Stress and Strain – Axial .Loading

Contents

Stress & Strain: Axial Loading

- Suitability of a structure or machine may depend on the deformations in the structure as well as the stresses induced under loading. Statics analyses alone are not sufficient.

- Considering structures as deformable allows determination of member forces and reactions which are statically indeterminate.

- Determination of the stress distribution within a member also requires consideration of deformations in the member.

- Chapter 2 is concerned with deformation of a structural member under axial loading. Later chapters will deal with torsional and pure bending loads.

Normal Strain

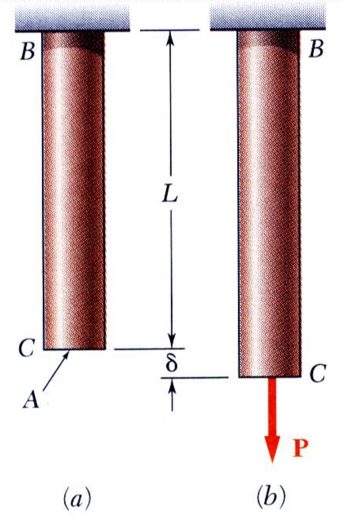


Fig. 2.1

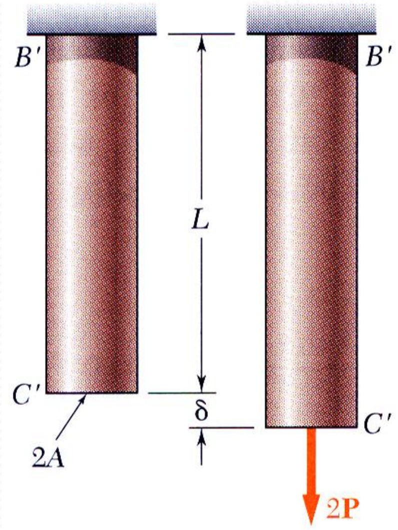


Fig. 2.3

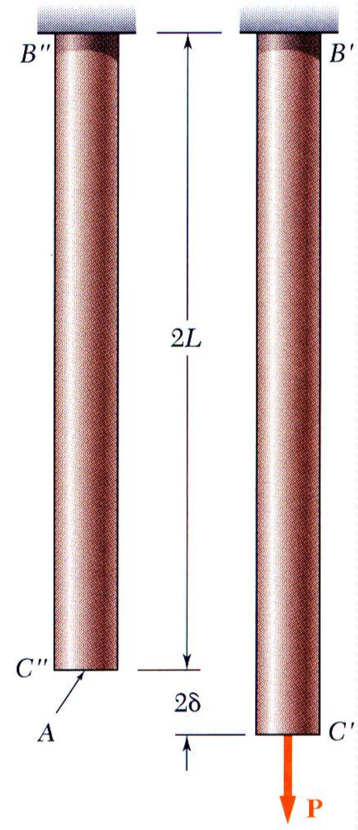


Fig. 2.4

$$\sigma = \frac{P}{A} = \text{stress}$$

$$\varepsilon = \frac{\delta}{L} = \text{normal strain}$$

$$\sigma = \frac{2P}{2A} = \frac{P}{A}$$

$$\varepsilon = \frac{\delta}{L}$$

$$\sigma = \frac{P}{A}$$

$$\varepsilon = \frac{2\delta}{2L} = \frac{\delta}{L}$$

Stress-Strain Test

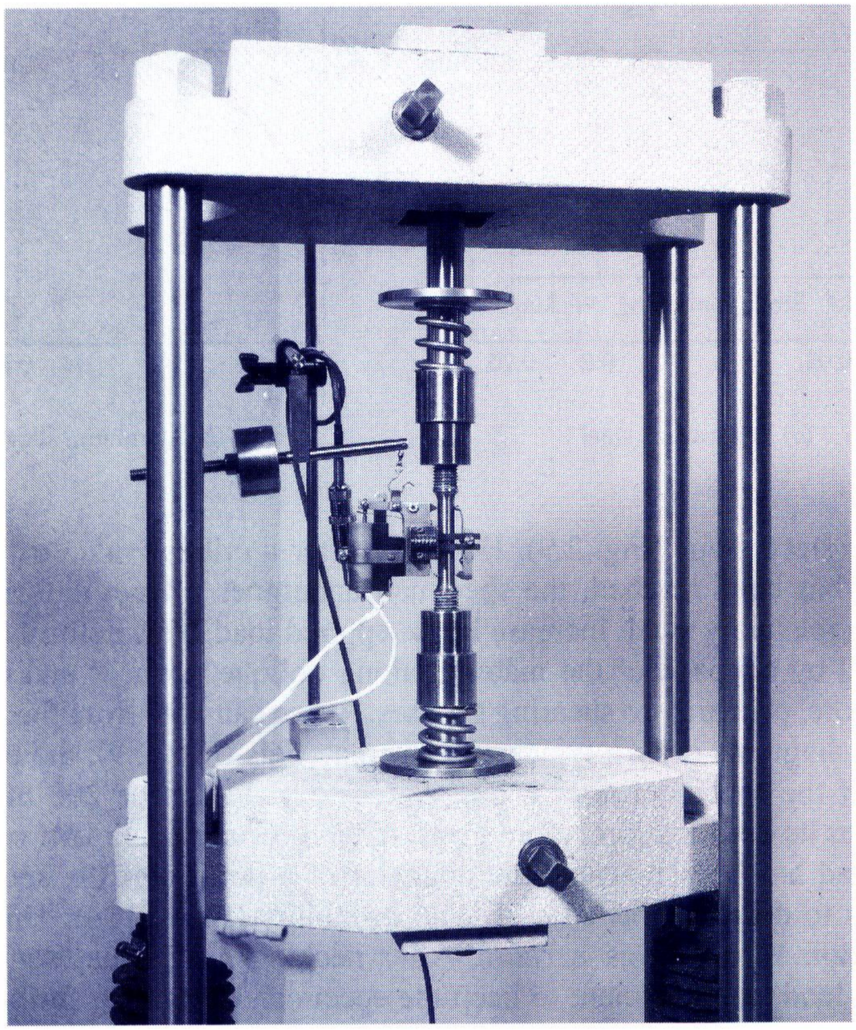


Fig. 2.7 This machine is used to test tensile test specimens, such as those shown in this chapter.

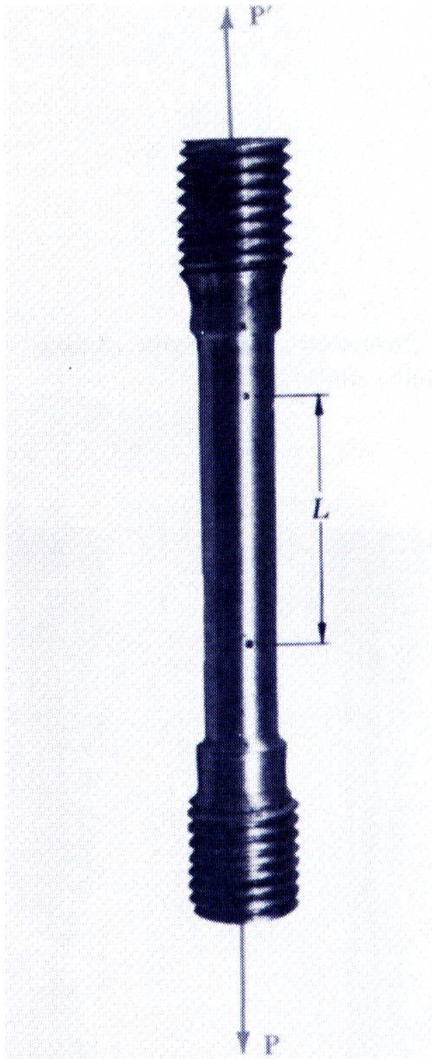
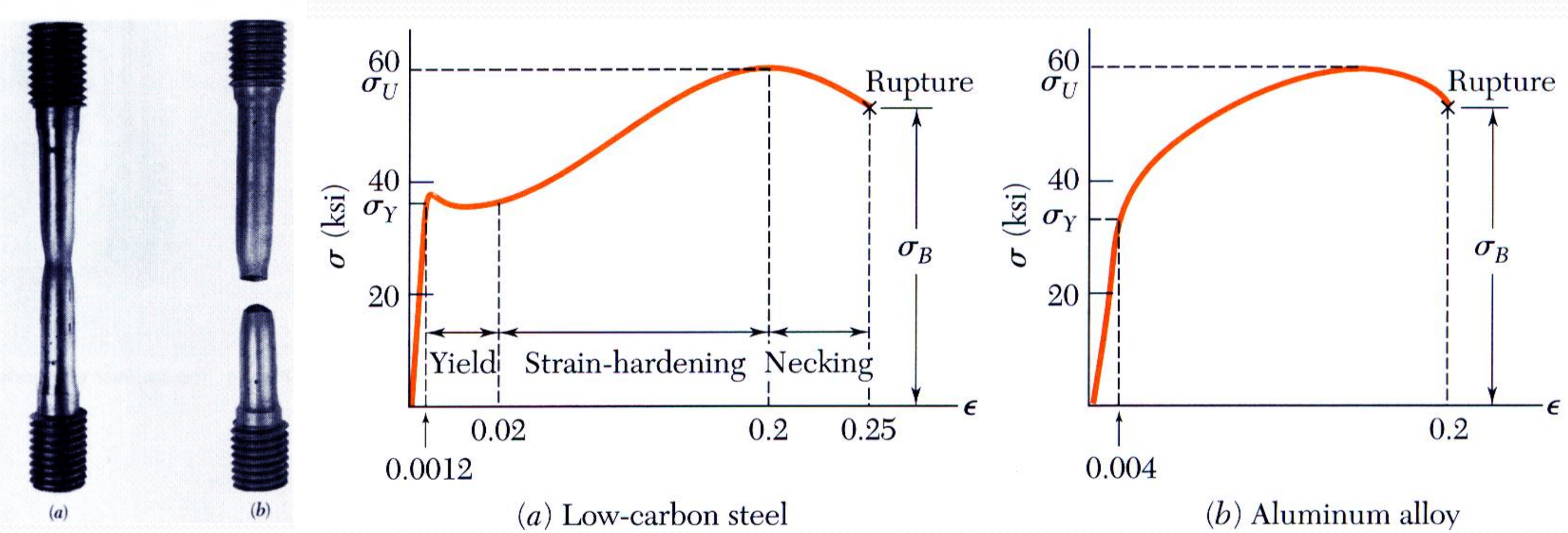
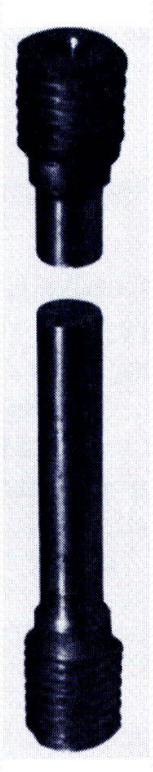


