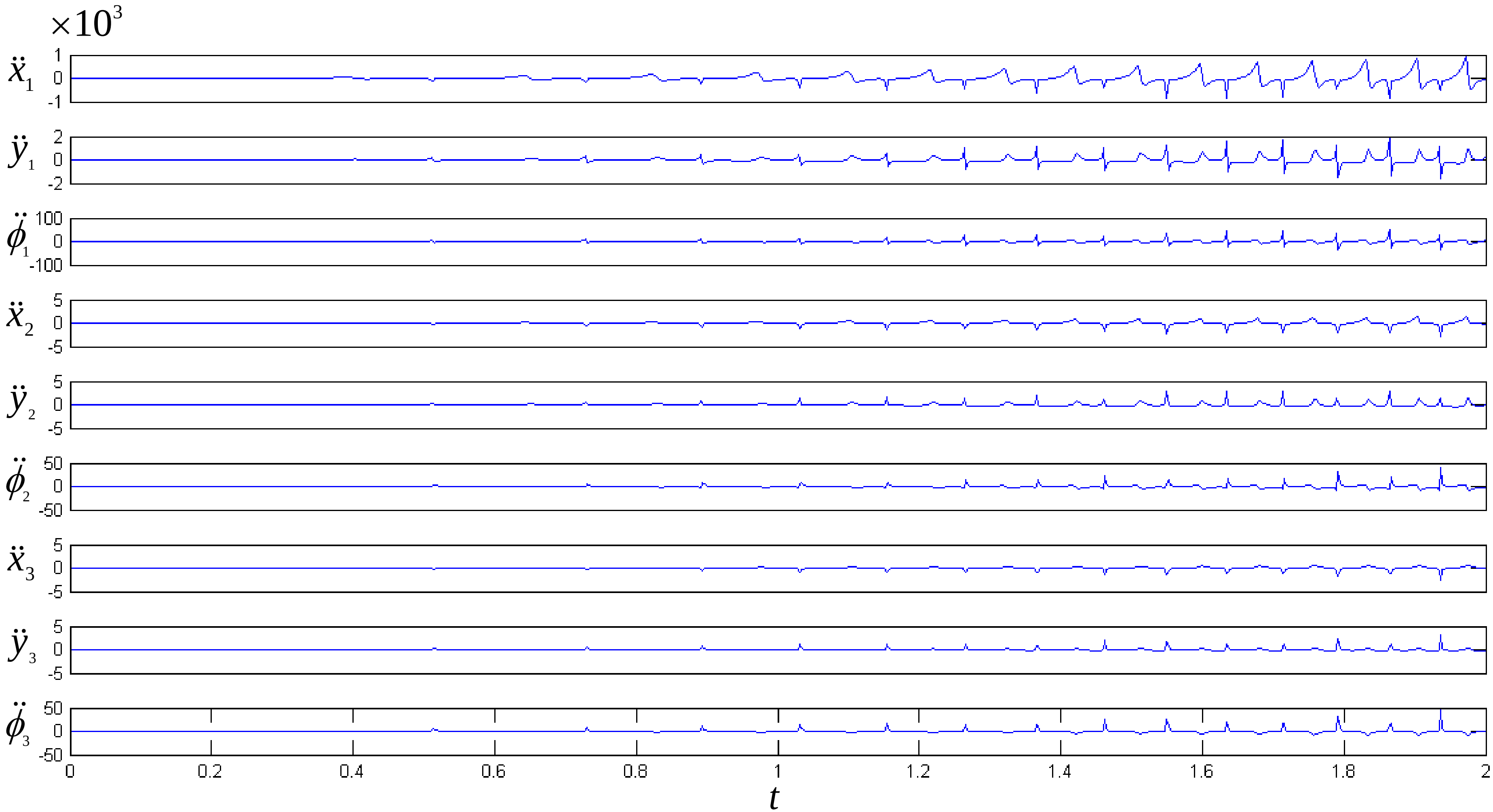
Time response of displacement



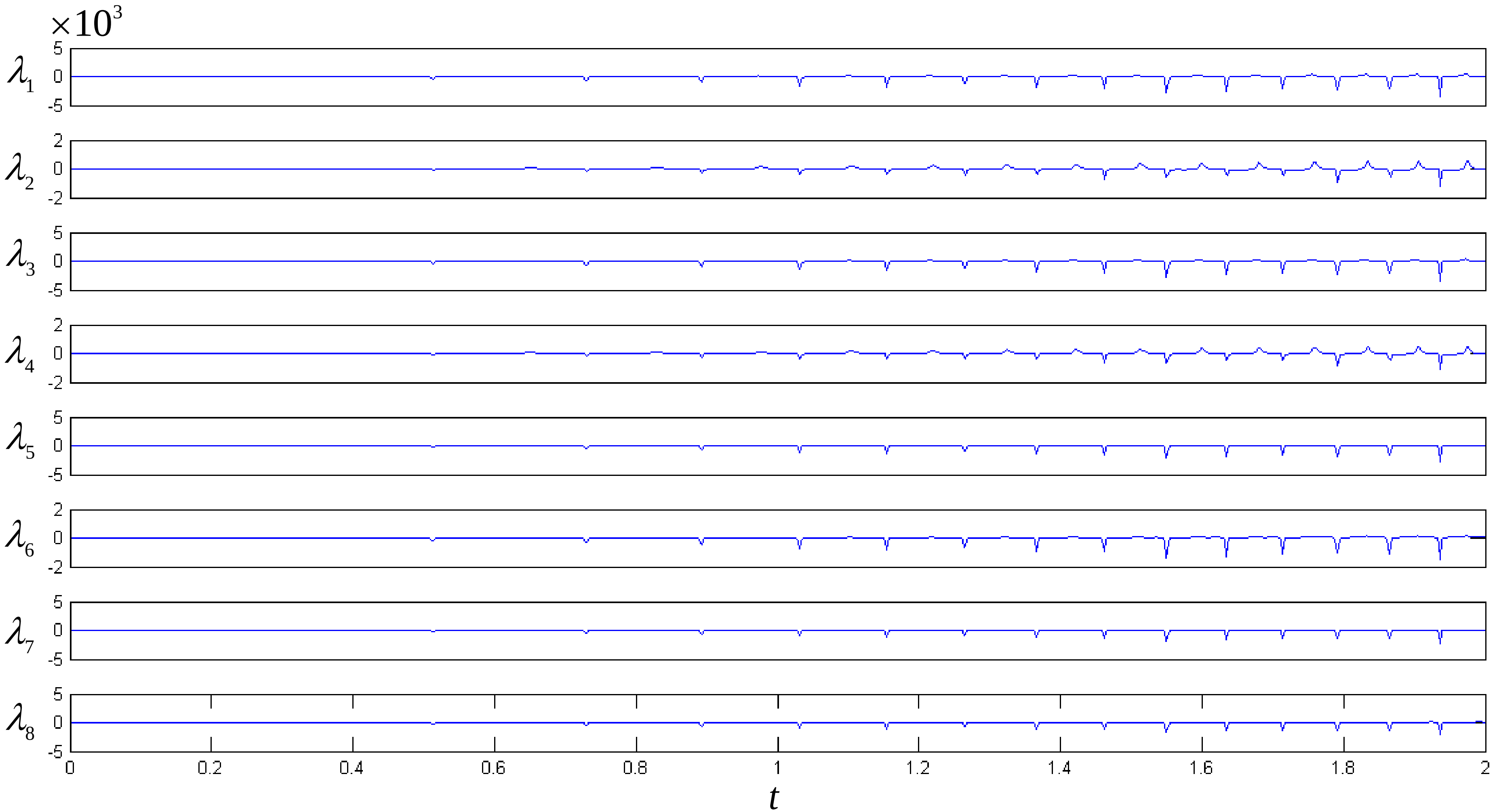
