Buckling of Columns

Buckling & Stability

Critical Load

Introduction

- In discussing the analysis and design of various structures in the previous chapters, we had two primary concerns:
  - the strength of the structure, i.e. its ability to support a specified load without experiencing excessive stresses;
  - the ability of the structure to support a specified load without undergoing unacceptable deformations.

Introduction

- Now we shall be concerned with stability of the structure,
  - with its ability to support a given load without experiencing a sudden change in its configuration.
- Our discussion will relate mainly to columns,
  - the analysis and design of vertical prismatic members supporting axial loads.

Introduction

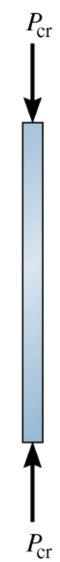
- Structures may fail in a variety of ways, depending on the :
  - Type of structure
  - Conditions of support
  - Kinds of loads
  - Material used

Introduction

- Failure is prevented by designing structures so that the maximum stresses and maximum displacements remain within tolerable limits.
- Strength and stiffness are important factors in design as we have already discussed
- Another type of failure is buckling

Introduction

- If a beam element is under a compressive load and its length is an order of magnitude larger than either of its other dimensions such a beam is called a ***columns***.
- Due to its size its axial displacement is going to be very small compared to its lateral deflection called *buckling*.



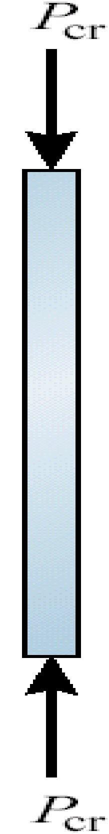
(a)

Introduction

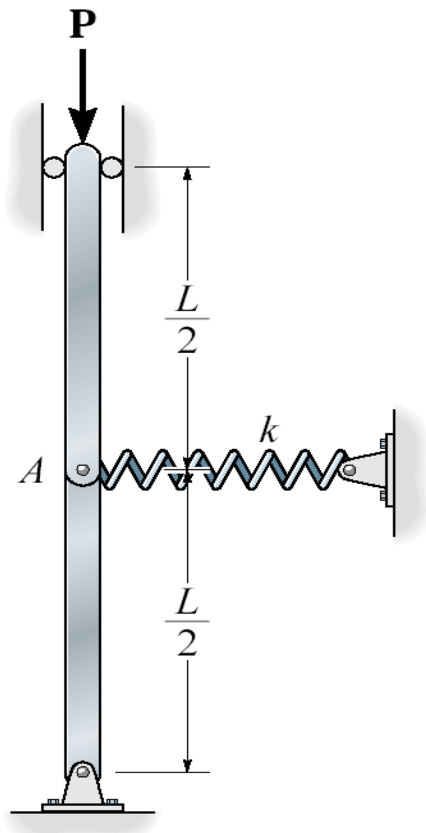
- Quite often the buckling of column can lead to sudden and dramatic failure. And as a result, special attention must be given to design of column so that they can safely support the loads.
- Buckling is not limited to columns.
  - Can occur in many kinds of structures
  - Can take many forms
  - Step on empty aluminum can
  - Major cause of failure in structures

Buckling & Stability

- Consider the figure
- Hypothetical structure
- Two rigid bars joined by a pin the center, held in a vertical position by a spring
- Is analogous to fig13-1 because both have simple supports at the end and are compressed by an axial load P.



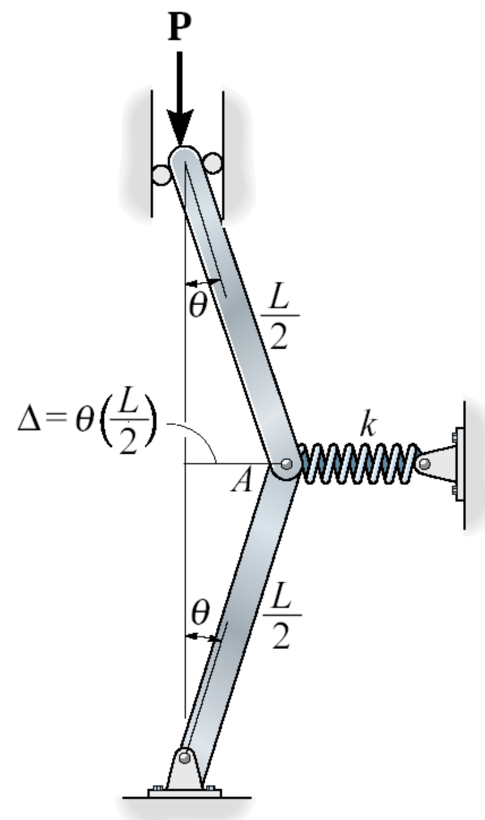
Buckling & Stability



- Elasticity of the buckling model is concentrated in the spring (real model can bend throughout its length
- Two bars are perfectly aligned
- Load P is along the vertical axis
- Spring is unstressed
- Bar is in direct compression