Fig. 2.8 Test specimen with tensile load.

Stress-Strain Diagram: Ductile Materials



Stress-Strain Diagram: Brittle Materials



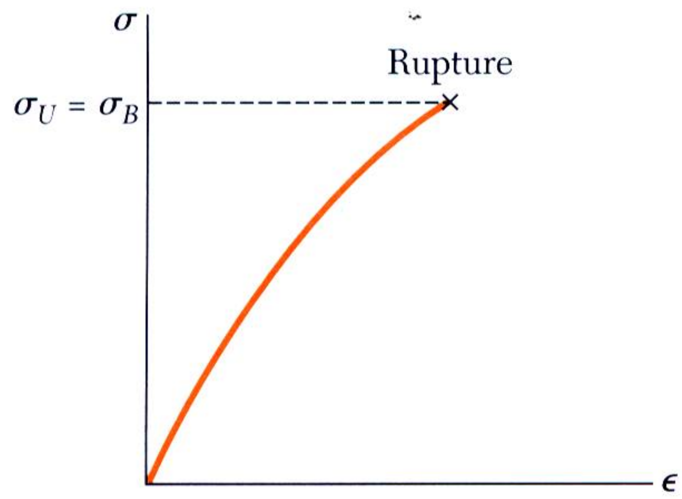


Fig. 2.11 Stress-strain diagram for a typical brittle material.

Hooke's Law: Modulus of Elasticity

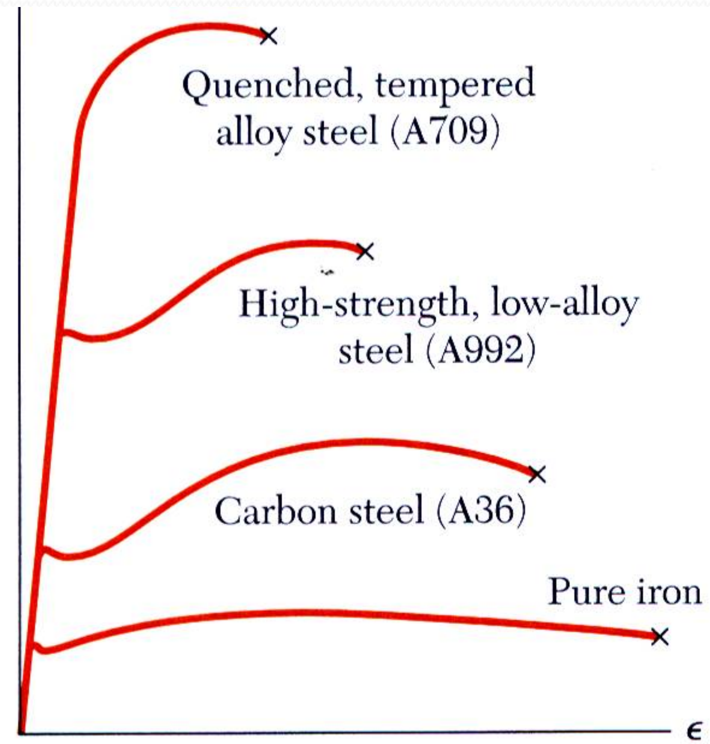


Fig. 2.16 Stress-strain diagrams for iron and different grades of steel.

- Below the yield stress

$$\sigma = E\varepsilon$$

E = Youngs Modulus or Modulus of Elasticity

- Strength is affected by alloying, heat treating, and manufacturing process but stiffness (Modulus of Elasticity) is not.

Elastic vs. Plastic Behavior

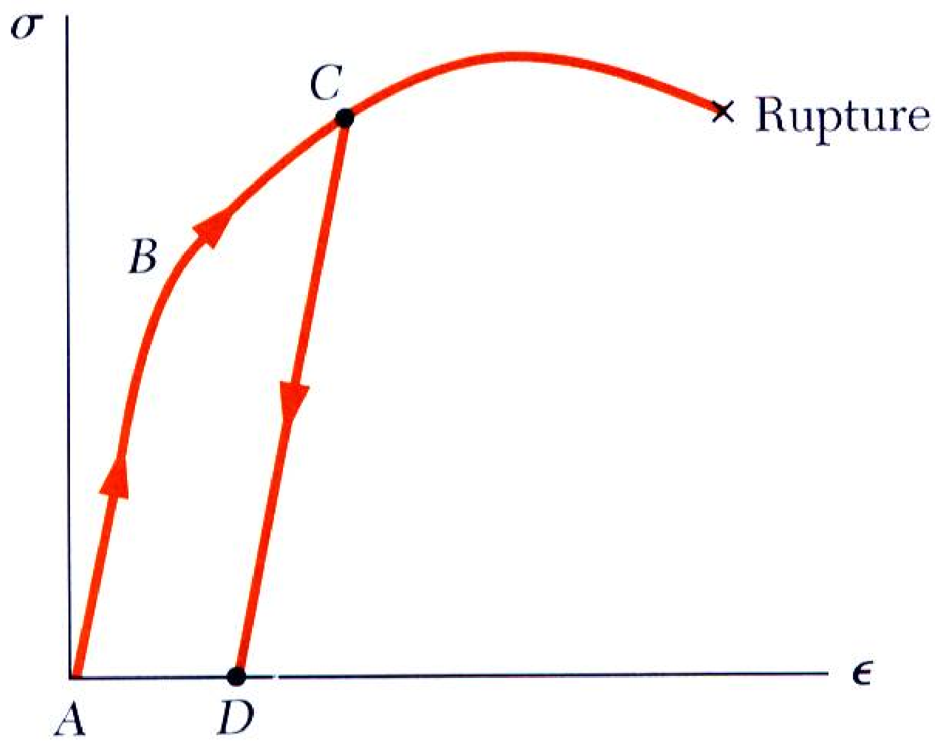


Fig. 2.18

- If the strain disappears when the stress is removed, the material is said to behave *elastically*.
- The largest stress for which this occurs is called the *elastic limit*.
- When the strain does not return to zero after the stress is removed, the material is said to behave *plastically*.