Time response of velocity



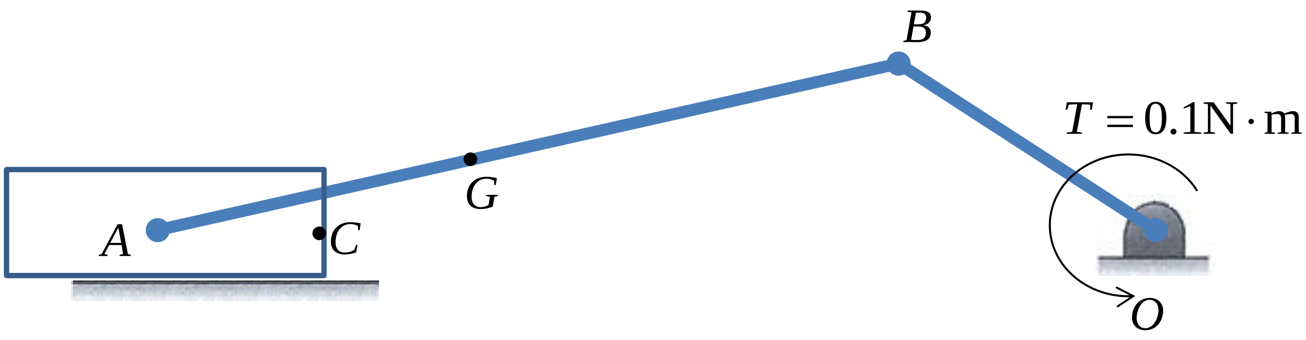
Time response of acceleration