Buckling & Stability

- Structure is disturbed by an exterr force that causes point A to move small distance laterally.
- Rigid bars rotate through small an θ
- Force develops in the spring
- Direction of the force tends to ret the structure to its original straigh position, called the Restoring Forc



Buckling & Stability

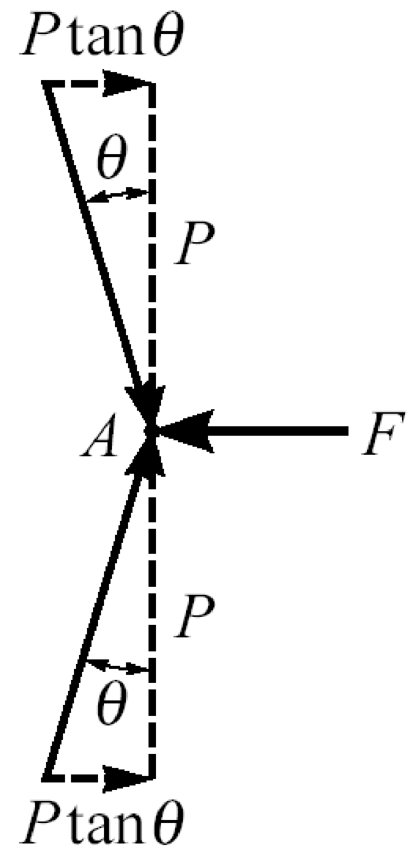
- At the same time, the tendency of the axial compressive force is to increase the lateral displacement.
- These two actions have opposite effects
  - Restoring force tends to decrease displacement
  - Axial force tends to increase displacement.

Buckling & Stability

- Now remove the disturbing force.
- If P is small, the restoring force will dominate over the action of the axial force and the structure will return to its initial straight position
  - Structure is called Stable
- If P is large, the lateral displacement of A will increase and the bars will rotate through larger and larger angles until the structure collapses
  - Structure is unstable and fails by lateral buckling

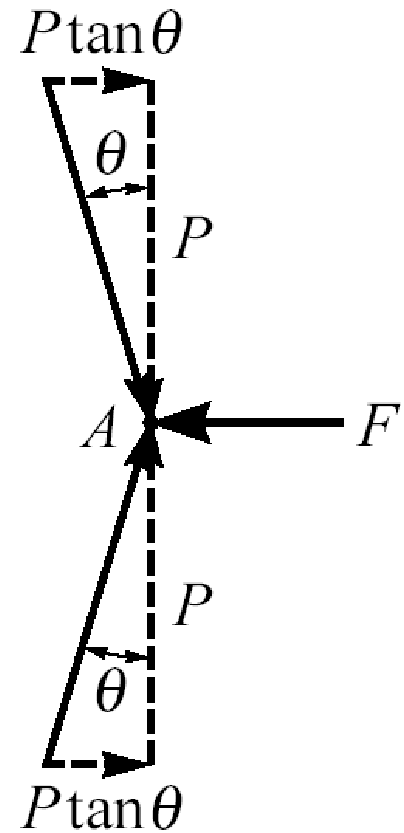
Critical Load

- Transition between stable and unstable conditions occurs at value of the axial force called t *Critical Load* P_{cr}.
- Find the critical load by consid the structure in the disturbed position and consider equilibri
- Consider the entire structure a FBD and sum the forces in the direction



Critical Load

- Next, consider the upper bar a free body
  - Subjected to axial forces P and fc in the spring
  - Force is equal to the stiffness k t the displacement Δ, F = kΔ
  - Since θ is small, the lateral displacement of point A is θL/2
  - Applying equilibrium and solving P_{cr}=kL/4