Fatigue

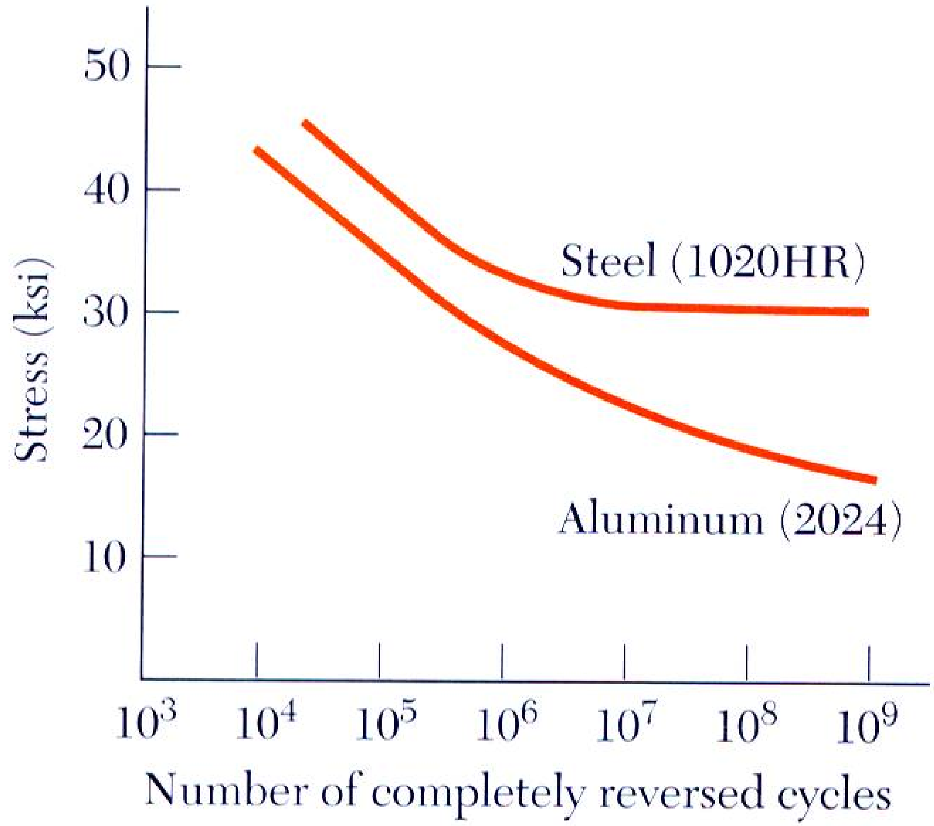


Fig. 2.21

- Fatigue properties are shown on S-N diagrams.
- A member may fail due to *fatigue* at stress levels significantly below the ultimate strength if subjected to many loading cycles.
- When the stress is reduced below the *endurance limit*, fatigue failures do not occur for any number of cycles.

Deformations Under Axial Loading

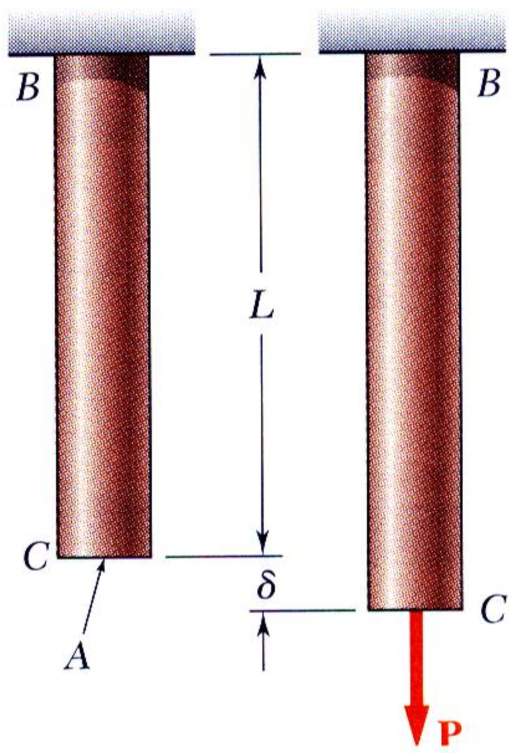


Fig. 2.22

- From Hooke's Law:

$$\sigma = E\varepsilon \qquad \varepsilon = \frac{\sigma}{E} = \frac{P}{AE}$$

- From the definition of strain:

$$\varepsilon = \frac{\delta}{L}$$

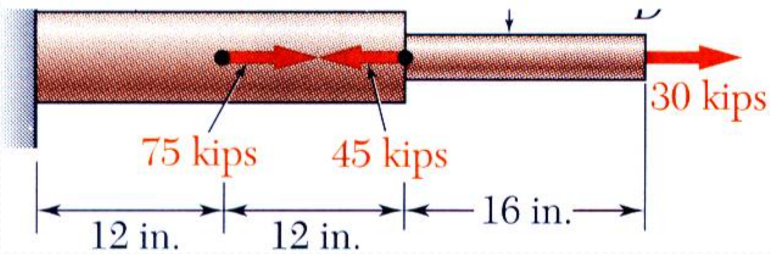
- Equating and solving for the deformation,

$$\delta = \frac{PL}{AE}$$

- With variations in loading, cross-section or material properties,

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i}$$

Example 2.01



$$E = 29\times10^{-6}\,\text{psi}$$

$$D = 1.07\,\text{in.} \quad d = 0.618\,\text{in.}$$

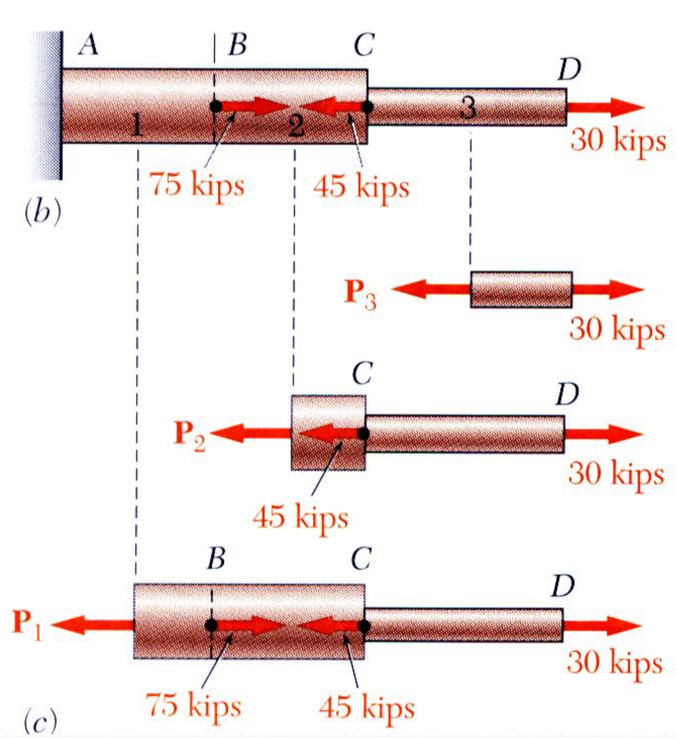
Determine the deformation of the steel rod shown under the given loads.

SOLUTION:

- Divide the rod into components at the load application points.
- Apply a free-body analysis on each component to determine the internal force
- Evaluate the total of the component deflections.

SOLUTION:

- Divide the rod into three components:



$L_1 = L_2 = 12\text{ in.}$ $\quad L_3 = 16\text{ in.}$

$A_1 = A_2 = 0.9\text{ in}^2$ $\quad A_3 = 0.3\text{ in}^2$

- Apply free-body analysis to each component to determine internal forces,

$$P_1 = 60\times10^3\text{ lb}$$

$$P_2 = -15\times10^3\text{ lb}$$

$$P_3 = 30\times10^3\text{ lb}$$

- Evaluate total deflection,

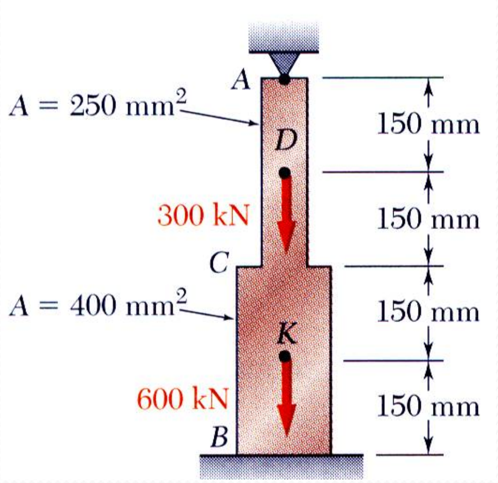
$$\delta = \sum_i \frac{P_i L_i}{A_i E_i} = \frac{1}{E}\left(\frac{P_1L_1}{A_1} + \frac{P_2L_2}{A_2} + \frac{P_3L_3}{A_3}\right)$$

