Deflections of Beams and Shafts

Deflection and slope are sometimes the limiting factors in beam and shaft design.

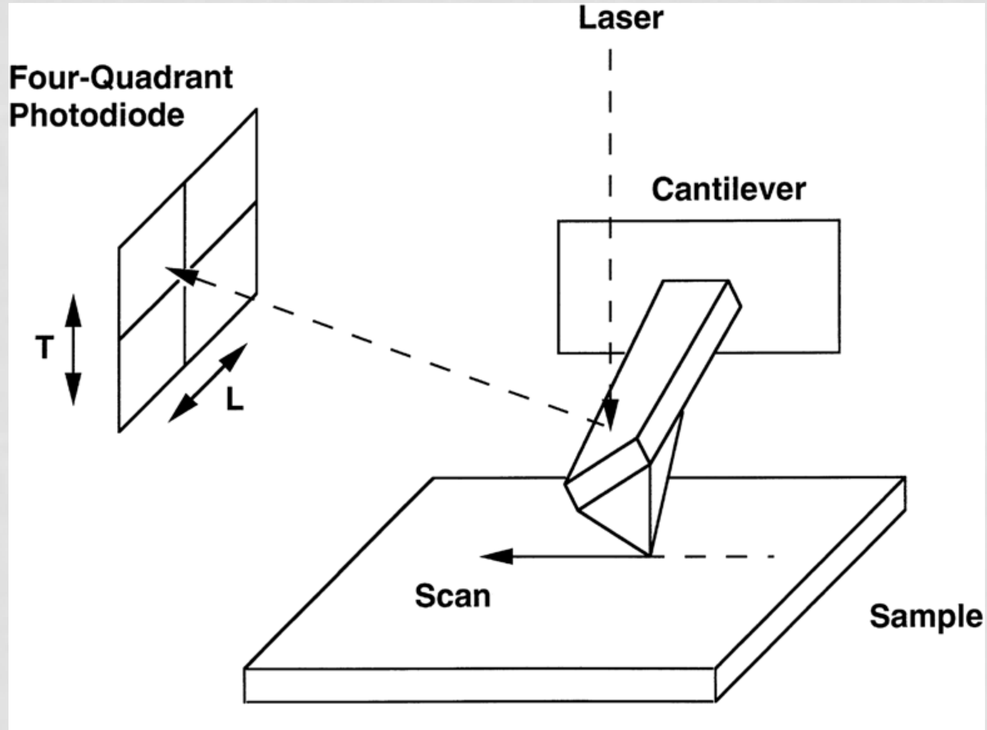
Methods for solving deflection and problems include:

- Integration Methods ☆
- Discontinuity (Singularity) Functions
- Moment-Area Method – ??
- Energy Methods (Castigliano's Theorem)
- Superposition Methods ☆
- Charts & tables ☆

Scanning tunnel microscope

STM
AFM
FFM
MFM

scanning probe microscopes



Source: Overney, 1995

Schematic drawing of a cantilever used in a typical scanning probe microscope

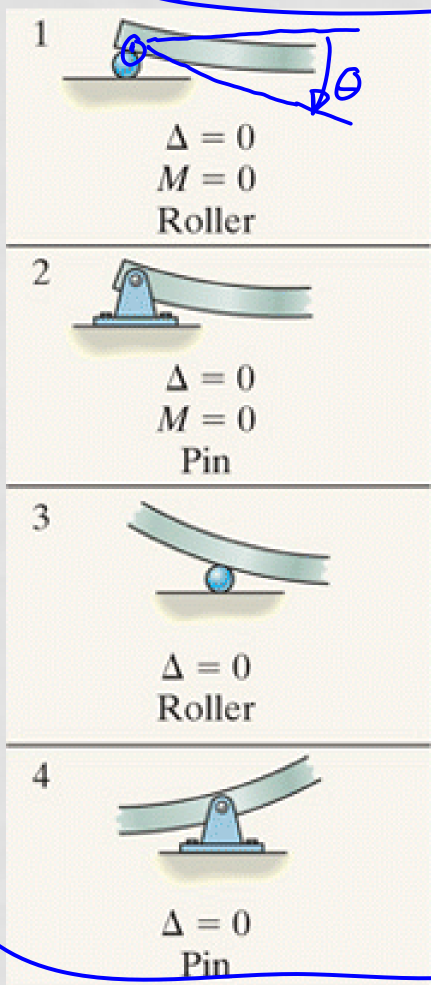
Visualization of Elastic Curves

It is useful to visualize elastic curves for beams subjected to known loads.

vertical

The schematics below can help you determine whether or not forces (V), moments (M), displacements (Δ), or slopes (θ) are zero for various types of supports.