Critical Load

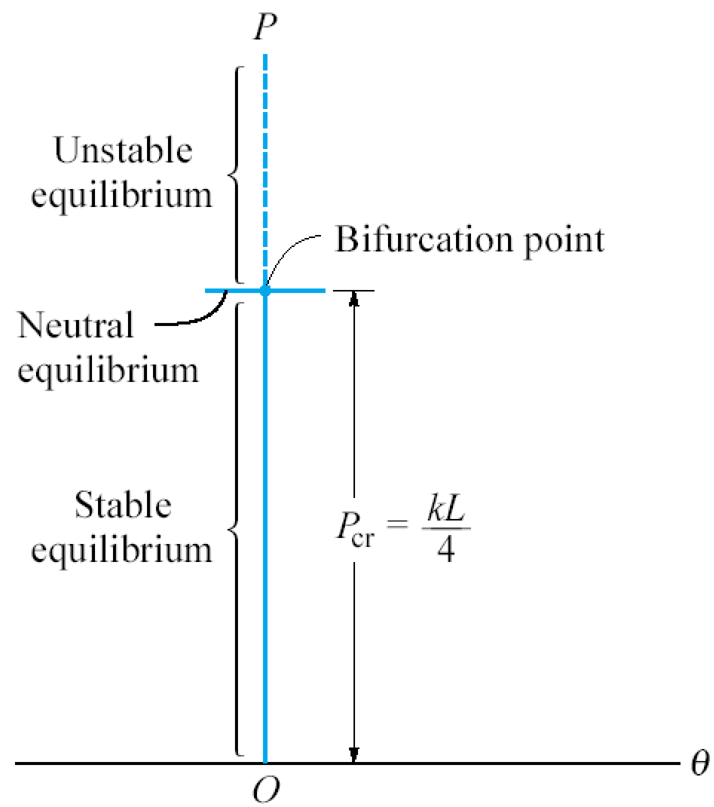
- Which is the critical load
  - At this value the structure is in equilibrium regardless of the magnitude of the angle (provided it stays small)
  - Critical load is the only load for which the structure will be in equilibrium in the disturbed position
  - At this value, restoring effect of the moment in the spring matches the buckling effect of the axial load
  - Represents the boundary between the stable and unstable conditions.

Critical Load

- If the axial load is less than P_{cr} the effect of the moment in the spring dominates and the structure returns to the vertical position after a small disturbance – stable condition.
- If the axial load is larger than P_{cr} the effect of the axial force predominates and the structure buckles – unstable condition.

Critical Load

- The boundary between stability and instability is called neutral equilibrium.
- The critical point, after which the deflections of the member become very large, is called the "bifurcation point" of the system



Critical Load

- This is analogous to a ball placed on a smooth surface
  - If the surface is concave (inside of a dish) the equilibrium is stable and the ball always returns to the low point when disturbed
  - If the surface is convex (like a dome) the ball can theoretically be in equilibrium on the top surface, but the equilibrium is unstable and the ball rolls away
  - If the surface is perfectly flat, the ball is in neutral equilibrium and stays where placed.

