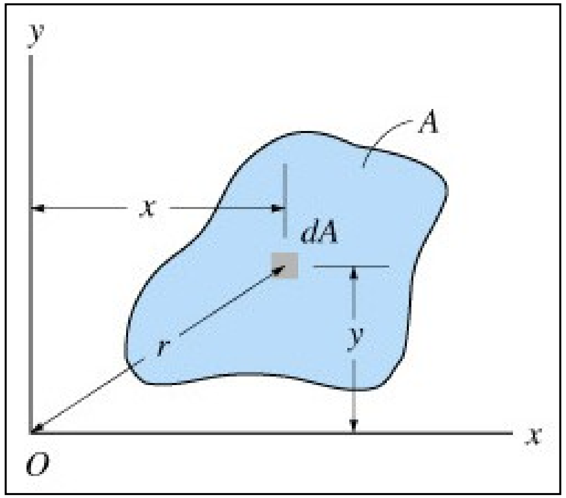
Lecture Notes

Module 5: Area Mment of Inertia and Internal Forces

Moments of Inertia and Radius of Gyration

- In general case

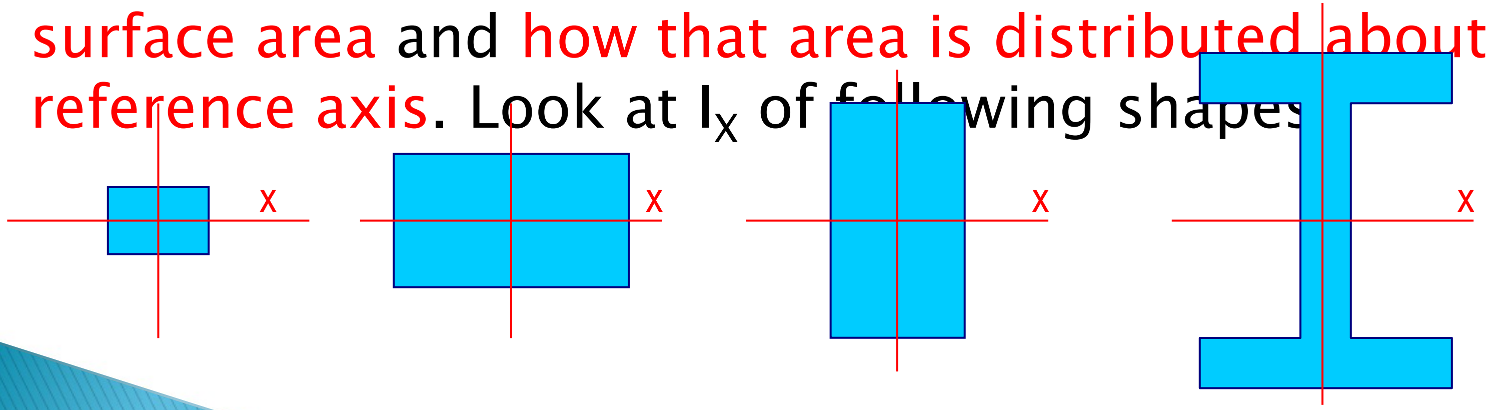


$$I_X = \int_A y^2 dA \qquad k_X = \sqrt{\frac{I_X}{A}}$$

$$I_y = \int_A x^2 dA \qquad k_y = \sqrt{\frac{I_y}{A}}$$

$$J_O = \int_A r^2 dA = I_X + I_Y \qquad k_O = \sqrt{\frac{J_O}{A}}$$

- It is a mathematical property of a section concerned with a surface area and how that area is distributed about the reference axis. Look at I_X of following shapes



Rectangular Area

- Moment of Inertia about centroidal axes for a

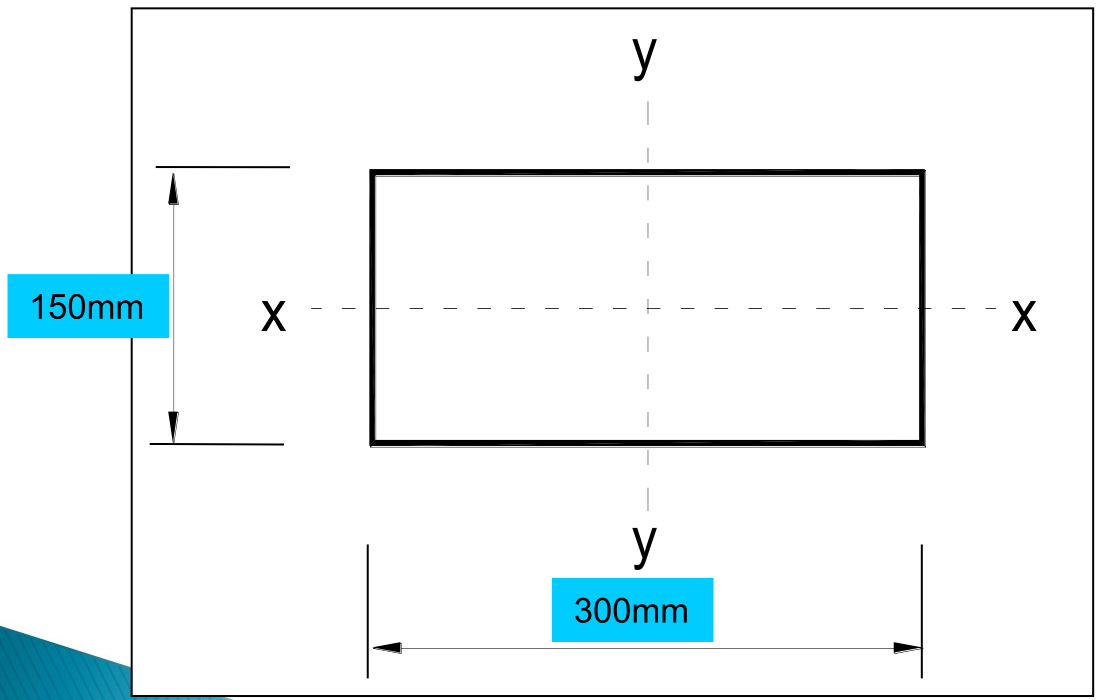
$$I_X = \frac{1}{12} bh^3 = \frac{1}{12}(300\,mm)(150\,mm)^3 = 562500\,mm^4$$

$$I_Y = \frac{1}{12} hb^3 = \frac{1}{12} hb^3 = (150\,mm)(300\,mm)^3 = 1125000\,mm^4$$

  - Which one is bigger?