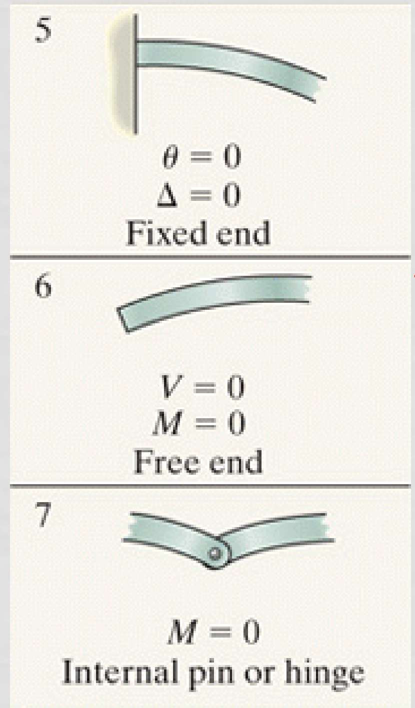
Δ



Displacements zero & moments zero if pin or roller is at end of shaft

$V \neq 0$

$\theta \neq 0$



moment transferred to wall $M \neq 0$

$V \neq 0$

$\Delta \neq 0$

$\theta \neq 0$

$\Delta \neq 0$

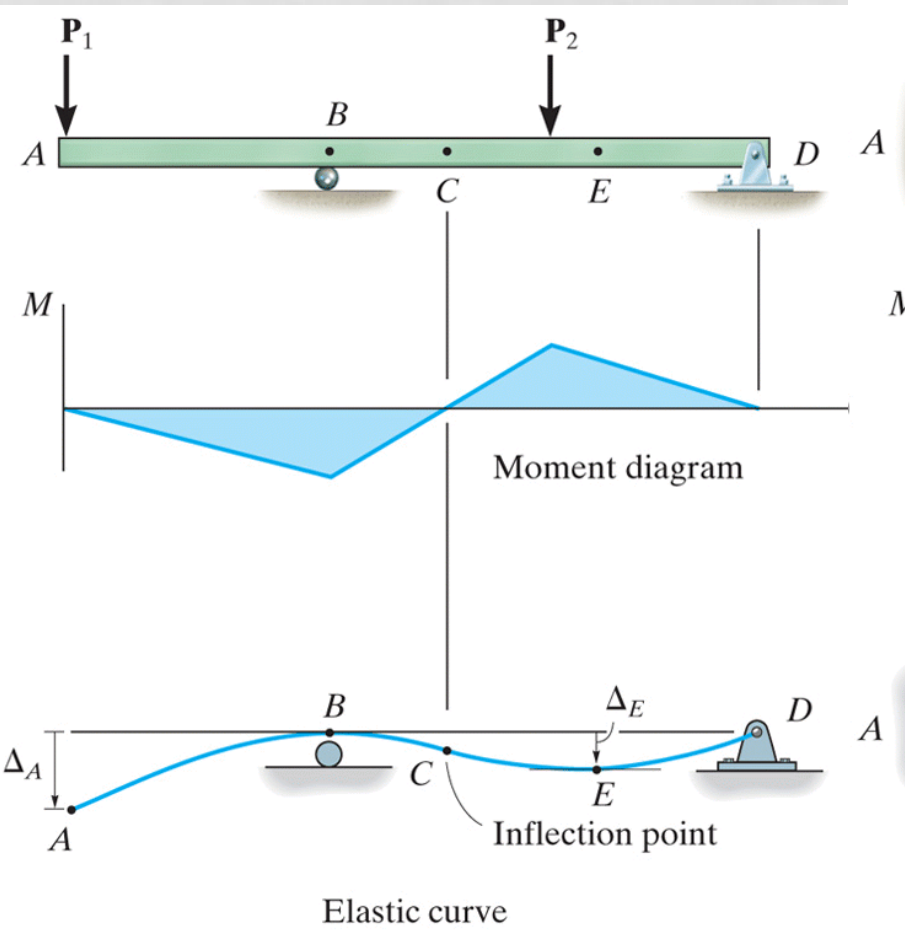
$\theta \neq 0$

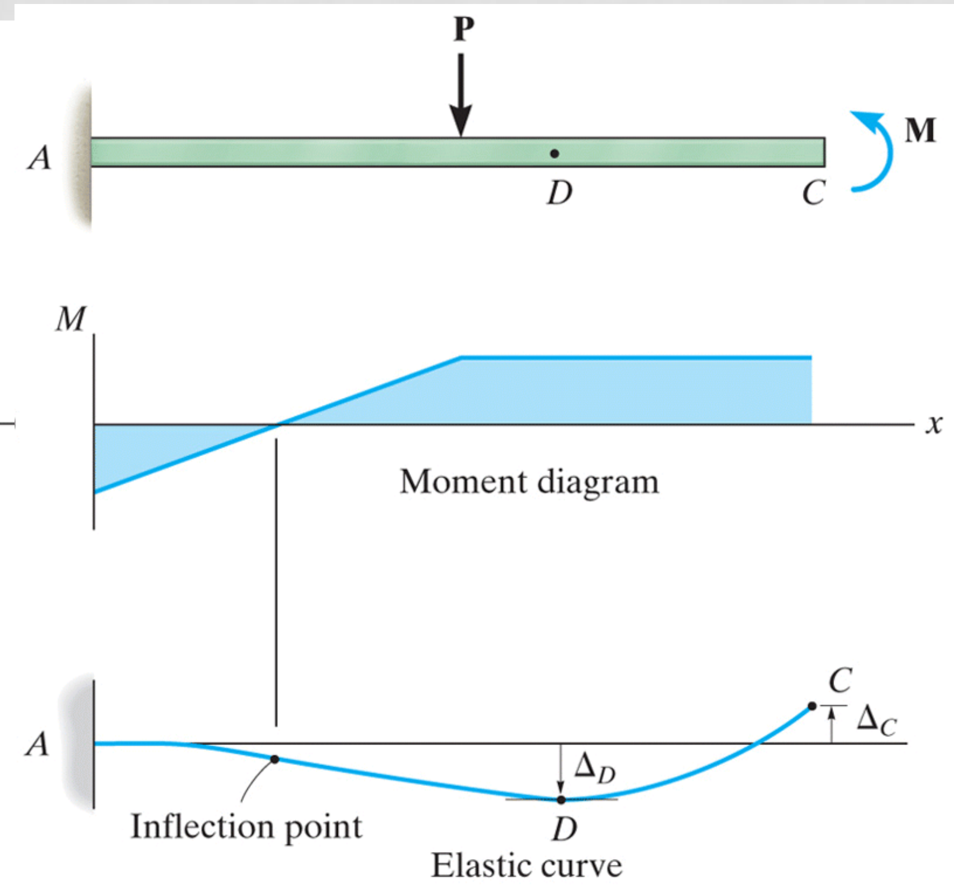
$V \neq 0$

Boundary Conditions

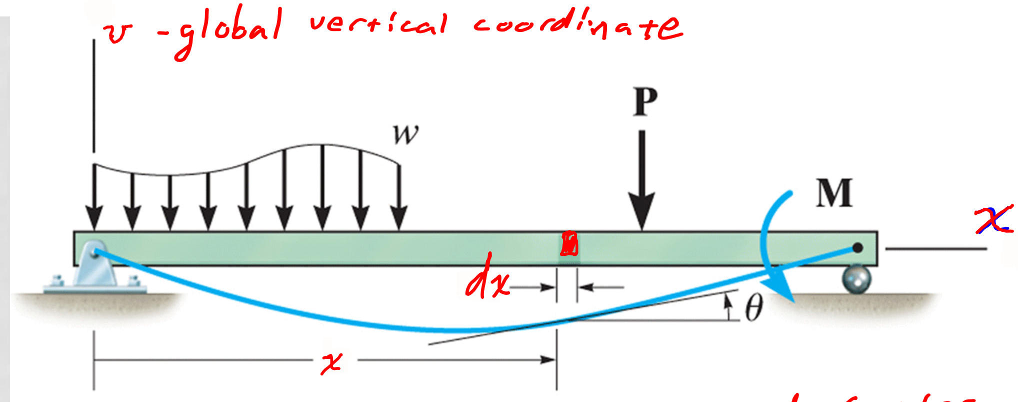
Visualization of Elastic Curves (continued)

Using bending moment diagrams together with an understanding of how the supports contribute to deflection and slope, elastic curves can be sketched as shown in the examples below:





Moment Curvature Relationship



ε is normal strain at a distance y from N. A.:

$$\varepsilon = \frac{ds' - ds}{ds} = \frac{(\rho - y)\cancel{d\theta} - \rho \cancel{d\theta}}{\rho \cancel{d\theta}}$$

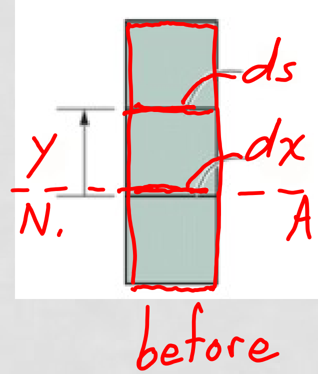
$$\varepsilon = \cancel{1} - \frac{y}{\rho} \cancel{-1} \; ; \quad \varepsilon = -\frac{y}{\rho}$$

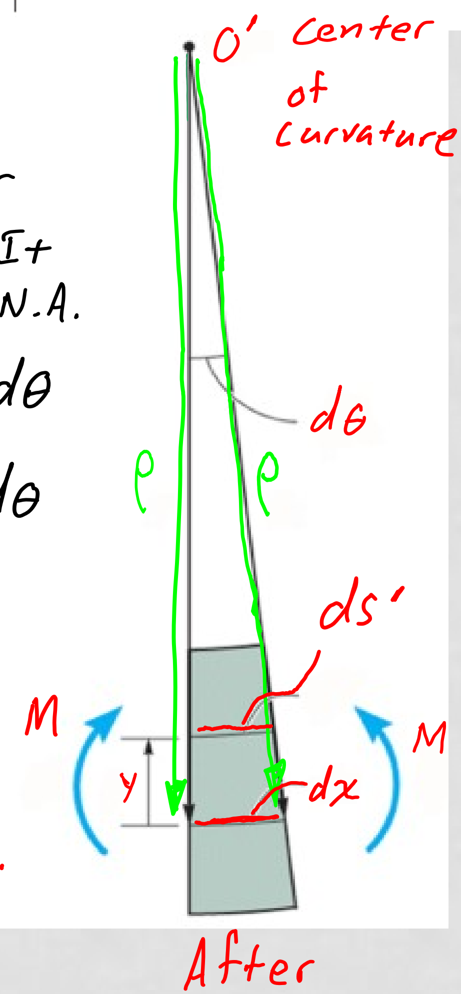
$$\boxed{\frac{1}{\rho} = -\frac{\varepsilon}{y}}$$ where $\frac{1}{\rho}$ is called "curvature"

dx does not change after deformation. It is along the N.A.