Critical Load

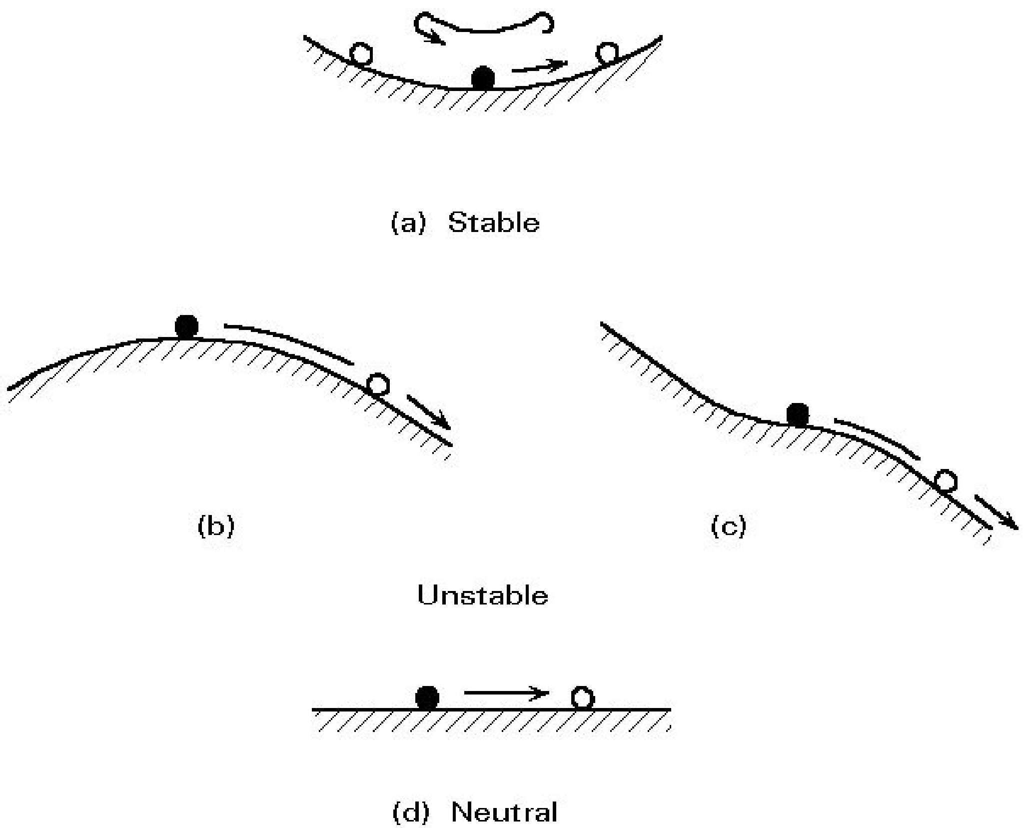


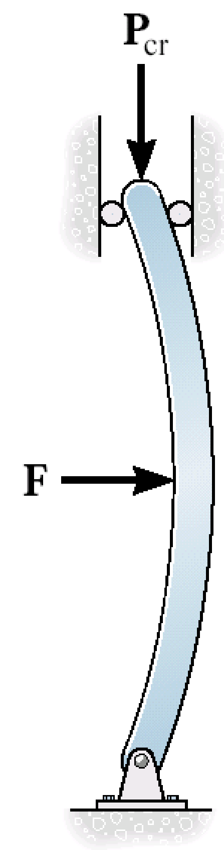
Figure 1 The three states of equilibrium

Critical Load

- In looking at columns under this type of loading we are only going to look at three different types of supports:
  - pin-supported,
  - doubly built-in and
  - cantilever.

Pin Supported Column

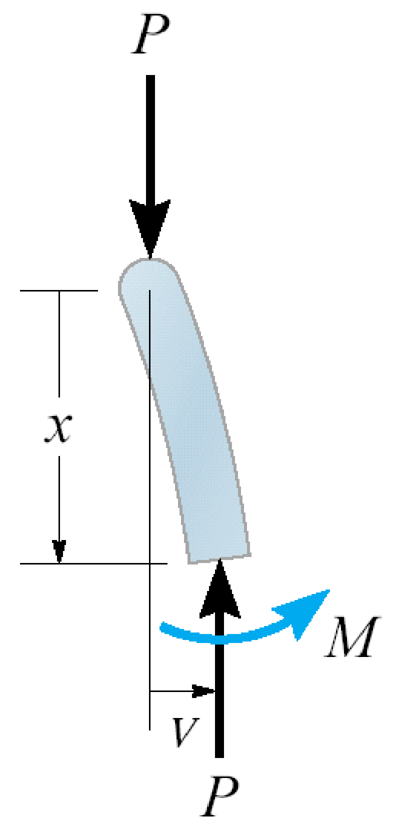
- Due to imperfections no column i really straight.
- At some critical compressive load will buckle.
- To determine the maximum compressive load (**Buckling Load**) assume that buckling has occurre



Pin Supported Column

- Looking at the FBD of the top o the beam
- Equating moments at the cut e M(x)=-Pv
- Since the deflection of the bea related with its bending mome distribution

$$EI\frac{d^2v}{dx^2} = -Pv$$



Pin Supported Column

- This equation simplifies to: $$\frac{d^2v}{dx^2} + \left(\frac{P}{EI}\right)v = 0$$
- P/EI is constant.
- This expression is in the form of a second order differential equation of the type $$\frac{d^2v}{dx^2} + \alpha^2 v = 0$$
- Where $\alpha^2 = \frac{P}{EI}$
- The solution of this equation is:

$$v = A\cos(\alpha x) + B\sin(\alpha x)$$

  – A and B are found using boundary conditions

Pin Supported Column

- Boundary Conditions
  - At x=0, v=0, therefore A=0
  - At x=L, v=0, then $0=B\sin(\alpha L)$
- If B=0, no bending moment exists, so the only logical solution is for $\sin(\alpha L)=0$ and the only way that can happen is if $\alpha L=n\pi$
- Where n=1,2,3,