Time response of $\boldsymbol{\lambda}$



A slider-crank mechanism



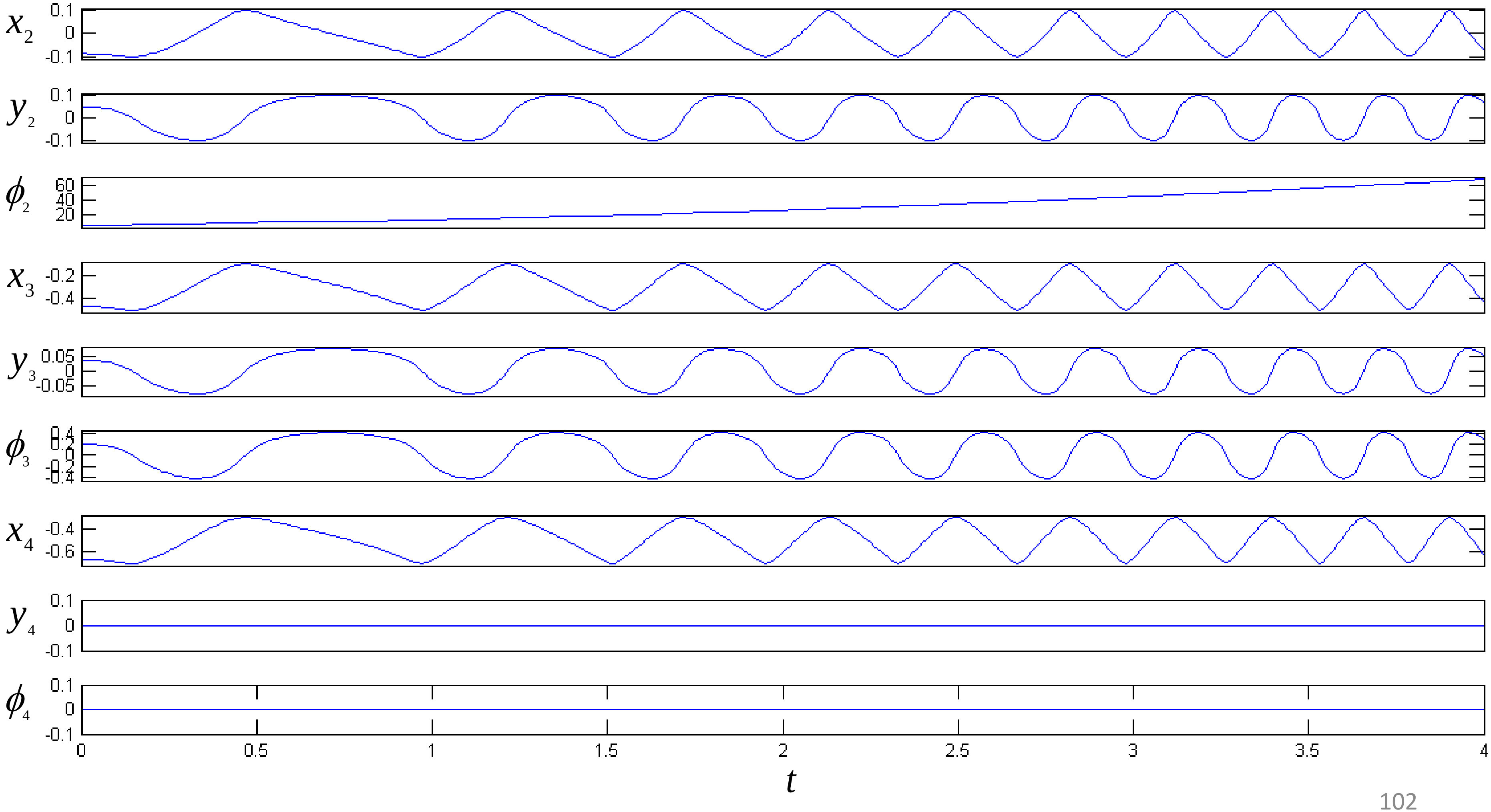
$\overline{AC} = 0.1\text{m}$ $M_{\overline{AC}} = 0.1\text{kg}$ $I_{slider} = 6.67\times10^{-4}\,\text{kg}\cdot\text{m}^2$