$$dx = ds = \rho \, d\theta$$

$$ds' = (\rho - y)\, d\theta$$





for a linear-elastic, homogeneous material

$$E = \frac{\sigma}{\varepsilon} \quad \text{or} \quad \varepsilon = \frac{\sigma}{E}$$

flexure eq: $\sigma = \frac{-My}{I}$

$$\left.\begin{matrix} E = \frac{\sigma}{\varepsilon} \text{ or } \varepsilon = \frac{\sigma}{E} \\ \sigma = \frac{-My}{I} \end{matrix}\right\} -\frac{\varepsilon}{y} = \frac{M}{EI}$$

$$\boxed{\frac{1}{\rho} = \frac{M}{EI}}$$

EI is "flexural rigidity" or "stiffness"

E is elastic modulus

Slope and Displacement by Integration

The coordinates of the elastic curve for a beam or shaft can be determined using an equation that expresses v as a function of x, where v is the vertical deflection and x is the horizontal position where that deflection occurs.

To begin, start with an equation for the curvature ($1/\rho$) expressed in terms of v and x. The derivation of this equation can be found in most Calculus books:

$$\frac{1}{\rho} = \frac{d^2v/dx^2}{\left[1+(dv/dx)^2\right]^{3/2}}$$

since most engineering designs ~~limit~~ limit deflections, the slope $\frac{dv}{dx}$ will be small, and $\left(\frac{dv}{dx}\right)^2$ is really small.

$$\frac{d^2v/dx^2}{\left[1+\underbrace{(dv/dx)^2}_{\sim 0}\right]^{3/2}} = \frac{M}{EI}$$

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

$$M(x) = EI\,\frac{d^2v}{dx^2}$$

To draw shear and bending moment diagrams for beams, the following equations could be used:

shear, function of x

moment, function of x

$$-w(x) = \frac{dV}{dx} \qquad V(x) = \frac{dM}{dx}$$

distributed load acting down

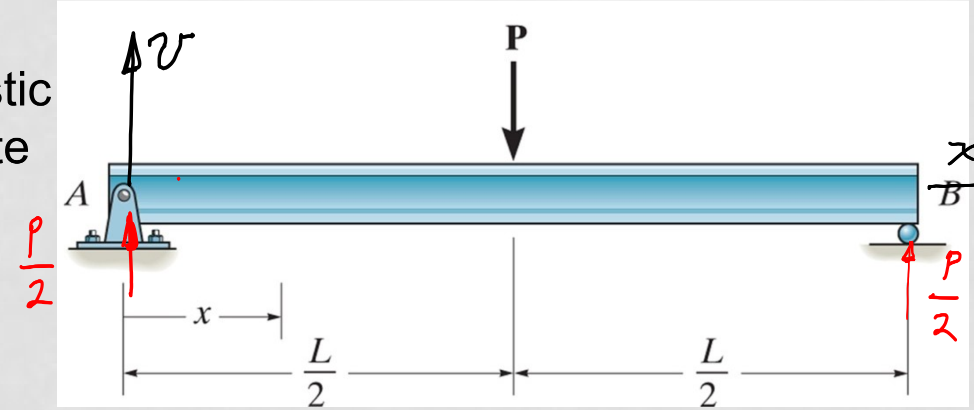
Using the simplified equation for curvature, the following equations can be derived for the moment *M*(*x*), shear $V(x)$, and distributed load $w(x)$:

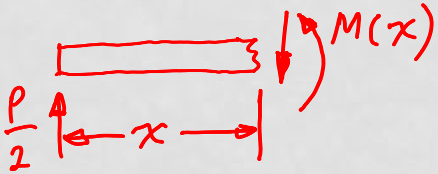
$$M(x) = EI\frac{d^2v}{dx^2}$$

$$V(x) = \frac{d(M(x))}{dx} = \frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right) = EI\frac{d^3v}{dx^3}$$