New Unit!

To Remember



1. Standard sections (C-shapes, Wide flange beams, Hollow steel sections): Check tables (last 3 pages of complementary notes)

2. Rectangular shapes: Calculate yourself, memorize equations.

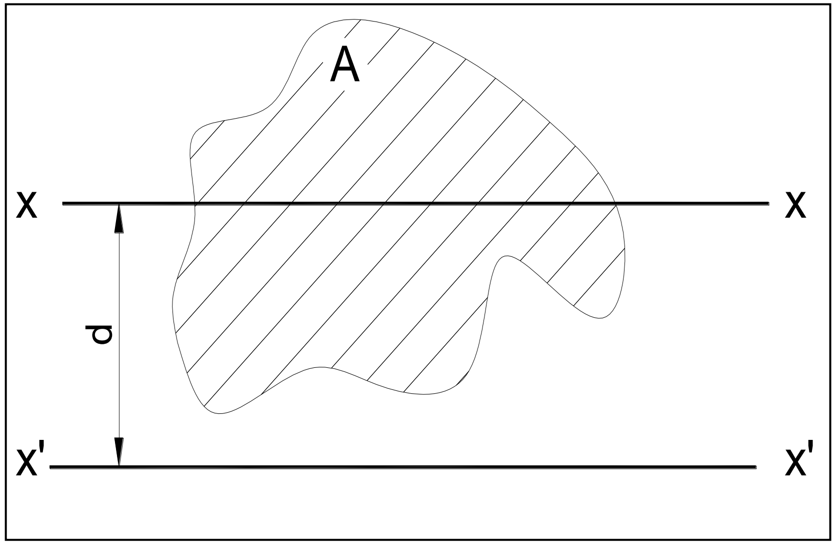
3. Other shapes (circles, triangles, etc..): The equation or value will be given to you.

Parallel-Axis Theorem for an Area

- Used to calculate the moment of inertia about other axis.
- The moment of inertia of an area about an axis is equal to the moment of inertia of the area about a parallel axis passing through the area's centroid plus the product of the area and the square of the perpendicular distance between the axes.

$$I_{x'} = I_x + Ad^2$$

- Ix': Moment of Inertia about new axis
- Ix: Moment of Inertia about original axis
- A: Area of shape
- d: Perpendicular distance from new to original axis



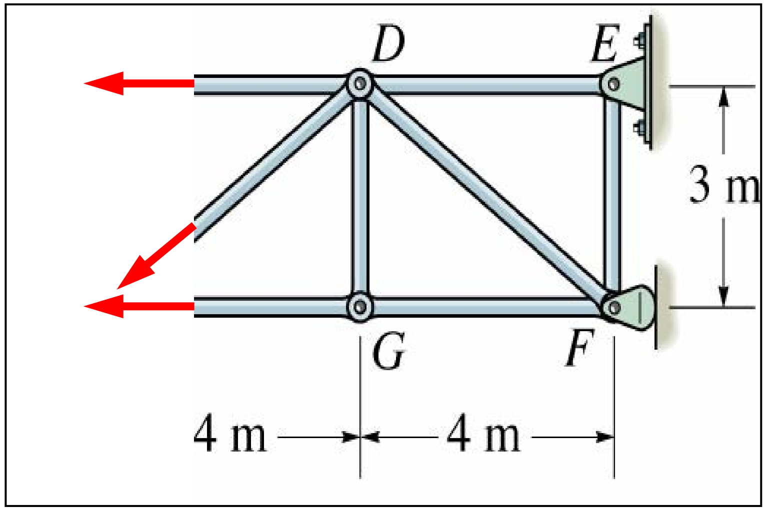
Moments of Inertia for Composite Areas

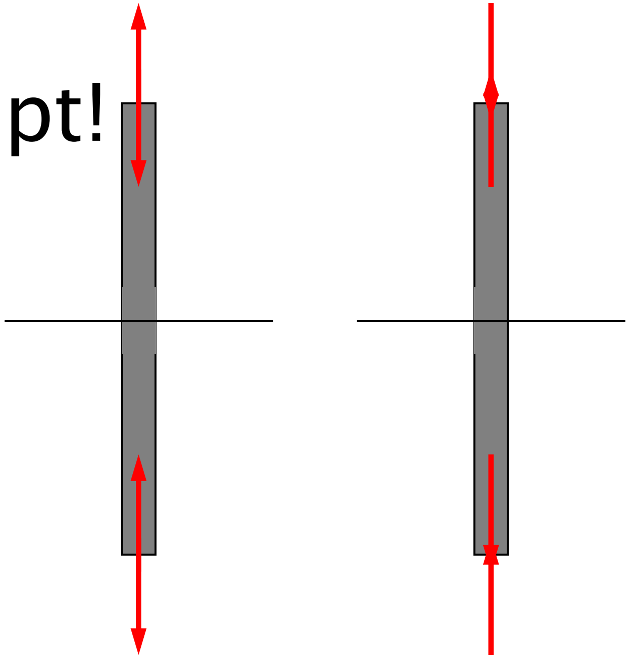
- If the moment of inertia of each simpler area (a part of the composite area) is known or can be determined about a common axis, then the moment of inertia of the composite area equals the algebraic sum of the moments of inertia of all its parts.