Pin Supported Column

- But since

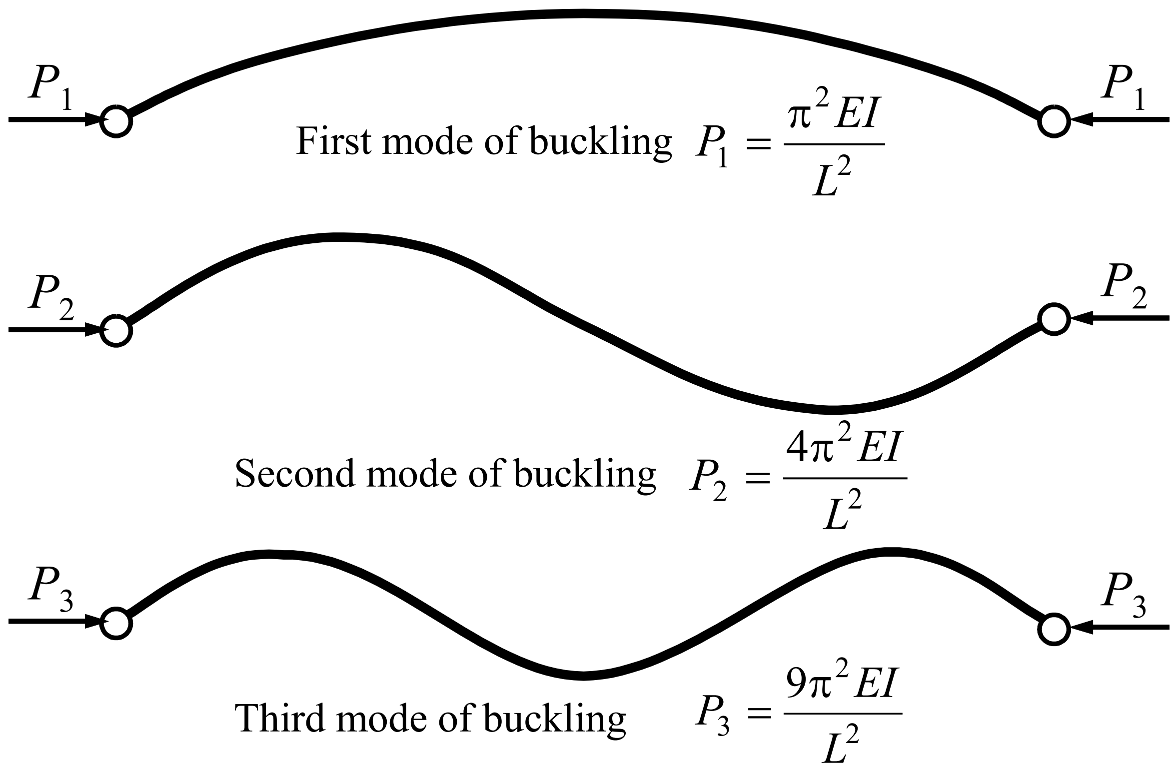
$$\alpha^2 = \frac{P}{EI} = \left(\frac{n\pi}{L}\right)^2$$

- Then we get that buckling load is:

$$P = n^2 \frac{\pi^2 EI}{L^2}$$

Pin Supported Column

- The values of n defines the buckling mode shapes



Critical Buckling Load

- Since $P_1 < P_2 < P_3$, the column buckles at P_1 and never gets to P_2 or P_3 unless bracing is place at the points where v=0 to prevent buckling at lower loads.
- The critical load for a pin ended column is then:

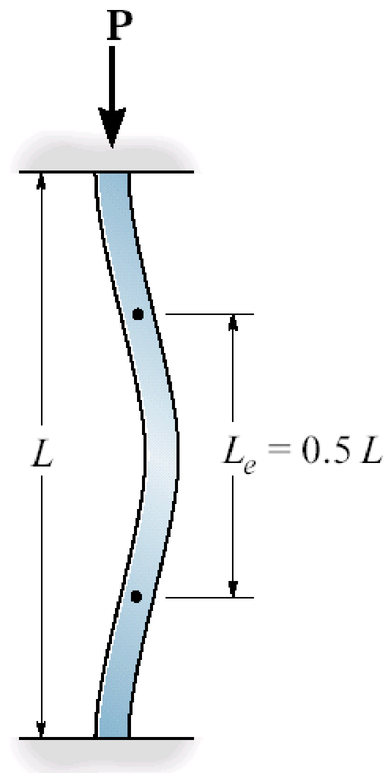
$$P_{Crit} = \frac{\pi^2 EI}{L^2}$$

- Which is called the Euler Buckling Load

Built-In Column

- The critical load for other column ca
expressed in terms of the critical bu
load for a pin-ended column.
- From symmetry conditions at the po
inflection occurs at ¼ L.
- Therefore the middle half of the colu
can be taken out and treated as a pi
ended column of length L_E=L/2
- Yielding:

$$P_{Crit} = \frac{4\pi^2 EI}{L^2}$$

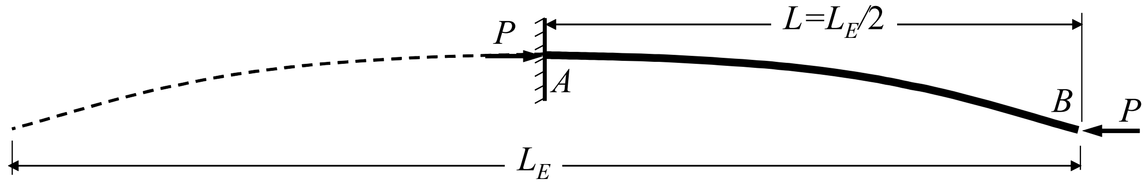


Fixed ends

Cantilever Column

- This is similar to the previous case.
- The span is equivalent to ½ of the Euler span L_E

$$P_{Crit} = \frac{\pi^2 EI}{L_E^2} = \frac{\pi^2 EI}{4L^2}$$

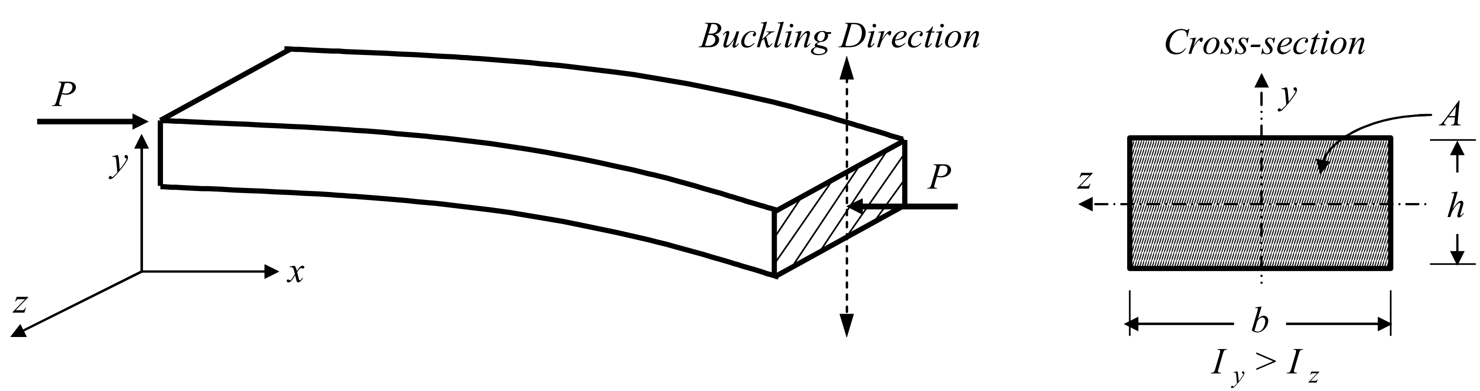


Therefore:

$$L_e = \begin{cases} L & pin - pin \\ 0.7L & fixed - pin \\ 0.5L & fixed - fixed \\ 2L & fixed - free \end{cases}$$

Note on Moment of Inertia

- Since P_{crit} is proportional to I, the column will buckle in the direction corresponding to the minimum value of I



Critical Column Stress

- A column can either fail due to the material yielding, or because the column buckles, it is of interest to the engineer to determine when this point of transition occurs.
- Consider the Euler buckling equation

$$P_E = \frac{\pi^2 EI}{L^2}$$

Critical Column Stress

- Because of the large deflection caused by buckling, the least moment of inertia *I* can be expressed as $I = Ar^2$
- where: *A* is the cross sectional area and *r* is the ***radius of gyration*** of the cross sectional area, i.e. . $r = \sqrt{I/A}$
- Note that the *smallest* radius of gyration of the column, i.e. the *least* moment of inertia *I* should be taken in order to find the critical stress.