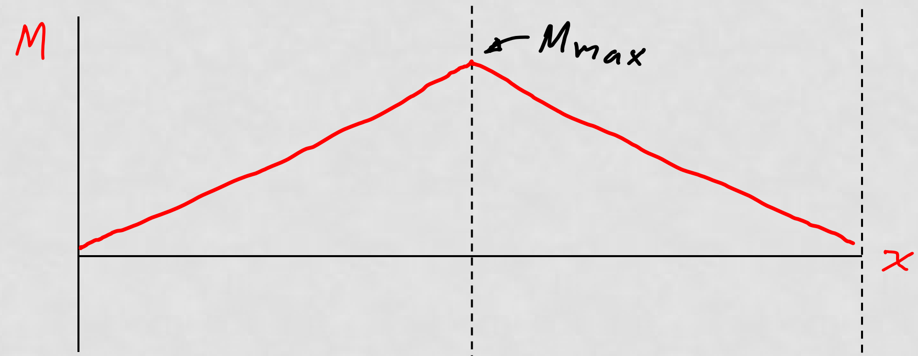
$$-w(x) = \frac{d(V(x))}{dx} = \frac{d}{dx}\left(EI\frac{d^3v}{dx^3}\right) = EI\frac{d^4v}{dx^4}$$

12-3. Determine the equation of the elastic curve for the beam using the *x* coordinate that is valid for $0 \leq x < L/2$. Specify the slope at *A* and the beam's maximum deflection. *EI* is constant.





$$M(x) = \frac{P}{2}x$$



$$EI\frac{d^2v}{dx^2} = M(x)$$

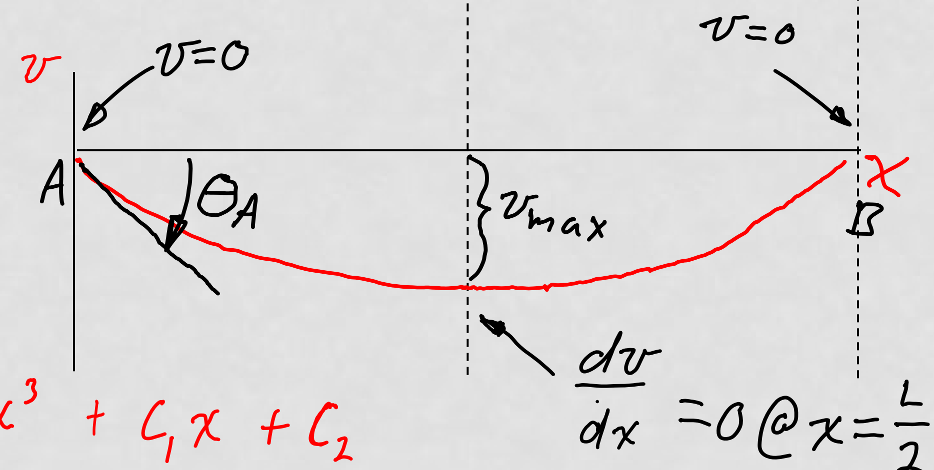
$$EI\frac{d^2v}{dx^2} = \frac{P}{2}x$$

$$EI\int d\left(\frac{dv}{dx}\right) = \int \frac{P}{2}x\,dx$$

$$EI\frac{dv}{dx} = \frac{P}{2}\frac{x^2}{2} + C_1 = \frac{P}{4}x^2 + C_1$$

$$EI\int dv = \int \frac{P}{4}x^2\,dx + \int C_1\,dx$$

$$EIv = \frac{P}{4}\frac{x^3}{3} + C_1x + C_2 = \frac{P}{12}x^3 + C_1x + C_2$$



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Dimensions and Geometric Characteristics of Standard Structural Shapes such as I-Beams, Channel Sections, & Angle Bars – see Appendix B

W-shapes: I-Beams

C-shapes: channel sections

L-shapes: angle bars

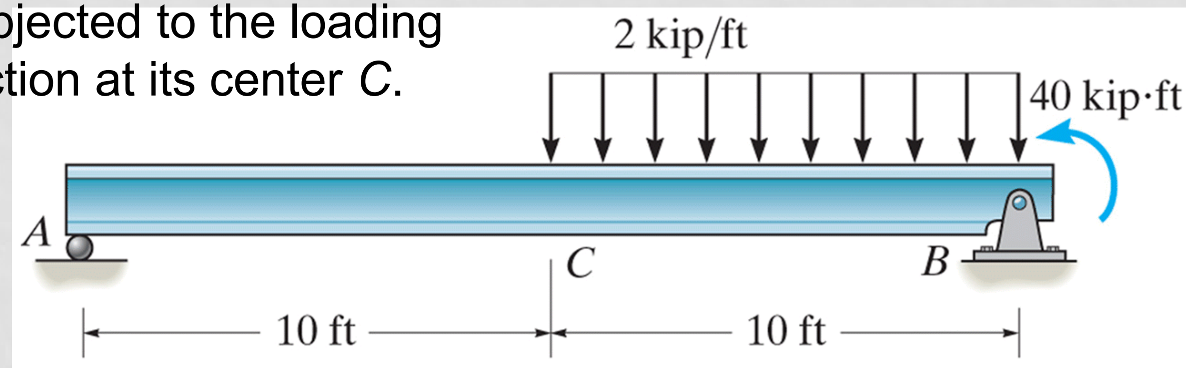
Elastic Curves, Slope & Deflection Equations for Beams – see Appendix C

Superposition can be used to solve more complicated slope and deflection problems using equations shown in Appendix C for typical simply-supported and fixed beams under a range of loading conditions.