- Steps to follow:
  - Divide the composite area into smaller and simpler parts
  - Find the Centroid of each part
  - Calculate the moment of inertia (I) for each part about its centroidal axis
  - Use parallel-axis theorem to calculate the moment of inertia of each part about the given axis (normally the centroidal axis of composite area, if this is the case, very likely you need to find the centroid of that composite area first)
  - Take the algebraic sum of the moments of inertia of all parts to get the moment of inertia of the composite area

Internal Forces

- We have learnt this concept!

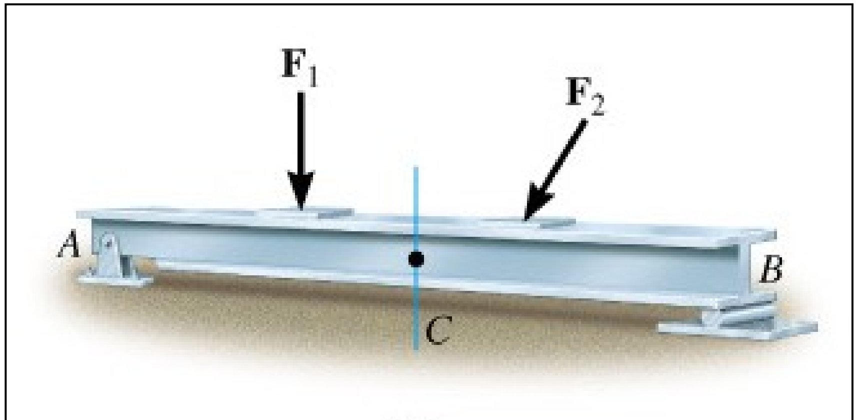




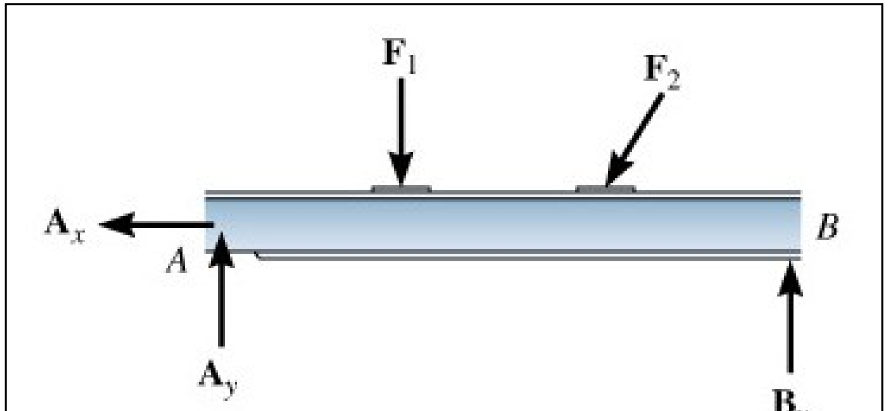
Internal Forces

- Now let's look at multi-force member!

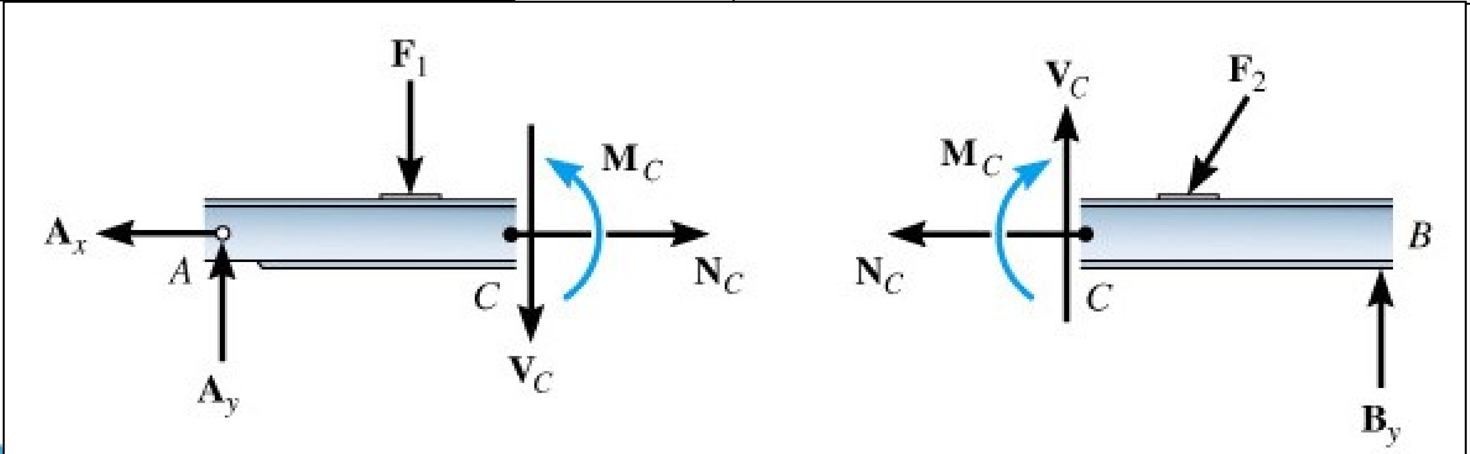
$\sum Fx = 0$
$\sum Fy = 0$
$\sum Mo = 0$



(a)