$\overline{AG} = 0.2\text{m}$ $M_{\overline{AB}} = 0.5\text{kg}$ $I_{\overline{AB}} = 1.04\times10^{-2}\,\text{kg}\cdot\text{m}^2$

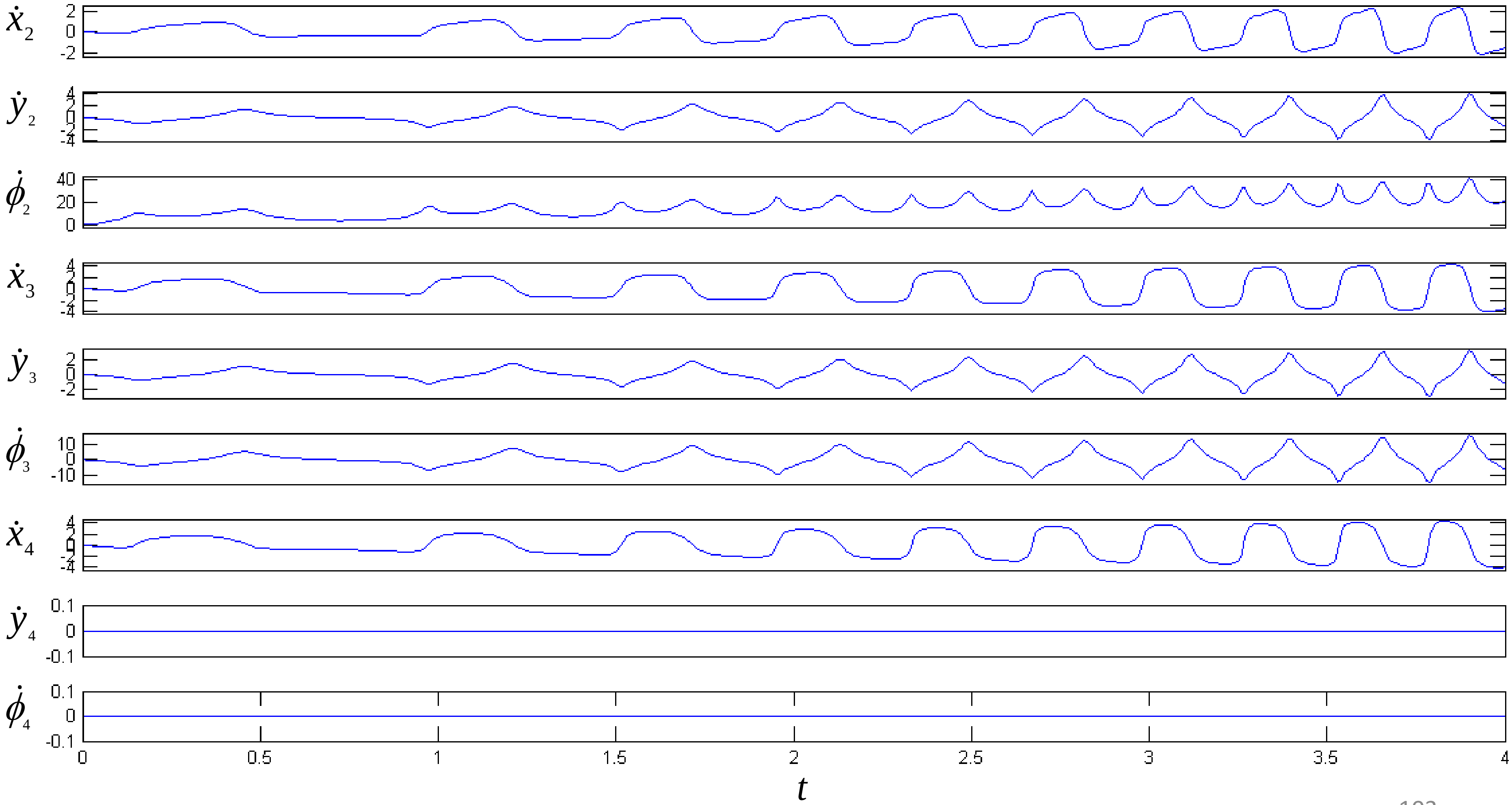
$\overline{GB} = 0.3\text{m}$ $M_{\overline{BO}} = 0.2\text{kg}$ $I_{\overline{BO}} = 6.67\times10^{-4}\,\text{kg}\cdot\text{m}^2$