Other references such as the *AISC Handbook* and *Roark's Theories of Stress and Strain* provide equations for a wider range of beams.

$E = 29 \times 10^3$ ksi, $I_x = 428$ in.^{4} from Appendix B

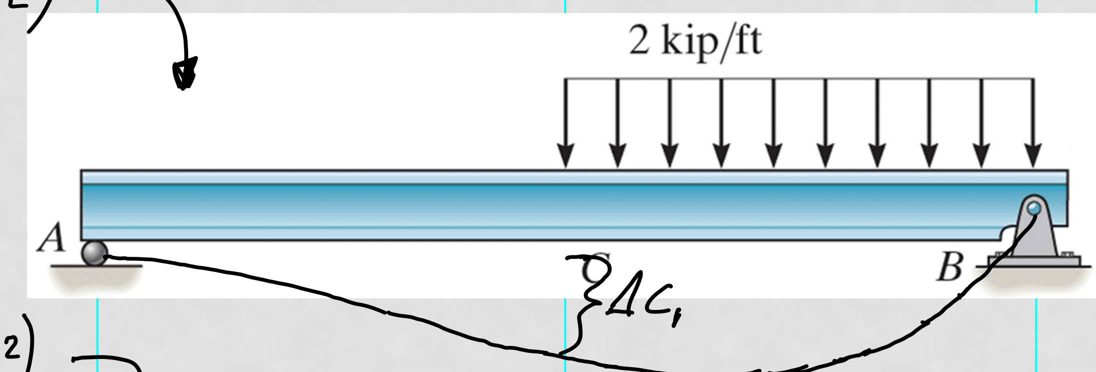
12-91. The W14 X 43 simply supported beam is made of A-36 steel and is subjected to the loading shown. Determine the deflection at its center *C*.



from Appendix C

$$v = \frac{-wx}{384EI}\left(16x^3 - 24Lx^2 + 9L^3\right)$$

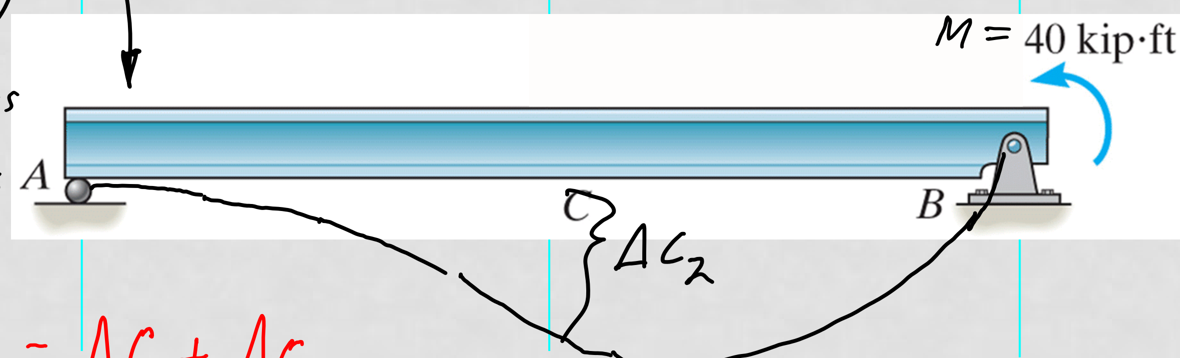
=



$$\Delta C_1 = v\Big|_{x=\frac{L}{2}} = \frac{-5wL^4}{768EI}$$

+

$$v = \frac{-Mx}{6EIL}\left(x^2 - 3Lx + 2L^2\right)$$

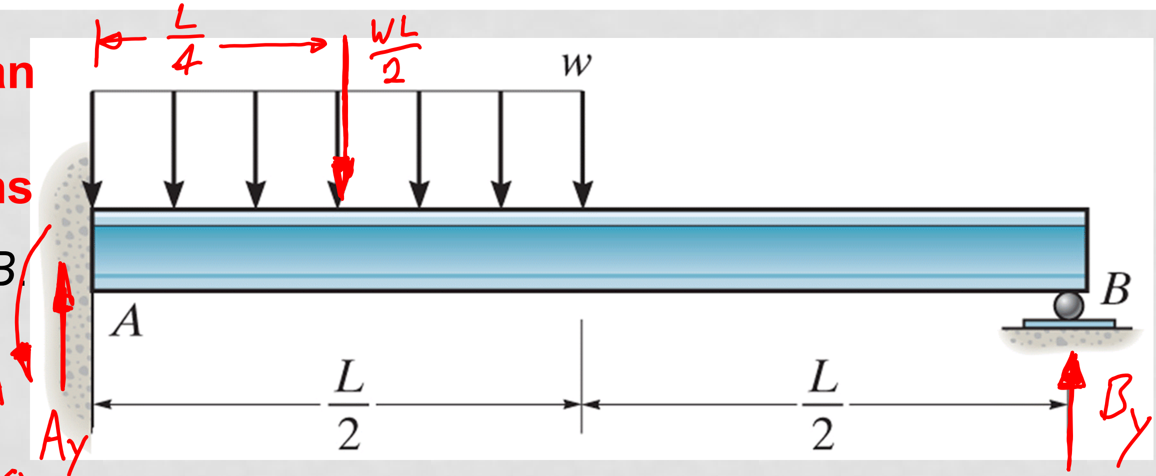


A cw moment at Pt. A causes same downward deflection as ccw moment at Pt. B.