Critical Column Stress

- Dividing the buckling equation by A, gives:

$$\sigma_E = \frac{P_E}{A} = \frac{\pi^2 E}{(L/r)^2}$$

- where:
  - σ_E is the compressive stress in the column and must not exceed the yield stress σ_Y of the material, i.e. $\sigma_E < \sigma_Y$,
  - L / r is called the ***slenderness ratio***, it is a measure of the column's flexibility.

Critical Buckling Load

- P_{crit} is the critical or maximum axial load on the column just before it begins to buckle
- E youngs modulus of elasticity
- I *least* moment of inertia for the columns cross sectional area.
- L unsupported length of the column whose ends are pinned.

EXAMPLE 13–2

The A-36 steel $W\,8 \times 31$ member shown in Fig. 13–10 is to be used as a pin-connected column. Determine the largest axial load it can support before it either begins to buckle or the steel yields.

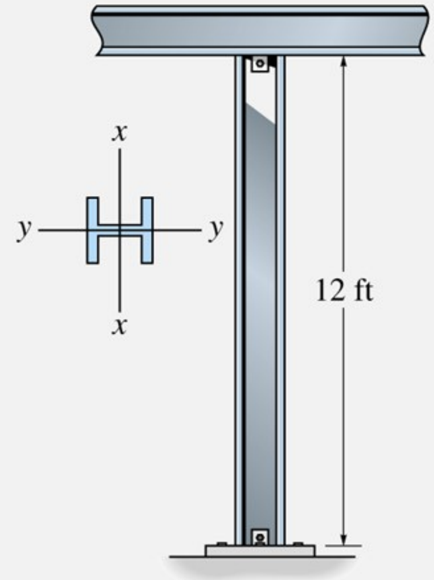


Fig. 13–10

SOLUTION

From the table in Appendix B, the column's cross-sectional area and moments of inertia are $A = 9.13\ \text{in}^2$, $I_x = 110\ \text{in}^4$, and $I_y = 37.1\ \text{in}^4$. By inspection, buckling will occur about the $y-y$ axis. Why? Applying Eq. 13–5, we have

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2[29(10^3)\ \text{kip/in}^2](37.1\ \text{in}^4)}{[12\ \text{ft}(12\ \text{in./ft})]^2} = 512\ \text{kip}$$

When fully loaded, the average compressive stress in the column is

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{512\ \text{kip}}{9.13\ \text{in}^2} = 56.1\ \text{ksi}$$

Since this stress exceeds the yield stress (36 ksi), the load P is determined from simple compression:

$$36\ \text{ksi} = \frac{P}{9.13\ \text{in}^2}; \qquad P = 329\ \text{kip} \qquad \textbf{\textit{Ans.}}$$

In actual practice, a factor of safety would be placed on this loading.

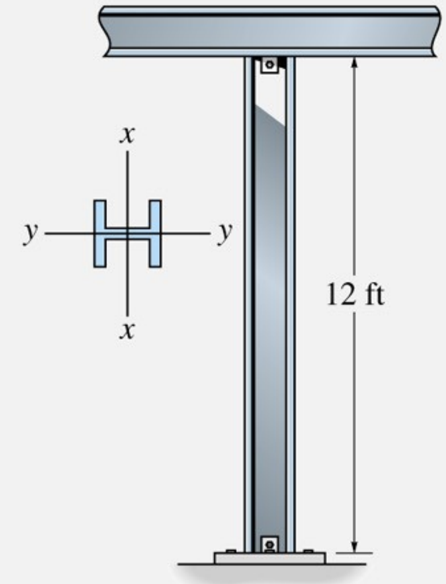
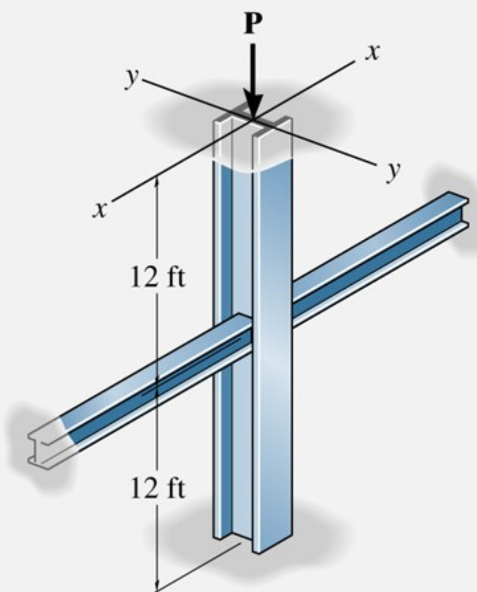