$\overline{BO} = 0.2\text{m}$ $g = 9.8\text{m/s}^2 \downarrow$

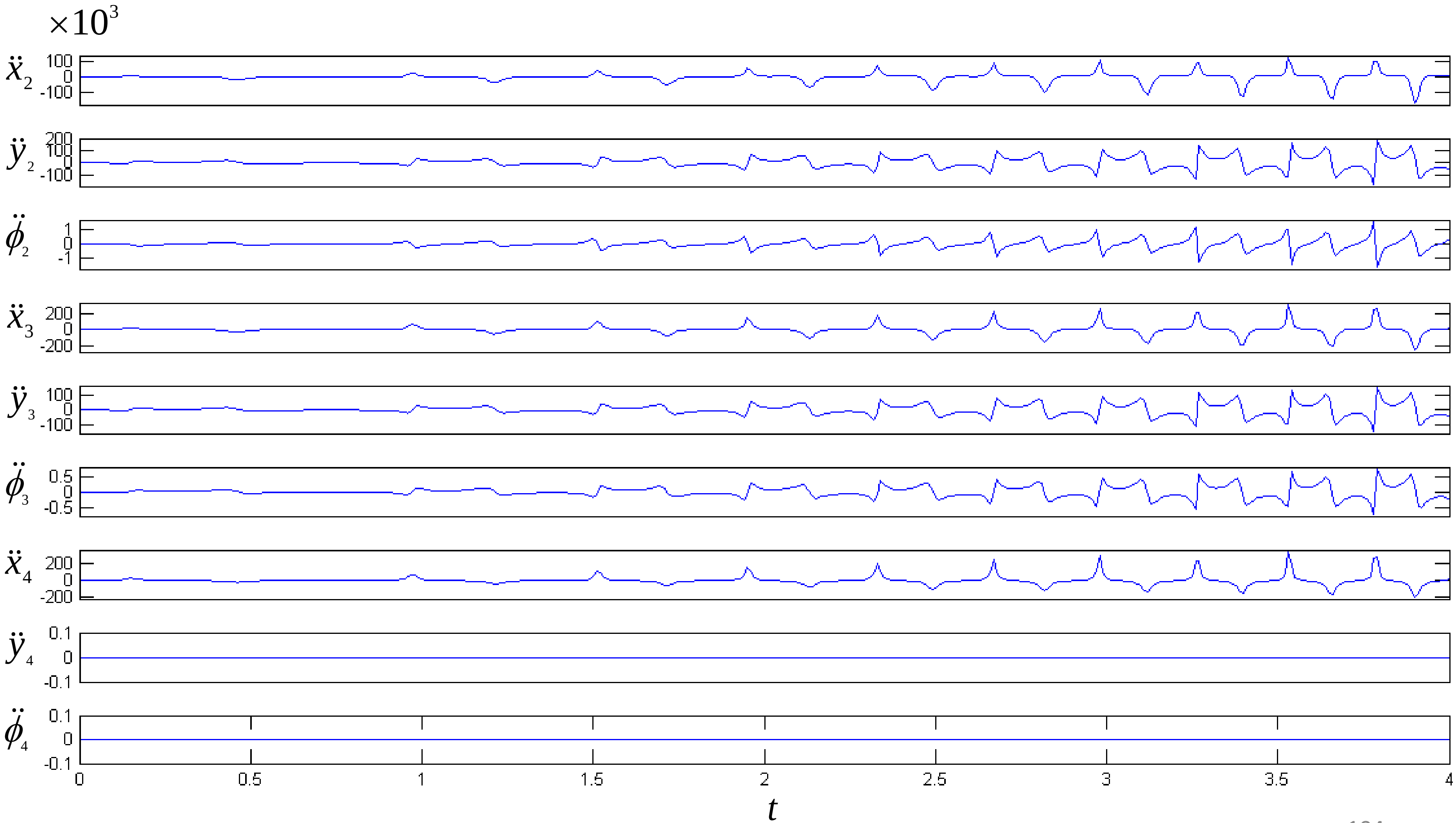
Time response of displacement



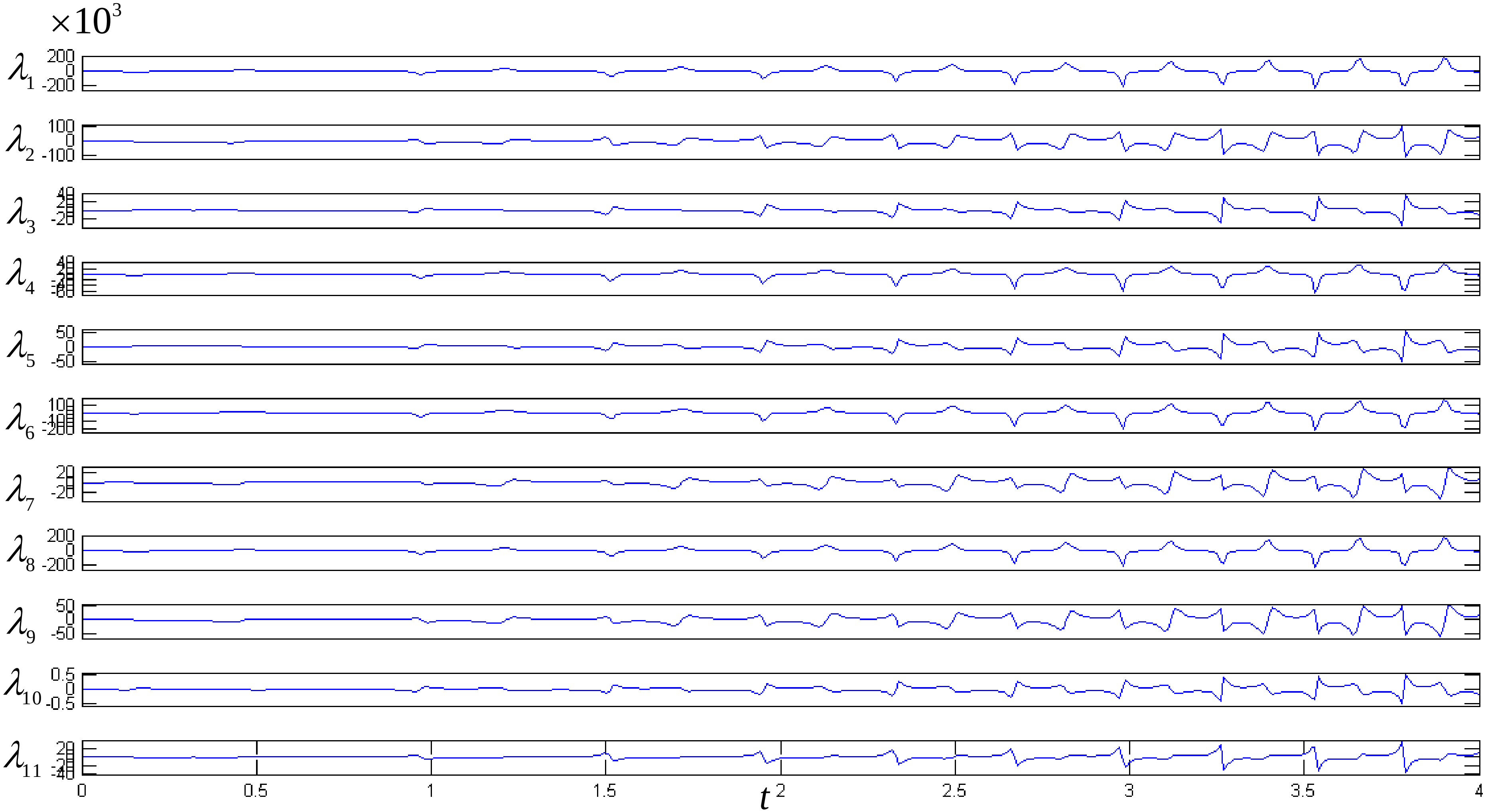
Time response of velocity