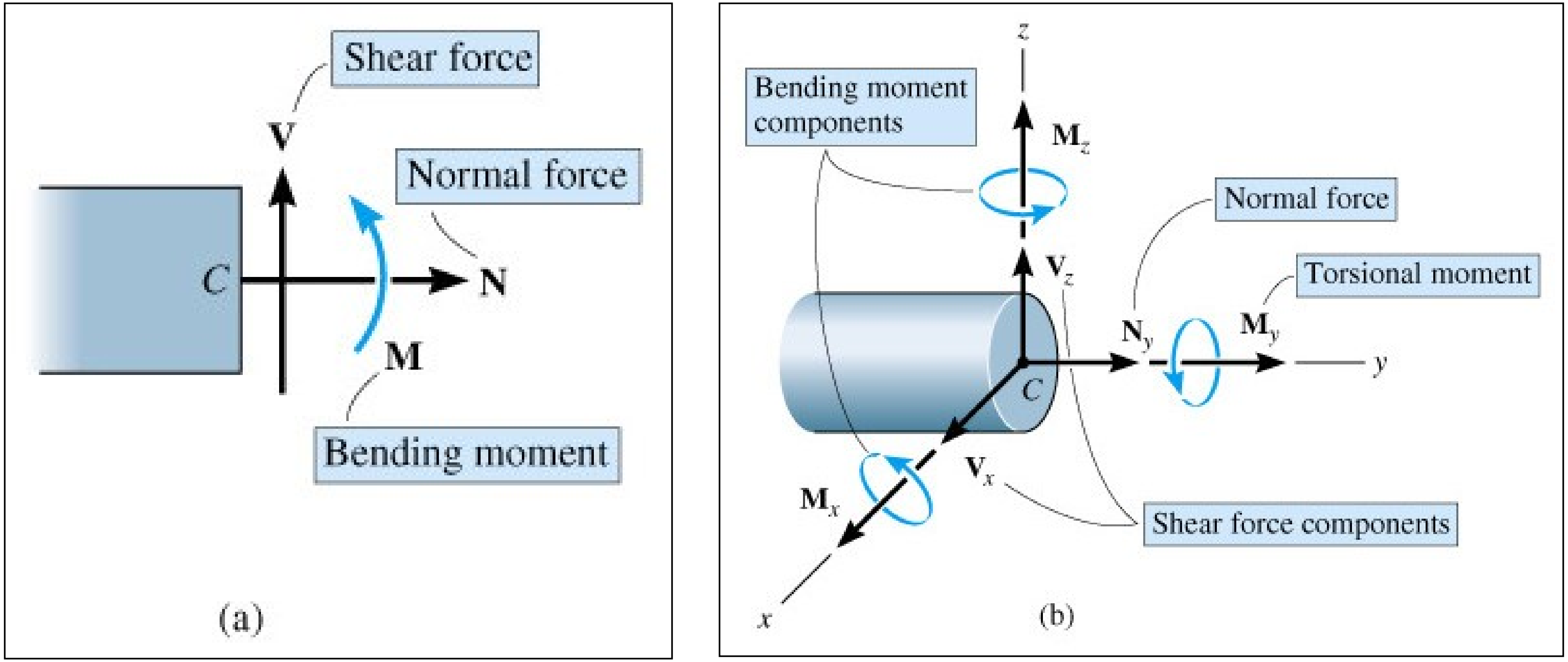
(b)



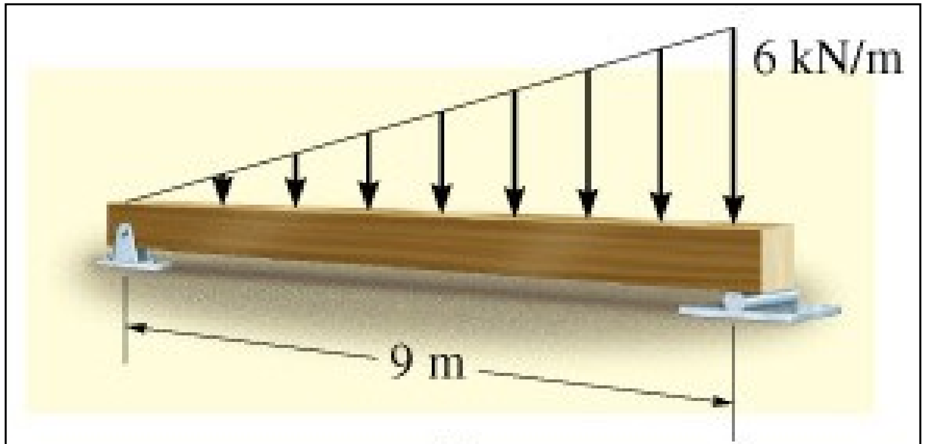
(c)