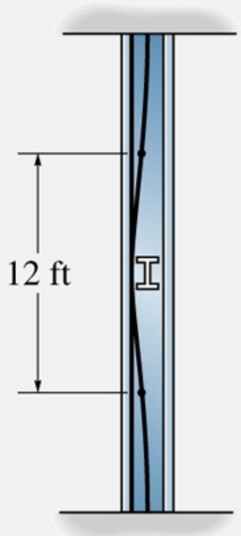
Fig. 13–10



(a)

EXAMPLE 13–3

A $W\ 6 \times 15$ steel column is 24 ft long and is fixed at its ends as shown in Fig. 13–13*a*. Its load-carrying capacity is increased by bracing it about the $y-y$ (weak) axis using struts that are assumed to be pin-connected to its midheight. Determine the load it can support so that the column does not buckle nor the material exceed the yield stress. Take $E_{st} = 29(10^3)$ ksi and $\sigma_Y = 60$ ksi.



x–x axis buckling

(b)

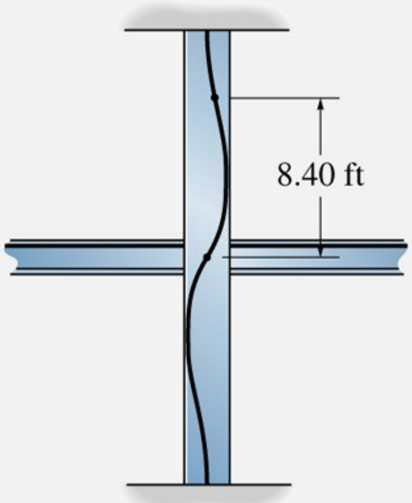
SOLUTION

The buckling behavior of the column will be *different* about the x and y axes due to the bracing. The buckled shape for each of these cases is shown in Figs. 13–13*b* and 13–13*c*. From Fig. 13–13*b*, the effective length for buckling about the $x-x$ axis is $(KL)_x = 0.5(24 \text{ ft}) = 12 \text{ ft} = 144 \text{ in.}$, and from Fig. 13–13*c*, for buckling about the $y-y$ axis, $(KL)_y = 0.7(24 \text{ ft}/2) = 8.40 \text{ ft} = 100.8 \text{ in.}$ The moments of inertia for a $W\,6 \times 15$ are determined from the table in Appendix B. We have $I_x = 29.1 \text{ in}^4$, $I_y = 9.32 \text{ in}^4$.

Applying Eq. 13–11, we have

$$(P_{\text{cr}})_x = \frac{\pi^2 E I_x}{(KL)_x^2} = \frac{\pi^2[29(10^3)\text{ ksi}]29.1\text{ in}^4}{(144\text{ in.})^2} = 401.7\text{ kip} \qquad (1)$$

$$(P_{\text{cr}})_y = \frac{\pi^2 E I_y}{(KL)_y^2} = \frac{\pi^2[29(10^3)\text{ ksi}]9.32\text{ in}^4}{(100.8\text{ in.})^2} = 262.5\text{ kip} \qquad (2)$$



y–y axis buckling

(c)

Fig. 13–13

By comparison, buckling will occur about the $y-y$ axis.

The area of the cross section is 4.43 in^2, so the average compressive stress in the column will be

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{262.5 \text{ kip}}{4.43 \text{ in}^2} = 59.3 \text{ ksi}$$

Since this stress is less than the yield stress, buckling will occur before the material yields. Thus,

$$P_{cr} = 263 \text{ kip}$$ ***Ans.***

Note: From Eq. 13–11 it can be seen that buckling will always occur about the column axis having the *largest* slenderness ratio, since a large slenderness ratio will give a small critical load. Thus, using the data for the radius of gyration from the table in Appendix B, we have

$$\left(\frac{KL}{r}\right)_x = \frac{144 \text{ in.}}{2.56 \text{ in.}} = 56.2$$

$$\left(\frac{KL}{r}\right)_y = \frac{100.8 \text{ in.}}{1.46 \text{ in.}} = 69.0$$

Hence, $y-y$ axis buckling will occur, which is the same conclusion reached by comparing Eqs. 1 and 2.