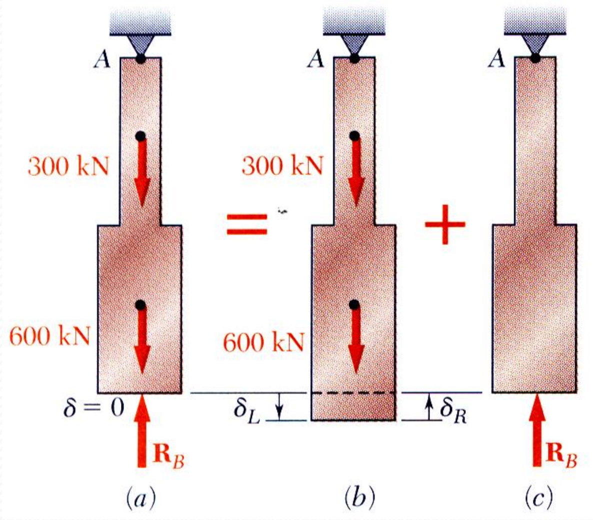
$$= \frac{1}{29\times10^6}\left[\frac{(60\times10^3)12}{0.9} + \frac{(-15\times10^3)12}{0.9} + \frac{(30\times10^3)16}{0.3}\right]$$

$$= 75.9\times10^{-3}\text{in.}$$

$$\boxed{\delta = 75.9\times10^{-3}\text{ in.}}$$

Static Indeterminacy

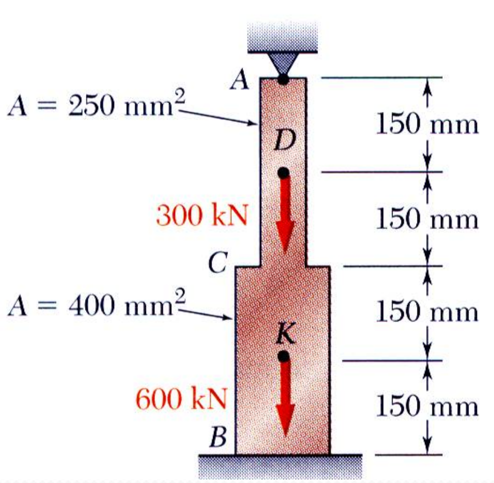


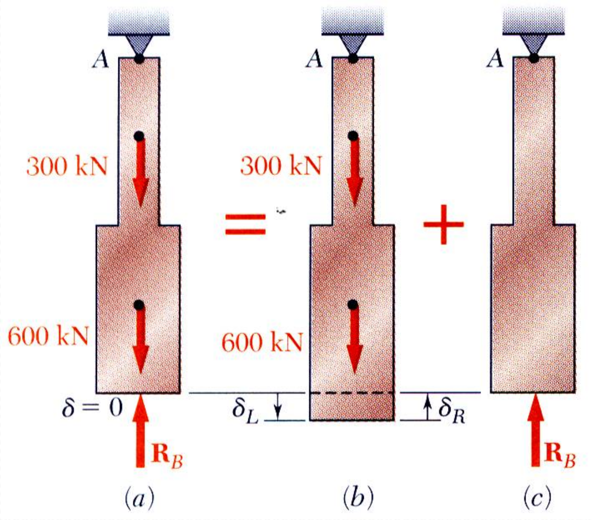


- Structures for which internal forces and reactions cannot be determined from statics alone are said to be *statically indeterminate*.
- A structure will be statically indeterminate whenever it is held by more supports than are required to maintain its equilibrium.
- Redundant reactions are replaced with unknown loads which along with the other loads must produce compatible deformations.
- Deformations due to actual loads and redundant reactions are determined separately and then added or *superposed*.

$$\delta = \delta_L + \delta_R = 0$$

Example 2.04



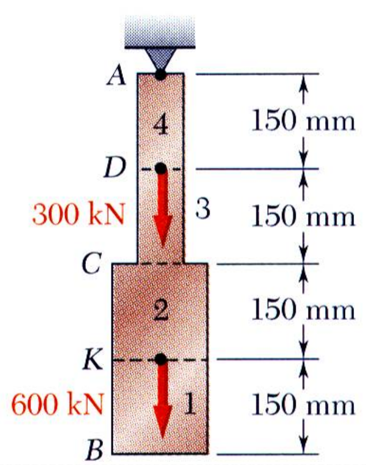


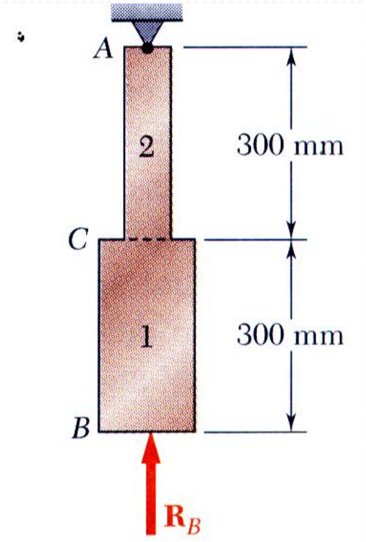
Determine the reactions at *A* and *B* for the steel bar and loading shown, assuming a close fit at both supports before the loads are applied.

SOLUTION:

- Consider the reaction at *B* as redundant, release the bar from that support, and solve for the displacement at *B* due to the applied loads.
- Solve for the displacement at *B* due to the redundant reaction at *B.*
- Require that the displacements due to the loads and due to the redundant reaction be compatible, i.e., require that their sum be zero.
- Solve for the reaction at *A* due to applied loads and the reaction found at *B.*

Example 2.04





SOLUTION:

- Solve for the displacement at *B* due to the applied loads with the redundant constraint released,

$$P_1 = 0 \quad P_2 = P_3 = 600\times10^3\,\text{N} \quad P_4 = 900\times10^3\,\text{N}$$

$$A_1 = A_2 = 400\times10^{-6}\,\text{m}^2 \quad A_3 = A_4 = 250\times10^{-6}\,\text{m}^2$$

$$L_1 = L_2 = L_3 = L_4 = 0.150\,\text{m}$$

$$\delta_\text{L} = \sum_i \frac{P_i L_i}{A_i E_i} = \frac{1.125\times10^9}{E}$$

- Solve for the displacement at *B* due to the redundant constraint,

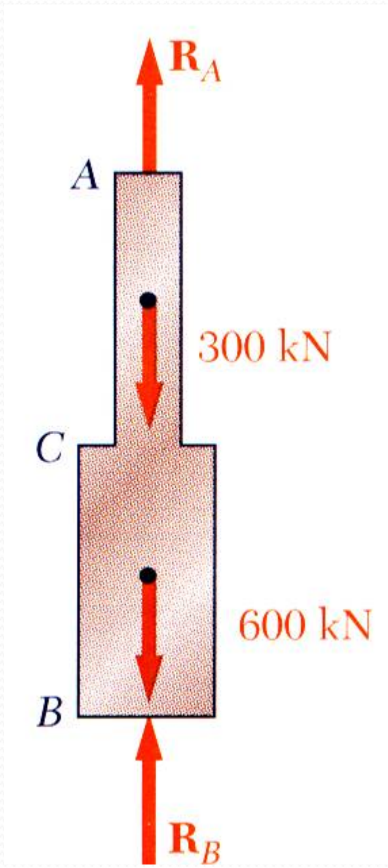
$$P_1 = P_2 = -R_B$$

$$A_1 = 400\times10^{-6}\,\text{m}^2 \quad A_2 = 250\times10^{-6}\,\text{m}^2$$

$$L_1 = L_2 = 0.300\,\text{m}$$

$$\delta_R = \sum_i \frac{P_i L_i}{A_i E_i} = -\frac{\left(1.95\times10^3\right)R_B}{E}$$

Example 2.04



- Require that the displacements due to the loads and due to the redundant reaction be compatible,

$$\delta = \delta_L + \delta_R = 0$$

$$\delta = \frac{1.125\times10^9}{E} - \frac{\left(1.95\times10^3\right)R_B}{E} = 0$$

$$R_B = 577\times10^3\,\text{N} = 577\,\text{kN}$$

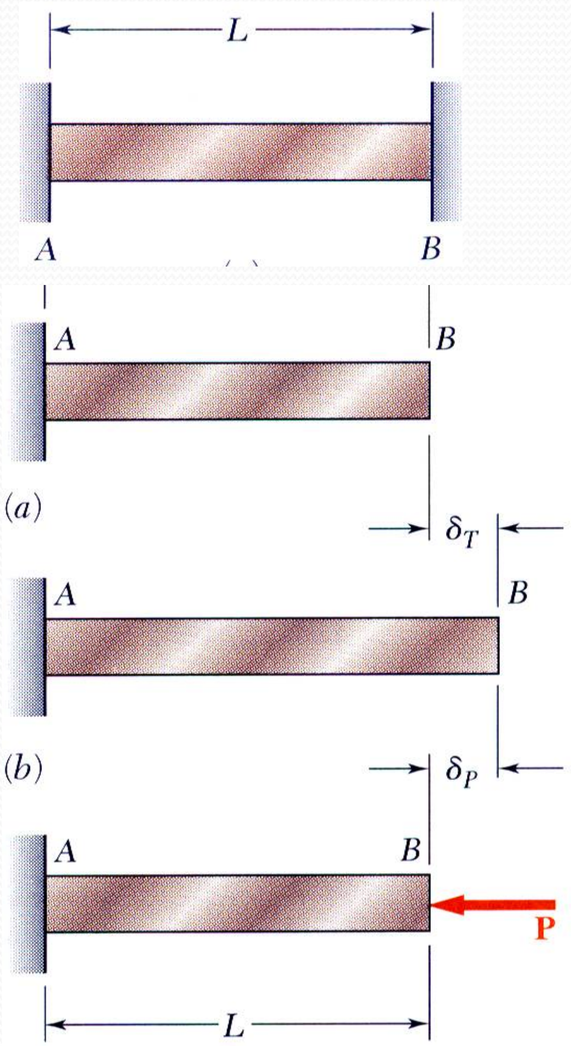
- Find the reaction at *A* due to the loads and the reaction at *B*

$$\sum F_y = 0 = R_A - 300\,\text{kN} - 600\,\text{kN} + 577\,\text{kN}$$

$$R_A = 323\,\text{kN}$$

$$R_A = 323\,\text{kN}$$
$$R_B = 577\,\text{kN}$$

Thermal Stresses



- A temperature change results in a change in length or *thermal strain*. There is no stress associated with the thermal strain unless the elongation is restrained by the supports.