Time response of acceleration



Time response of $\boldsymbol{\lambda}$



5.1 Euler Angles

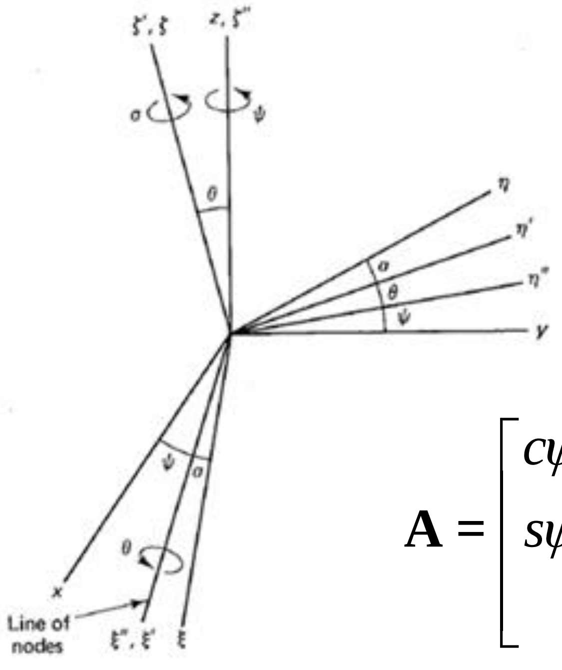


Figure 5.1 The rotations defining the Euler Angles.