$$\Delta C_2 = v\Big|_{x=\frac{L}{2}}$$

$$\Delta C_{tot} = \Delta C_1 + \Delta C_2$$

Superposition techniques can be used to solve statically indeterminate beam problems



12-127. Find reactions at *A* & *B*.

$\Sigma F_y = 0: \quad A_y + B_y - \frac{wL}{2} = 0$

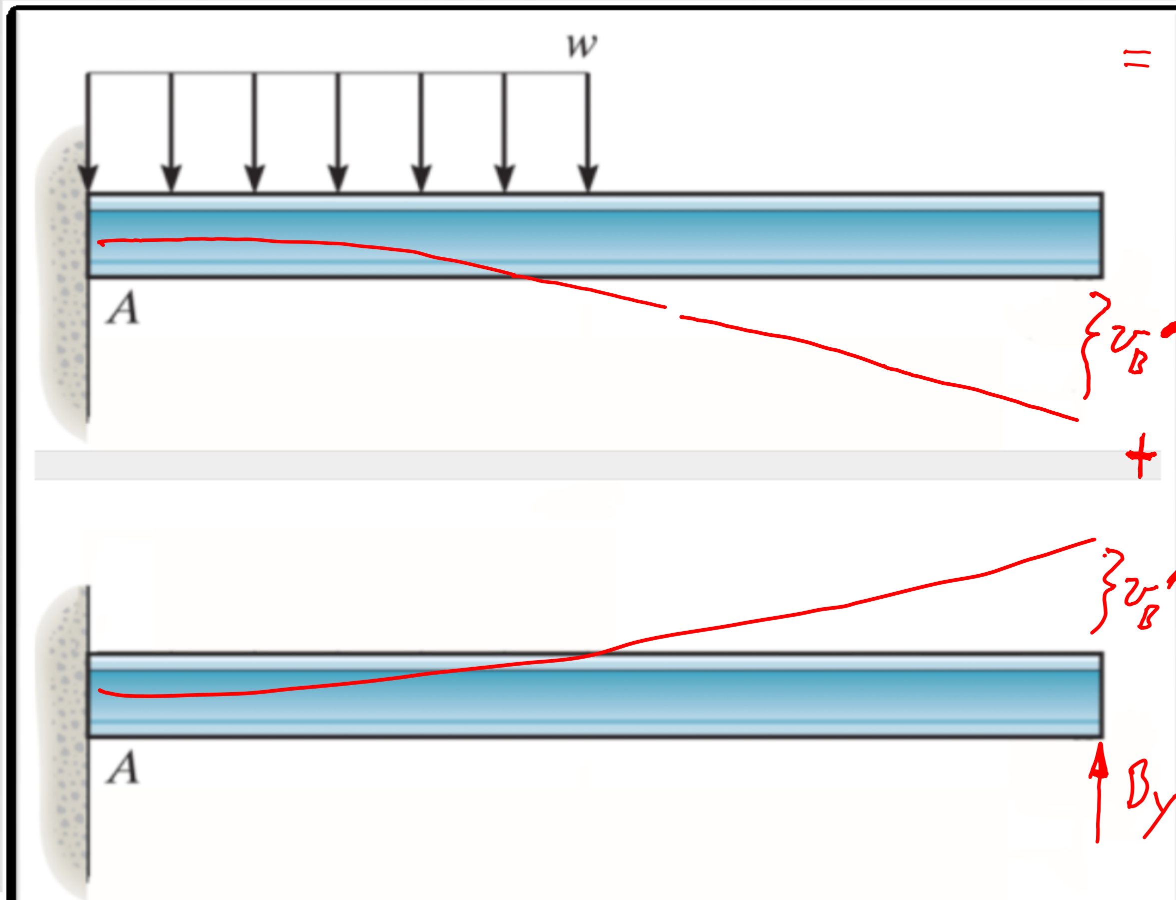
$\Sigma M_A = 0: \quad B_y(L) - \left(\frac{wL}{2}\right)\left(\frac{L}{4}\right) + M_A = 0$

compatibility: $v_B' + v_B'' = 0$

from Appendix C:

$$v_B' = \frac{7wL^4}{384EI} \downarrow$$

$$v_B'' = \frac{B_y L^3}{3EI} \uparrow$$



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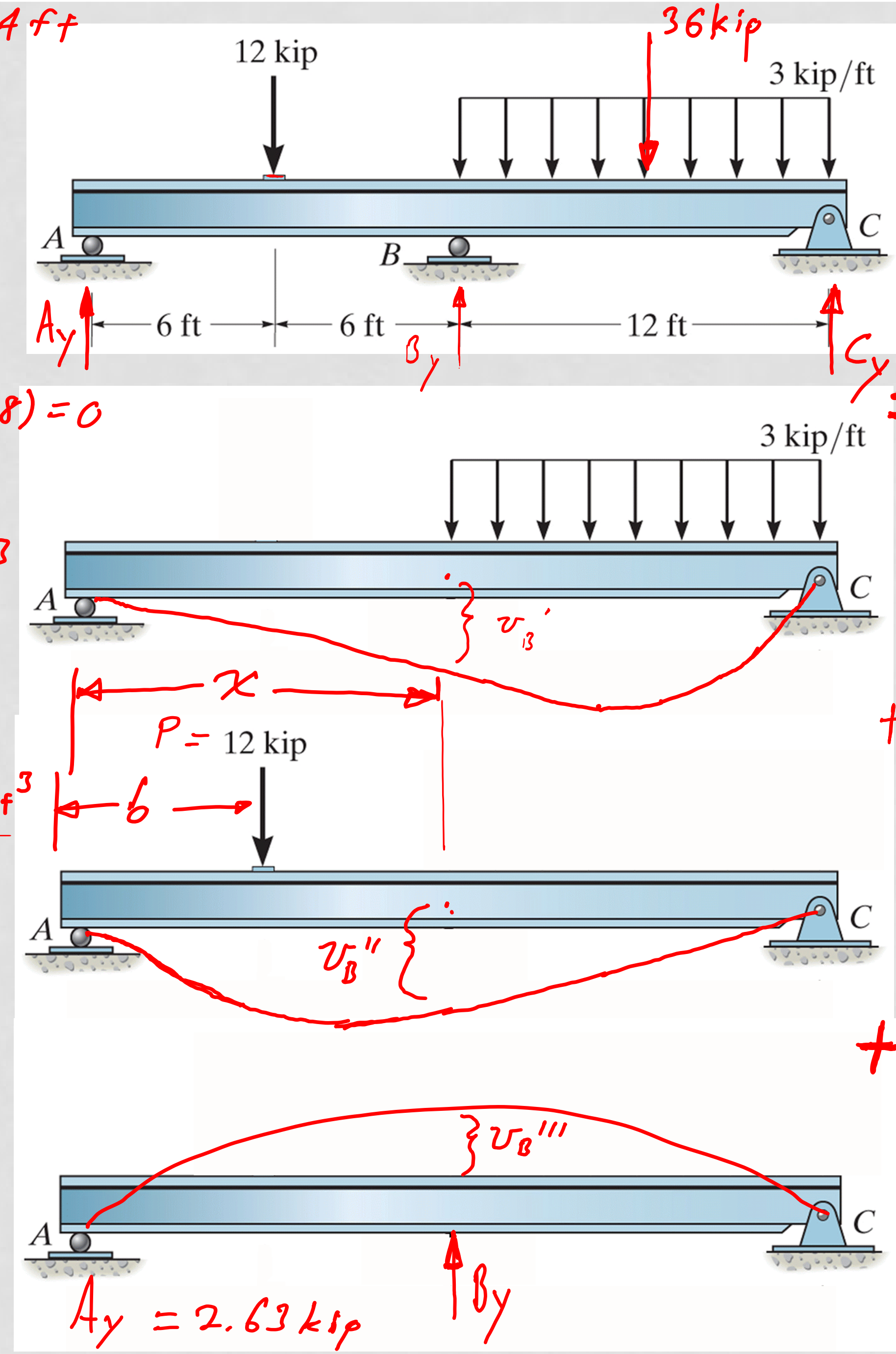
12-124. Determine the reactions at the supports *A*, *B*, & *C*, then draw the shear and moment diagrams. *EI* is constant.



$\Sigma F_y = 0;\quad A_y + B_y + C_y - 12 - 36 = 0$

$\circlearrowleft + \Sigma M_A = 0:\quad B_y(12) + C_y(24) - 12(6) - 36(18) = 0$

$v_B' + v_B'' + v_B''' = 0$; use Eqs. in App. C

$$v_B' = \frac{5wL^4}{768EI} = \frac{5(3)(24)^4}{768EI} = \frac{6480\ \text{kip}\cdot\text{ft}^3}{EI}$$

$$v_B'' = \frac{Pbx}{6EIL}(L^2 - b^2 - x^2)$$

$$= \frac{(12)(6)(12)}{6EI(24)}(24^2 - 6^2 - 12^2) = \frac{2376\ \text{kip}\cdot\text{ft}^3}{EI}$$

$$v_B''' = \frac{B_y L^3}{48EI} = \frac{B_y(24)^3}{48EI} = \frac{288 B_y\ \text{ft}^3}{EI}$$

$$+\downarrow\quad 0 = \frac{6480}{\cancel{EI}} + \frac{2376}{\cancel{EI}} - \frac{288B_y}{\cancel{EI}}$$

$B_y = 30.75$ kip $\qquad C_y = 14.63$ kip