Internal Forces

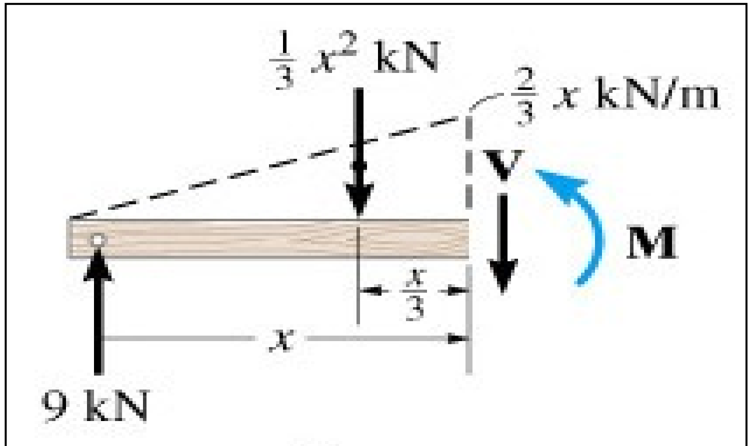
- Naming those internal forces



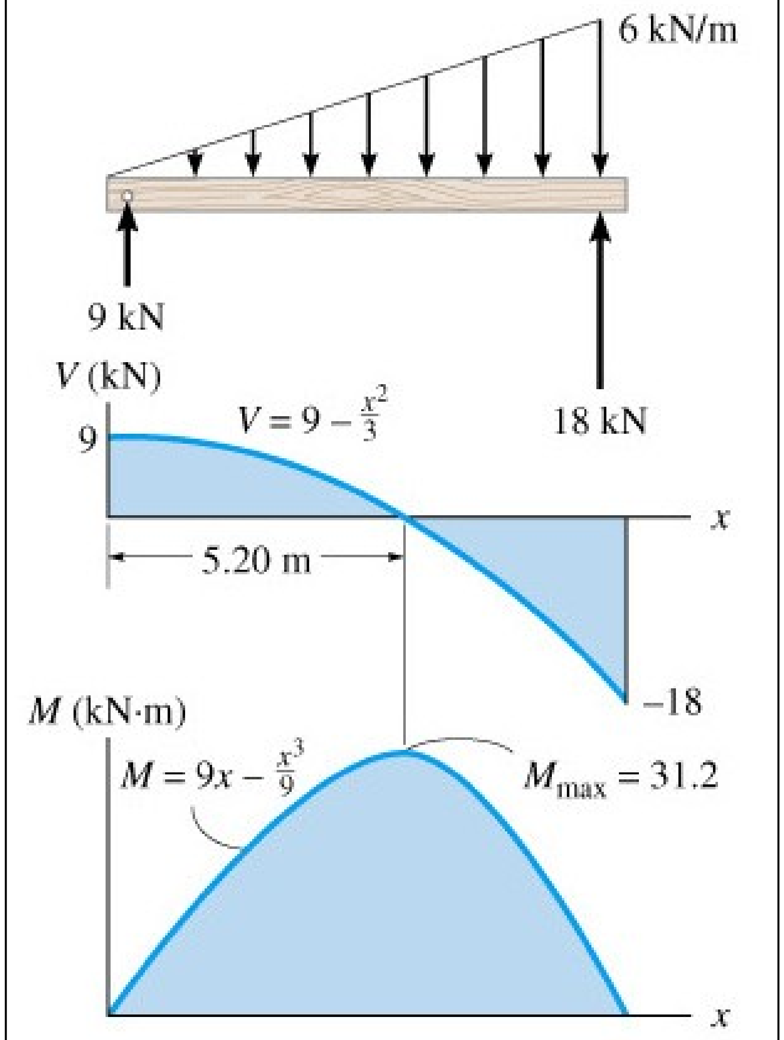
SFD and BMD



(a)



(b)



(c)

Objectives

- By the end of this Module you should be able to:
  - Understand how Shear force diagrams (SFD) and bending moment diagrams (BMD) relate to the internal forces in structures
  - Know how to draw SFD for beams
  - Know how to draw BMD for beams
  - Understand how different load types (concentrated load, distributed load, couple) affect the shape of SFD and BMD
  - Understand the design on tension and compression members

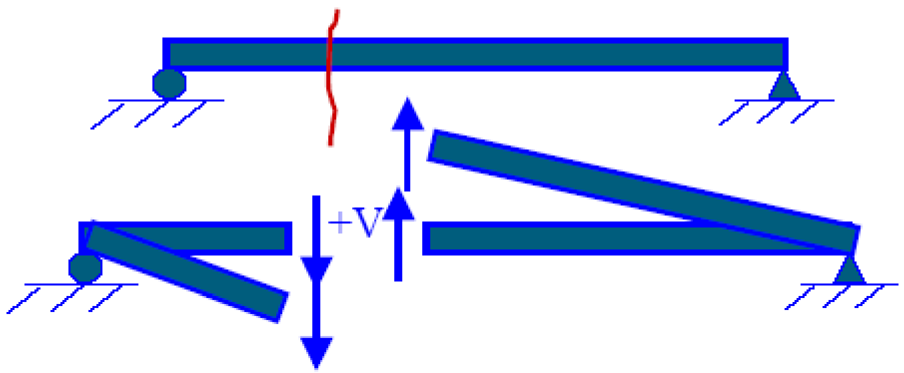
Remember These rules

shear diagram is like the force polygons you drew in Statics. If the force arrows are drawn tip to tail, they form a closed polygon.

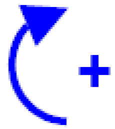
- When there is no load along a portion of the beam then:
  - The shear will be constant (may or may not be zero)
  - The moment will be at most linearly varying (may be zero or constant)
- If there is a uniform load along a portion of the beam then:
  - The shear will be linearly varying
  - The moment will be quadratic
- When there is a concentrated force applied at a point, either a load or a reaction then:
  - The shear diagram will take a sudden jump equal to the applied force at the point The diagram will move down if the force is directed downward, and vice versa
  - There will be a change in the slope of the moment diagram at that point
- The only thing that will cause a jump in the M diagram is a concentrated applied moment, and those are rare

Sign Convention–Axial Loads

Now Clockwise is +ve!