- Treat the additional support as redundant and apply the principle of superposition.

$$\delta_T = \alpha(\Delta T)L \qquad \delta_P = \frac{PL}{AE}$$

$$\alpha = \text{thermal expansion coef.}$$

- The thermal deformation and the deformation from the redundant support must be compatible.

$$\delta = \delta_T + \delta_P = 0$$

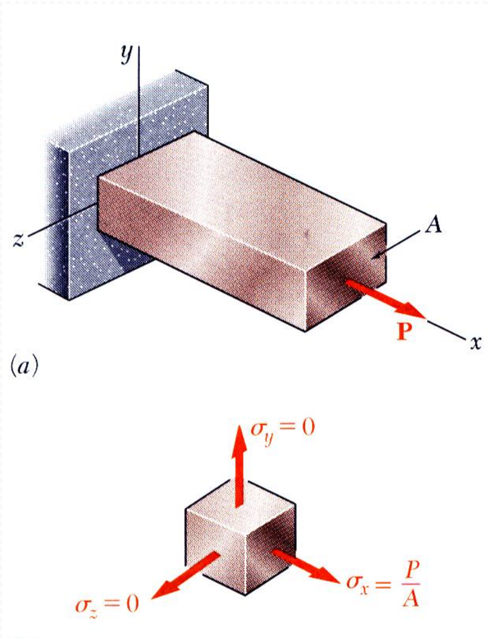
$$\alpha(\Delta T)L + \frac{PL}{AE} = 0$$

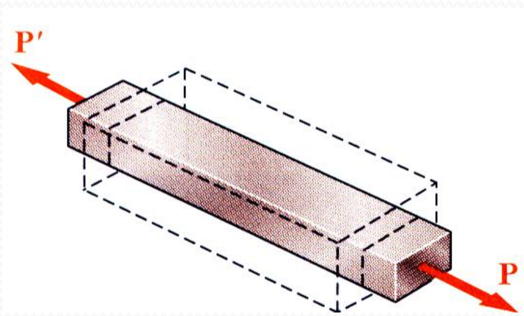
$$\delta = \delta_T + \delta_P = 0$$

$$P = -AE\alpha(\Delta T)$$

$$\sigma = \frac{P}{A} = -E\alpha(\Delta T)$$

Poisson's Ratio





- For a slender bar subjected to axial loading:

$$\varepsilon_x = \frac{\sigma_x}{E} \qquad \sigma_y = \sigma_z = 0$$

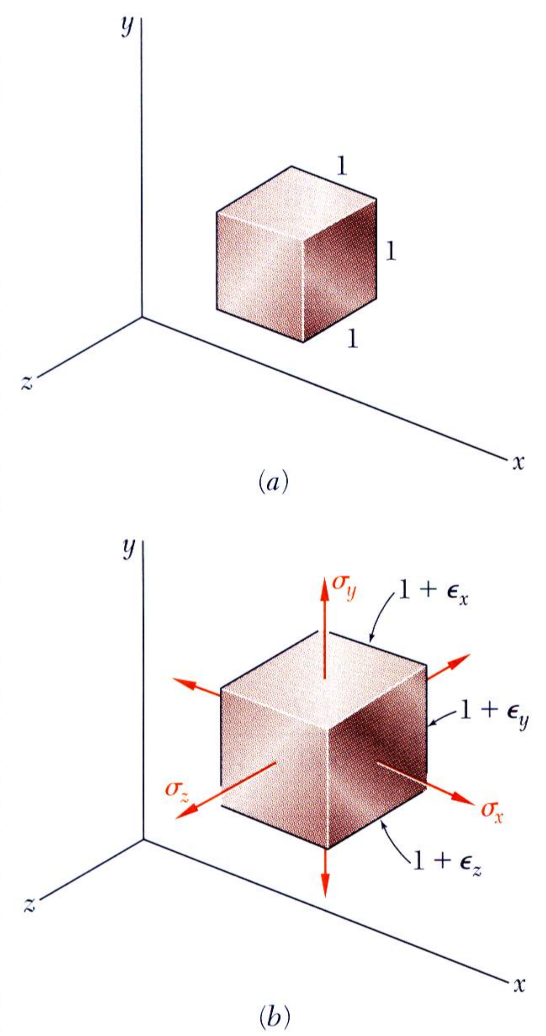
- The elongation in the x-direction is accompanied by a contraction in the other directions. Assuming that the material is isotropic (no directional dependence),

$$\varepsilon_y = \varepsilon_z \neq 0$$

- Poisson's ratio is defined as

$$\nu = \left| \frac{\text{lateral strain}}{\text{axial strain}} \right| = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x}$$

Generalized Hooke's Law



- For an element subjected to multi-axial loading, the normal strain components resulting from the stress components may be determined from the *principle of superposition*. This requires:

  1) strain is linearly related to stress
  2) deformations are small

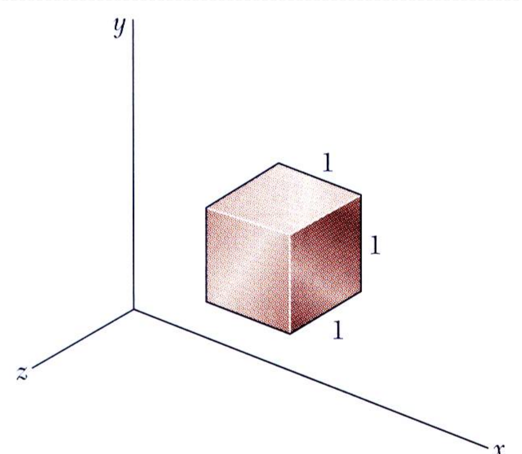
- With these restrictions:

$$\varepsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

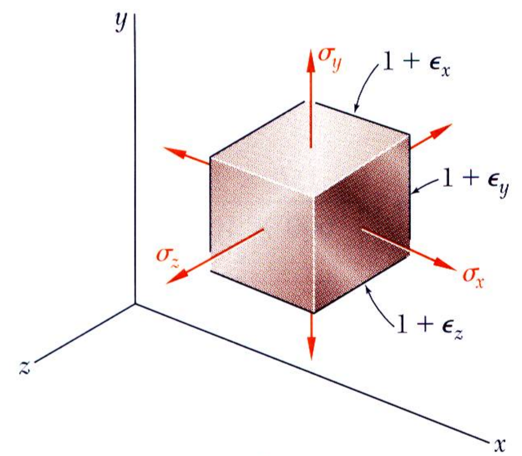
$$\varepsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\varepsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

Dilatation: Bulk Modulus



(a)



(b)

- Relative to the unstressed state, the change in volume is

$$e = 1 - \left[(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)\right] = 1 - \left[1 + \varepsilon_x + \varepsilon_y + \varepsilon_z\right]$$

$$= \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$= \frac{1-2\nu}{E}\left(\sigma_x + \sigma_y + \sigma_z\right)$$

$$= \text{dilatation (change in volume per unit volume)}$$

- For element subjected to uniform hydrostatic pressure,

$$e = -p\,\frac{3(1-2\nu)}{E} = -\frac{p}{k}$$

$$k = \frac{E}{3(1-2\nu)} = \text{bulk modulus}$$

- Subjected to uniform pressure, dilatation must be negative, therefore

$$0 < \nu < \tfrac{1}{2}$$

Shearing Strain

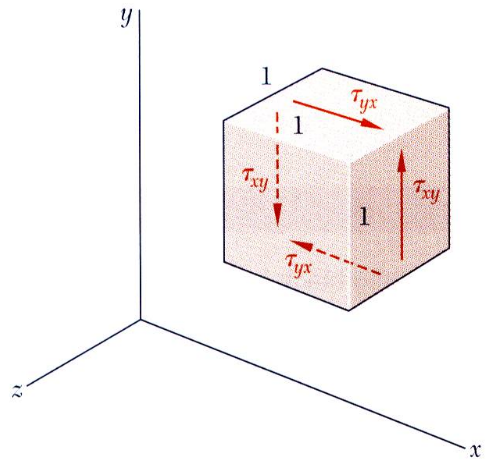


Fig. 2.46

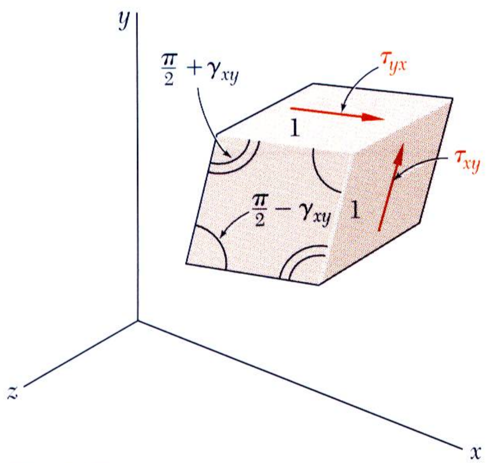


Fig. 2.47

- A cubic element subjected to a shear stress will deform into a rhomboid. The corresponding *shear* strain is quantified in terms of the change in angle between the sides,