$$\mathbf{A} = \begin{bmatrix} c\psi c\phi - s\psi c\theta s\phi & -c\psi s\phi - s\psi c\theta c\phi & s\psi s\theta \\ s\psi c\phi + c\psi c\theta s\phi & -s\psi s\phi + c\psi c\theta c\phi & -c\psi s\theta \\ s\theta s\phi & s\theta c\phi & c\theta \end{bmatrix}$$

$$\mathbf{A} \equiv \mathbf{DCB} \qquad \mathbf{D} = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Time Derivatives of Euler Angles

$$w_{(x)} = \dot{y}\sin q \sin f + \dot{q}\cos f$$

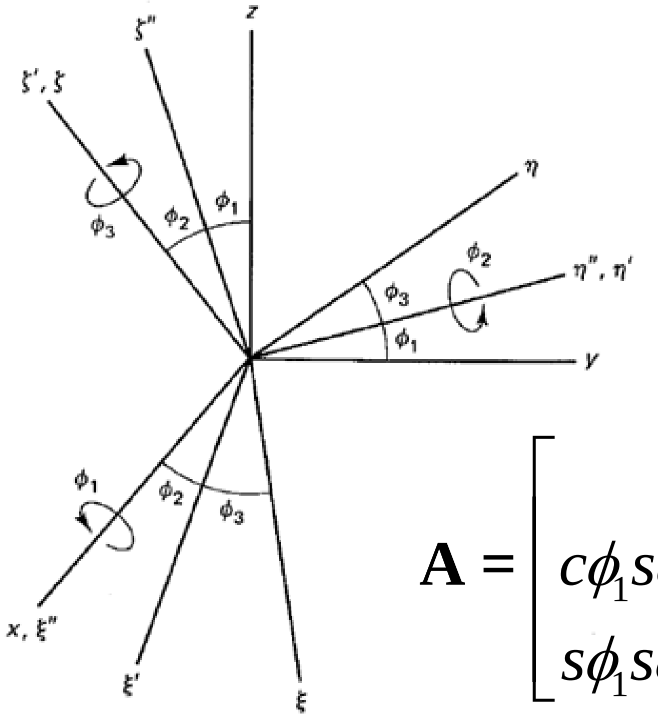
$$w_{(h)} = \dot{y}\sin q \cos f - \dot{q}\sin f$$

$$w_{(z)} = \dot{y}\cos q + \dot{f}$$

$$\begin{bmatrix} \omega_{(\xi)} \\ \omega_{(\eta)} \\ \omega_{(\zeta)} \end{bmatrix} = \begin{bmatrix} \sin\theta\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & -\sin\phi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}$$

Bryant Angles

$$\mathbf{A} = \mathbf{DCB} \quad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi_1 & -s\phi_1 \\ 0 & s\phi_1 & c\phi_1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} c\phi_2 & 0 & s\phi_2 \\ 0 & 1 & 0 \\ -s\phi_2 & 0 & c\phi_2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} c\phi_3 & -s\phi_3 & 0 \\ s\phi_3 & c\phi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\mathbf{A} = \begin{bmatrix} c\phi_2 c\phi_3 & -c\phi_2 s\phi_3 & s\phi_2 \\ c\phi_1 s\phi_3 + s\phi_1 s\phi_2 c\phi_3 & c\phi_1 c\phi_3 + s\phi_1 s\phi_2 s\phi_3 & -s\phi_1 c\phi_2 \\ s\phi_1 s\phi_3 + c\phi_1 s\phi_2 c\phi_3 & s\phi_1 c\phi_3 + c\phi_1 s\phi_2 s\phi_3 & c\phi_1 c\phi_2 \end{bmatrix}$$

Figure 5.4 Rotations defining Bryant angles.

Time Derivative of Bryant Angles

$$\begin{bmatrix} \omega_{(\xi)} \\ \omega_{(\eta)} \\ \omega_{(\zeta)} \end{bmatrix} = \begin{bmatrix} \cos\phi_1 \cos\phi_3 & \sin\phi_3 & 0 \\ -\cos\phi_2 \sin\phi_3 & \cos\phi_3 & 0 \\ \sin\phi_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{bmatrix}$$