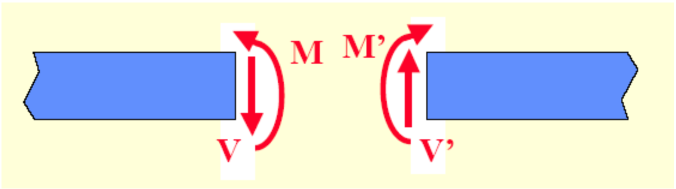


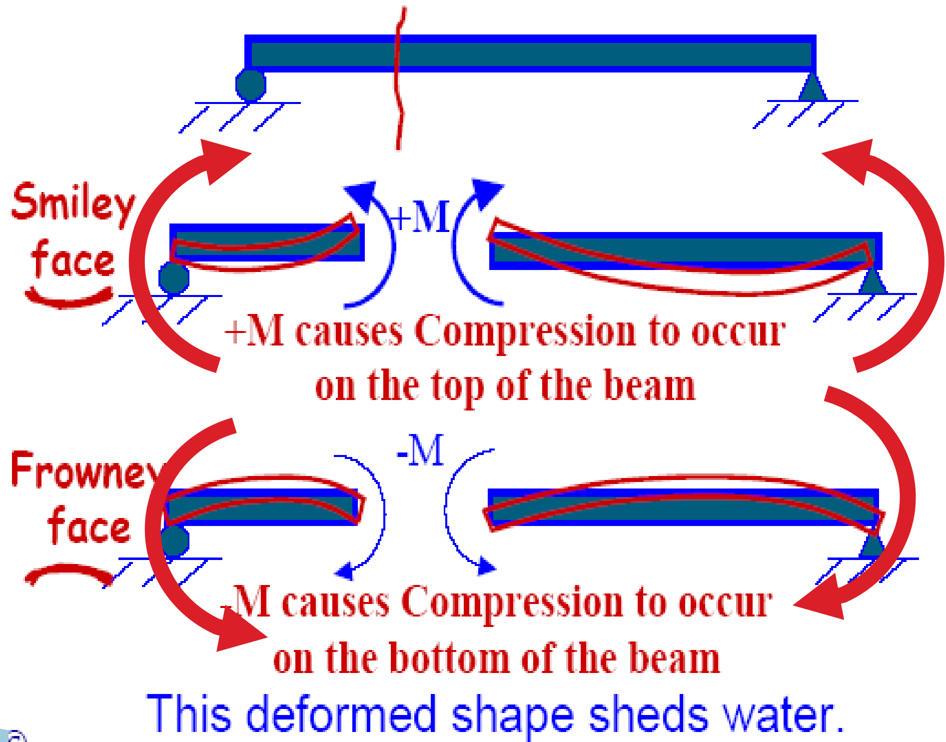
+V makes both FBD's tend to rotate Clock Wise.



Holding Water/the two hands rule

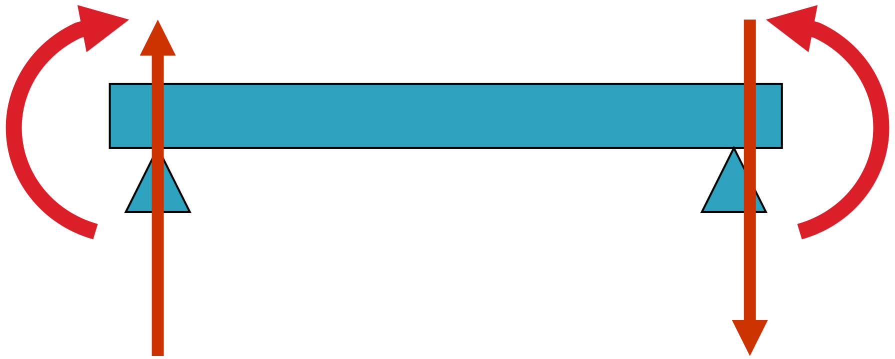
- In North America, if the moment tends to cause the beam to **curve upward** it is **positive**; if the moment tends to cause it to **curve downward** it is **negative.** This is the opposite of the conventions for most of the rest of the world.
- If the upper side is in compression, it is +ve
- If it holds water, it is positive
- If it smiles at you, it is positive
- Two thumbs out (not up) is +ve
    - Counter clockwise on the right
    - Clock wise on the left



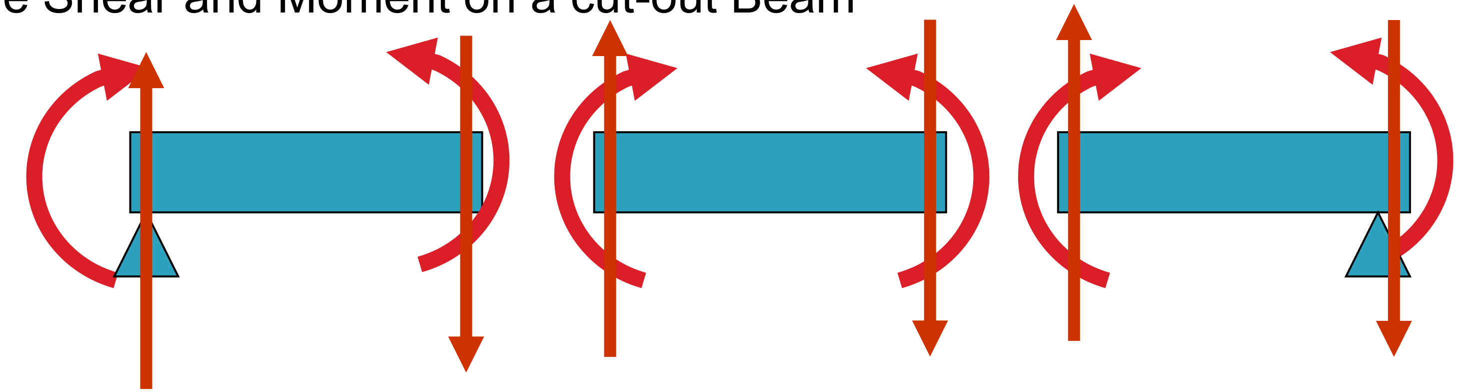


Sign Convention from another angle

Positive Shear and Moment on a whole Beam



Positive Shear and Moment on a cut-out Beam



Rules:

-- Coming from the left or the right makes all the difference

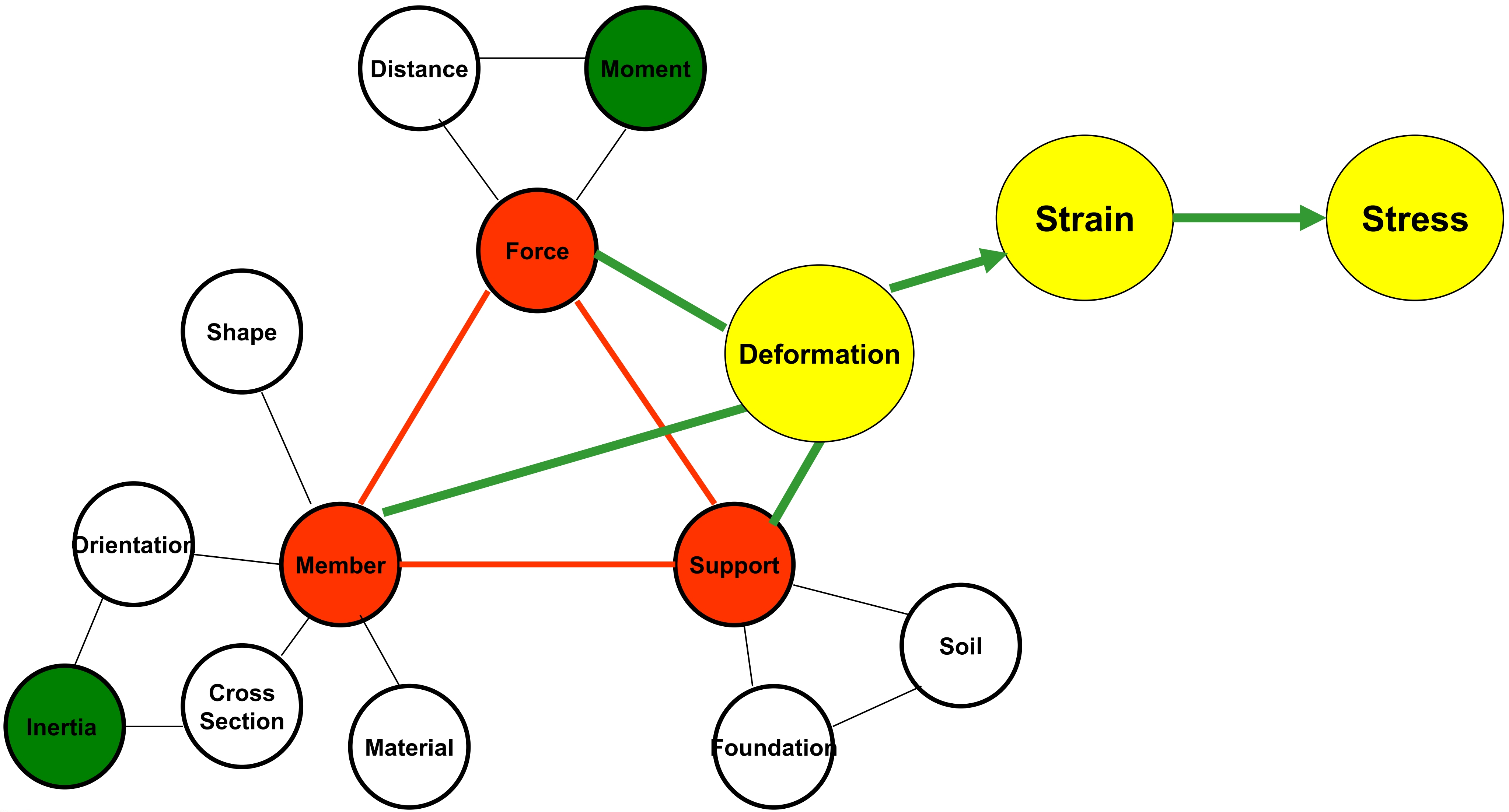
-- For bending Moment : use the "two hands rule"

-- For the Shear force: use the "clockwise rule" or the "up from the left and down from the right rule"

Internal forces in beams

- Internal forces are generated within loaded structural elements.
- These forces are generated within every type of element; if they were not developed, the structure would fail.
- These are known as **Shear**, **Moment**, and **Normal Forces** . The normal force is usually found in columns
- Shear and moment are found in beams and frames
- We will focus on **Shear(V)** and **Moment(M)** in beams