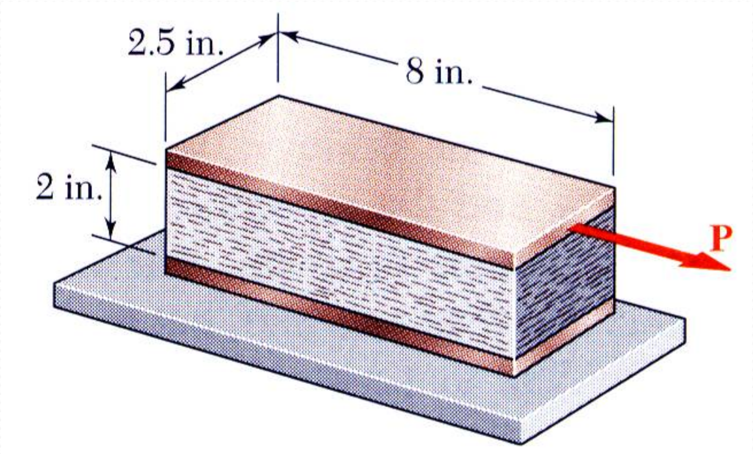
$$\tau_{xy} = f(\gamma_{xy})$$

- A plot of shear stress vs. shear strain is similar the previous plots of normal stress vs. normal strain except that the strength values are approximately half. For small strains,

$$\tau_{xy} = G\gamma_{xy} \quad \tau_{yz} = G\gamma_{yz} \quad \tau_{zx} = G\gamma_{zx}$$

where G is the modulus of rigidity or shear modulus.

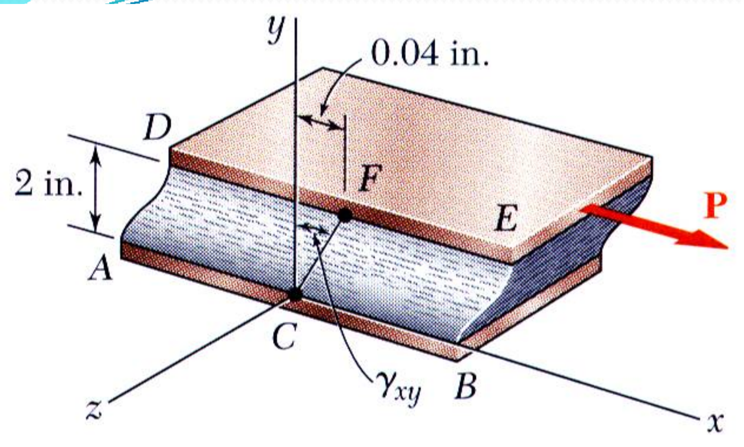
Example 2.10



A rectangular block of material with modulus of rigidity G = 90 ksi is bonded to two rigid horizontal plates. The lower plate is fixed, while the upper plate is subjected to a horizontal force P. Knowing that the upper plate moves through 0.04 in. under the action of the force, determine a) the average shearing strain in the material, and b) the force P exerted on the plate.

SOLUTION:

- Determine the average angular deformation or shearing strain of the block.
- Apply Hooke's law for shearing stress and strain to find the corresponding shearing stress.
- Use the definition of shearing stress to find the force P.



- Determine the average angular deformation or shearing strain of the block.

$$\gamma_{xy} \approx \tan\gamma_{xy} = \frac{0.04\,\text{in.}}{2\,\text{in.}} \qquad \gamma_{xy} = 0.020\,\text{rad}$$

- Apply Hooke's law for shearing stress and strain to find the corresponding shearing stress.

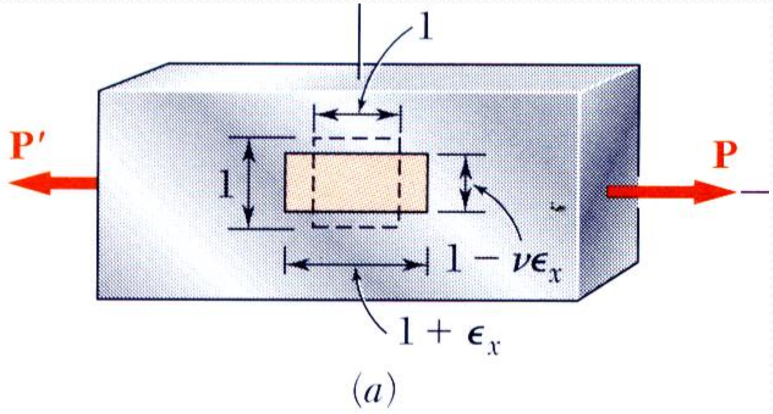
$$\tau_{xy} = G\gamma_{xy} = \left(90\times10^3\,\text{psi}\right)\left(0.020\,\text{rad}\right) = 1800\,\text{psi}$$

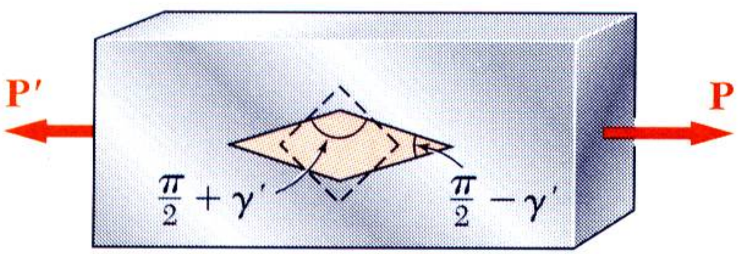
- Use the definition of shearing stress to find the force *P*.

$$P = \tau_{xy}A = \left(1800\,\text{psi}\right)\left(8\,\text{in.}\right)\left(2.5\,\text{in.}\right) = 36\times10^3\,\text{lb}$$

$$\boxed{P = 36.0\,\text{kips}}$$

Relation Among *E*, *ν*, and *G*

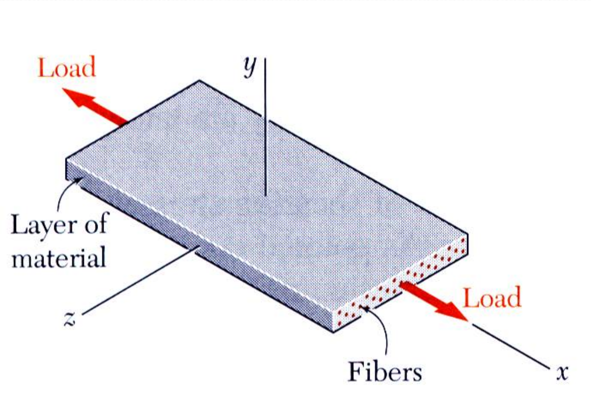




- An axially loaded slender bar will elongate in the axial direction and contract in the transverse directions.
- An initially cubic element oriented as in top figure will deform into a rectangular parallelepiped. The axial load produces a normal strain.
- If the cubic element is oriented as in the bottom figure, it will deform into a rhombus. Axial load also results in a shear strain.
- Components of normal and shear strain are related,

$$\frac{E}{2G} = (1+\nu)$$

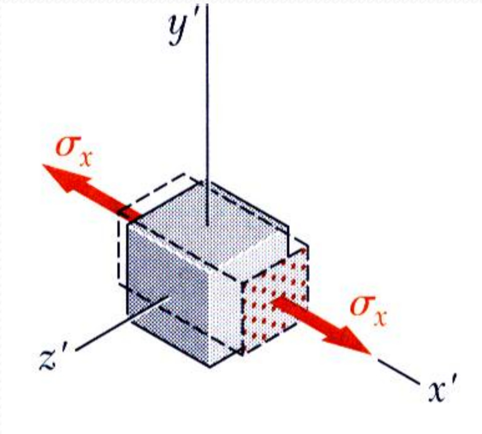
Composite Materials



- *Fiber-reinforced composite materials* are formed from *lamina* of fibers of graphite, glass, or polymers embedded in a resin matrix.

- Normal stresses and strains are related by Hooke's Law but with directionally dependent moduli of elasticity,

$$E_x = \frac{\sigma_x}{\varepsilon_x} \quad E_y = \frac{\sigma_y}{\varepsilon_y} \quad E_z = \frac{\sigma_z}{\varepsilon_z}$$

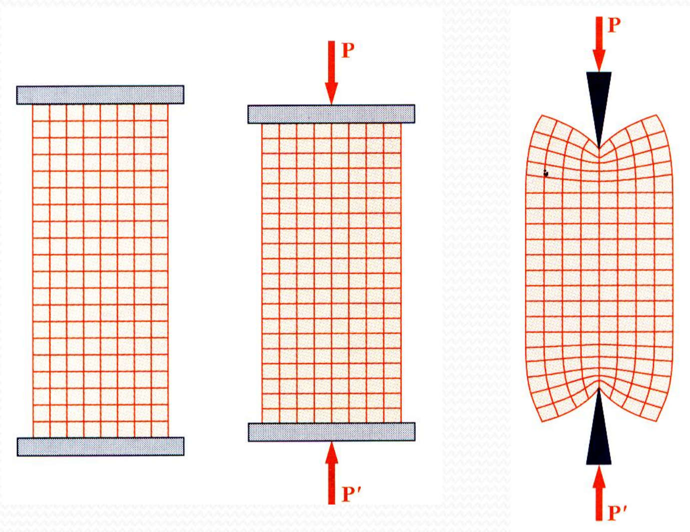


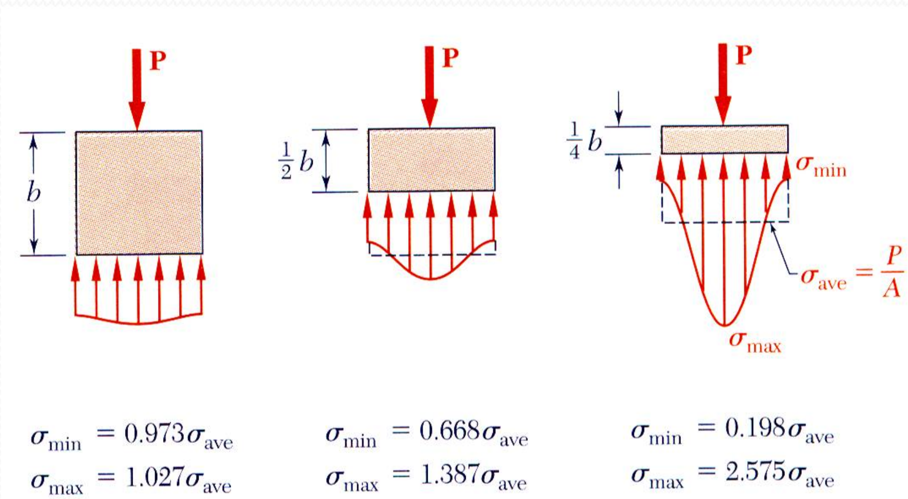
- Transverse contractions are related by directionally dependent values of Poisson's ratio, e.g.,

$$\nu_{xy} = -\frac{\varepsilon_y}{\varepsilon_x} \quad \nu_{xz} = -\frac{\varepsilon_z}{\varepsilon_x}$$

- Materials with directionally dependent mechanical properties are *anisotropic*.

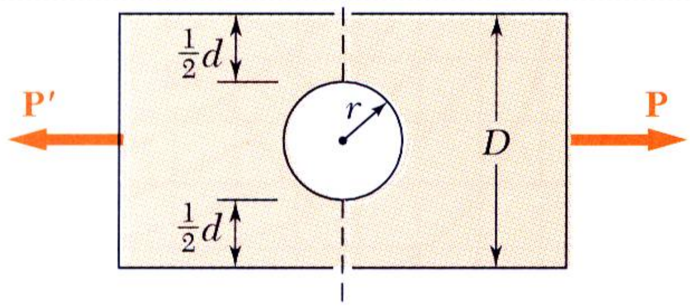
Saint-Venant's Principle

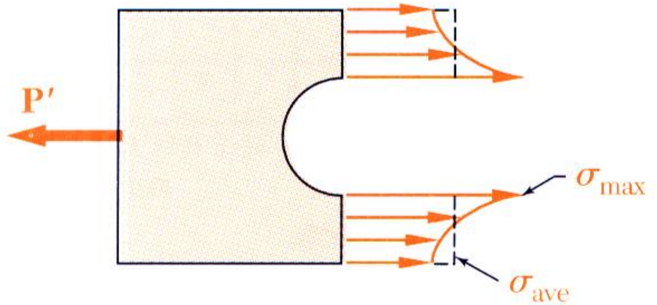


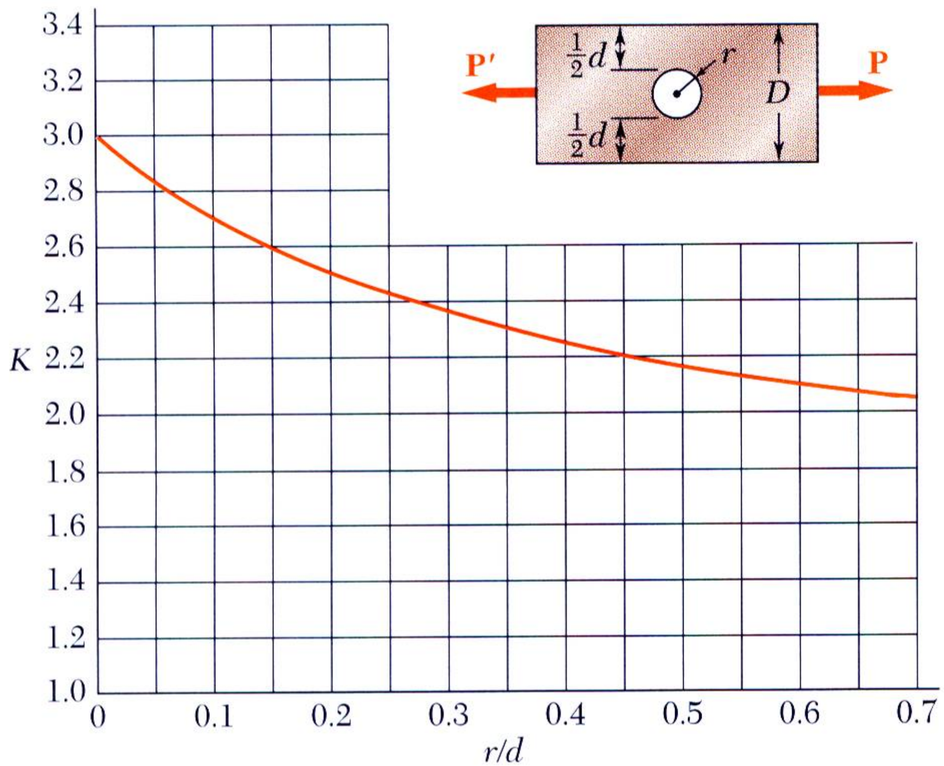


- Loads transmitted through rigid plates result in uniform distribution of stress and strain.
- Concentrated loads result in large stresses in the vicinity of the load application point.
- Stress and strain distributions become uniform at a relatively short distance from the load application points.
- **Saint-Venant's Principle:** Stress distribution may be assumed independent of the mode of load application except in the immediate vicinity of load application points.

Stress Concentration: Hole





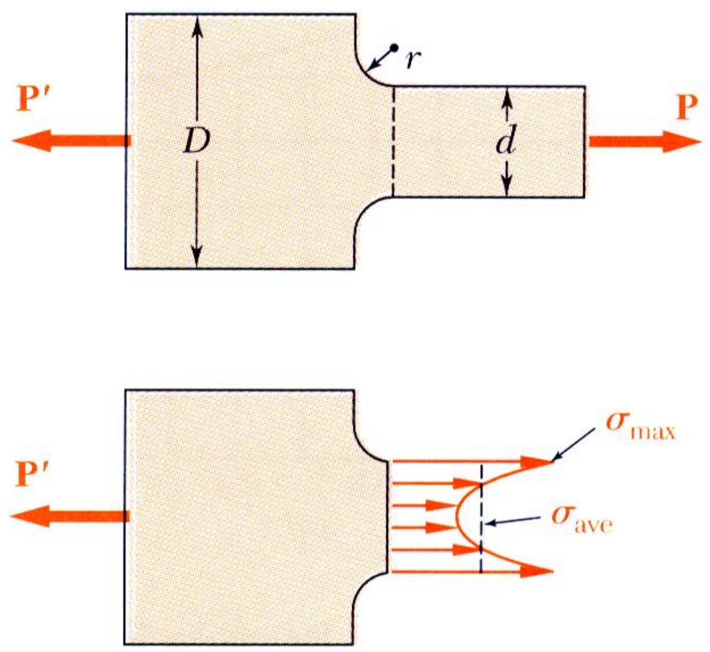


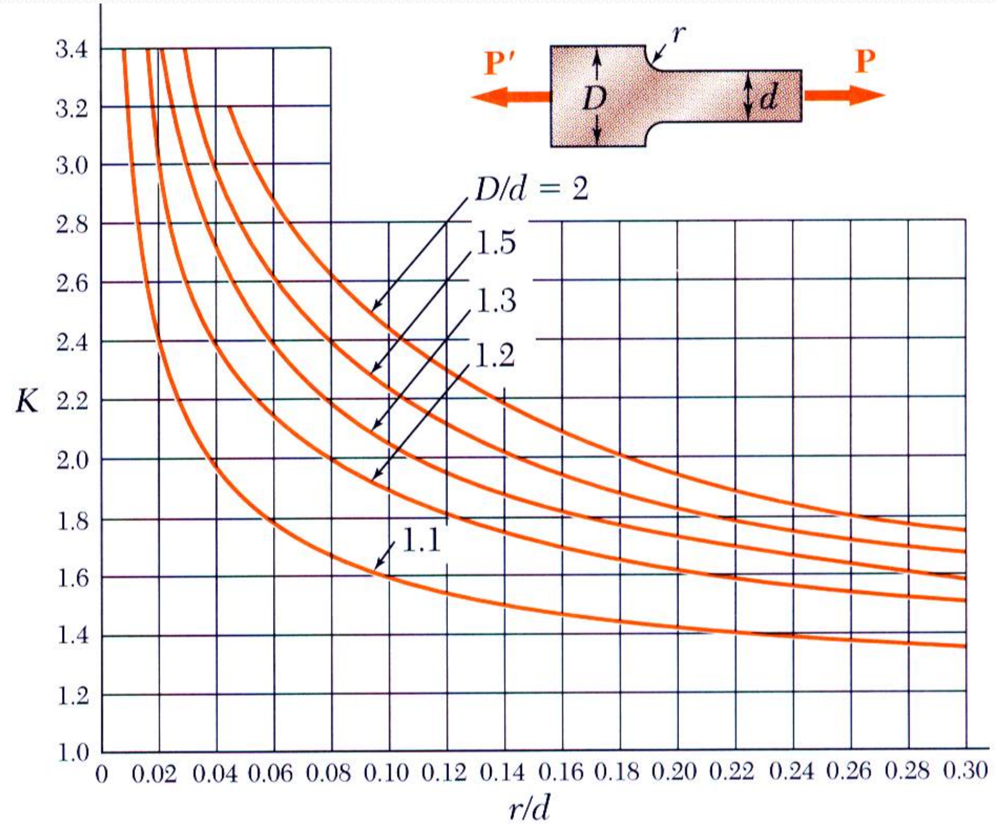
(*a*) Flat bars with holes

Discontinuities of cross section may result in high localized or *concentrated* stresses.

$$K = \frac{\sigma_{max}}{\sigma_{ave}}$$

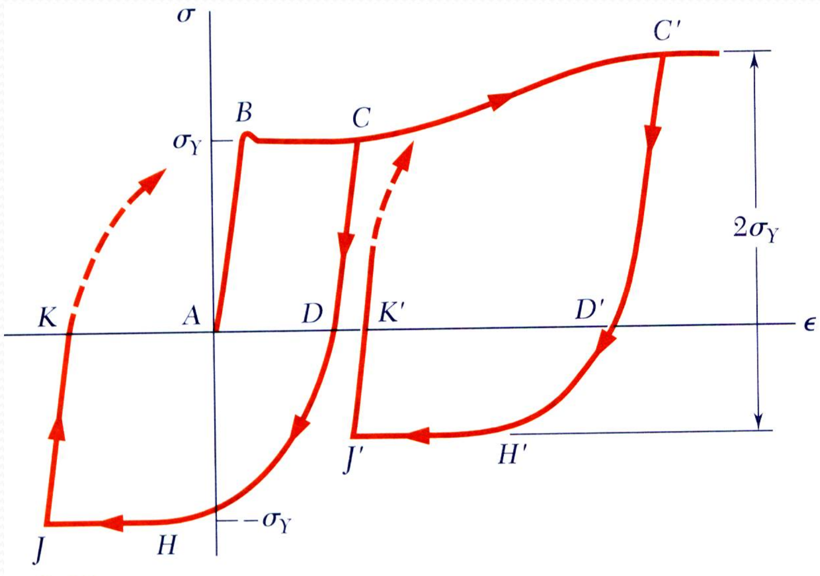
Stress Concentration: Fillet

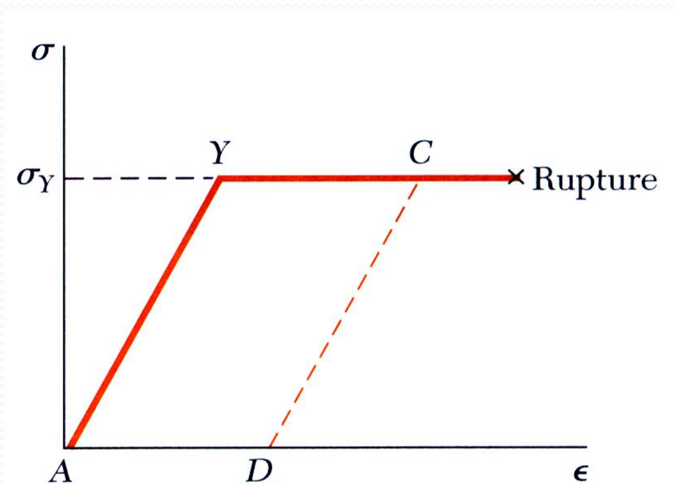




(*b*) Flat bars with fillets

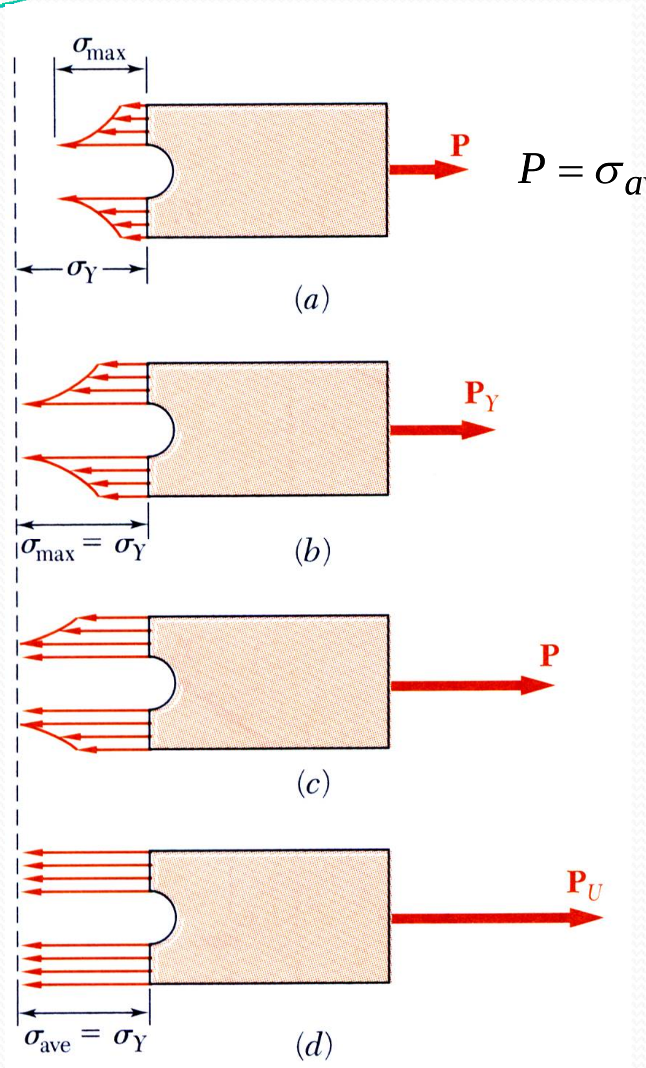
Elastoplastic Materials





- Previous analyses based on assumption of linear stress-strain relationship, i.e., stresses below the yield stress
- Assumption is good for brittle material which rupture without yielding
- If the yield stress of ductile materials is exceeded, then plastic deformations occur
- Analysis of plastic deformations is simplified by assuming an idealized *elastoplastic material*
- Deformations of an elastoplastic material are divided into elastic and plastic ranges
- Permanent deformations result from loading beyond the yield stress

Plastic Deformations



$$P = \sigma_{ave} A = \frac{\sigma_{max} A}{K}$$

- Elastic deformation while maximum stress is less than yield stress

$$P_Y = \frac{\sigma_Y A}{K}$$

- Maximum stress is equal to the yield stress at the maximum elastic loading

- At loadings above the maximum elastic load, a region of plastic deformations develop near the hole

$$P_U = \sigma_Y A = K P_Y$$

- As the loading increases, the plastic region expands until the section is at a uniform stress equal to the yield stress

Residual Stresses

- When a single structural element is loaded uniformly beyond its yield stress and then unloaded, it is permanently deformed but all stresses disappear. This is not the general result.
  - *Residual stresses* will remain in a structure after loading and unloading if
    - only part of the structure undergoes plastic deformation
    - different parts of the structure undergo different plastic deformations
  - Residual stresses also result from the uneven heating or cooling of structures or structural elements