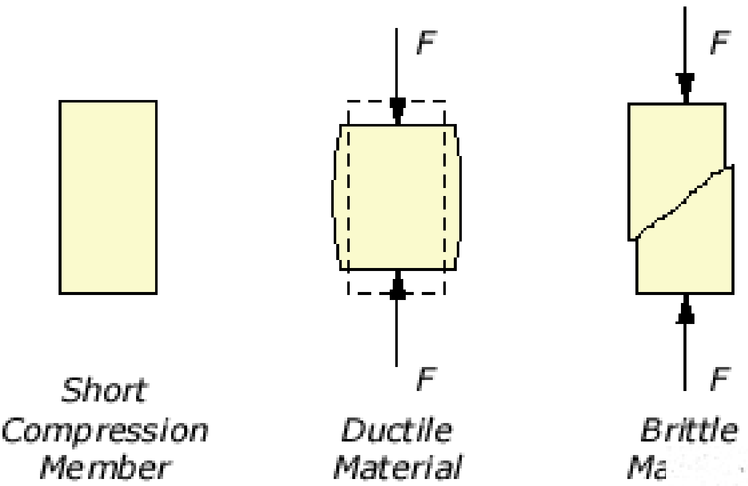
The Big Picture: Where we are

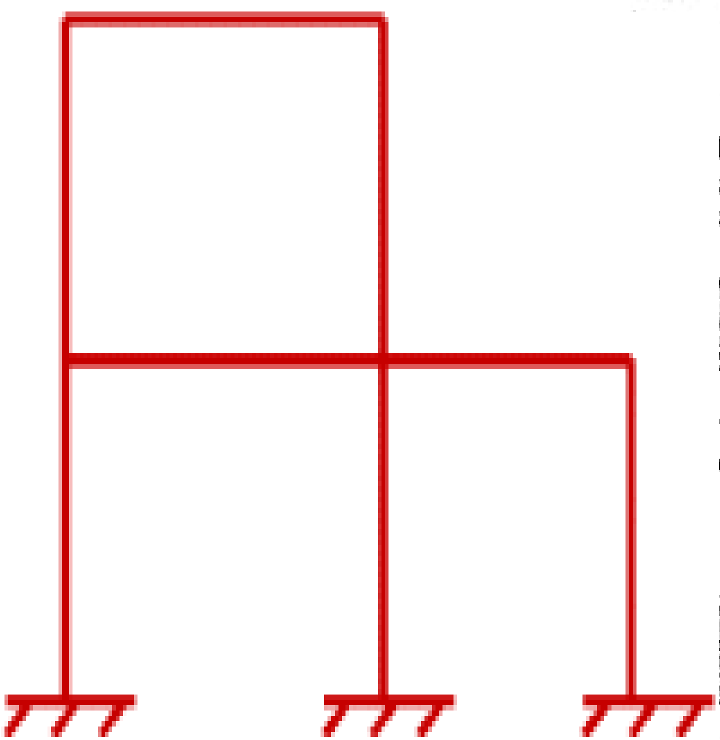


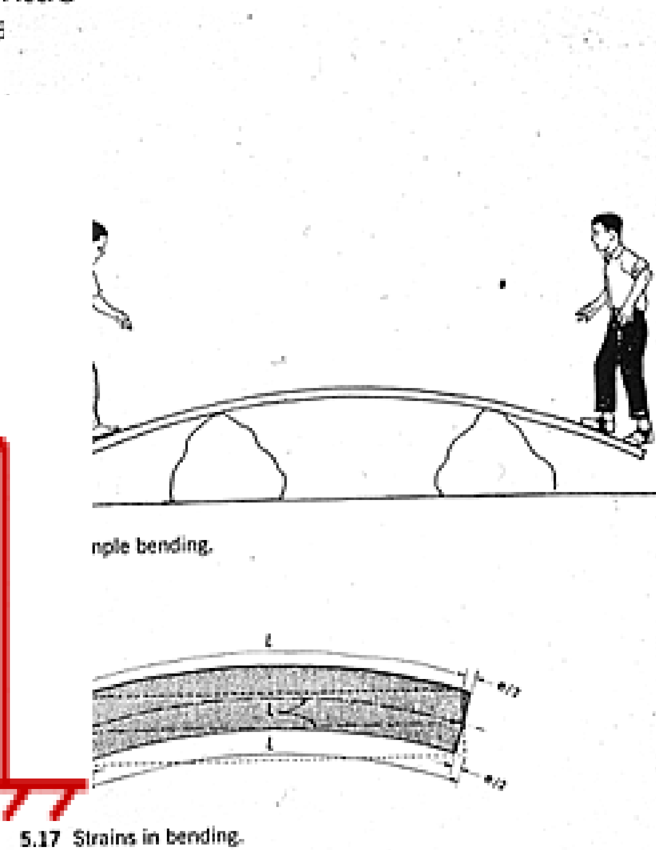
Strain-Stress Flavors

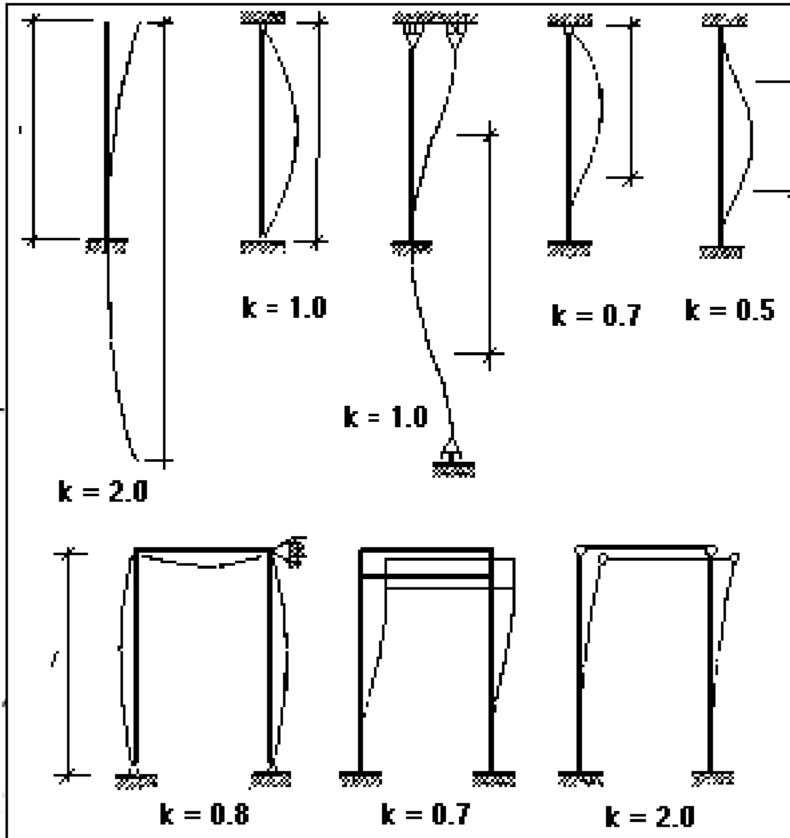
- There are three main types of strain. Consequently, there are three types of Stress
- Axial Strain (load is perpendicular to the surface)
- Shear Strain (load is parallel to the surface)
- Bending Strain (kind of combination of the other two and happens in beams and frames)

Deformation

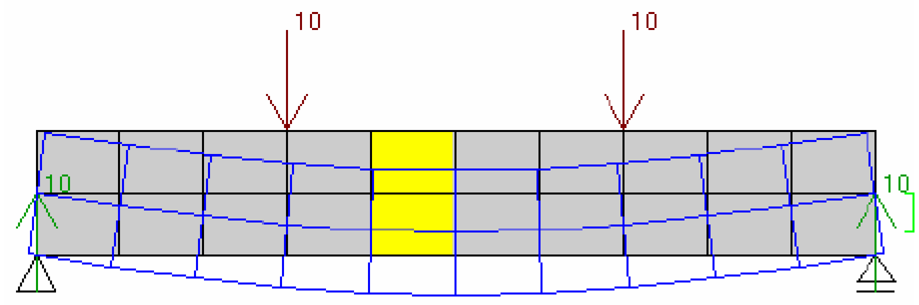


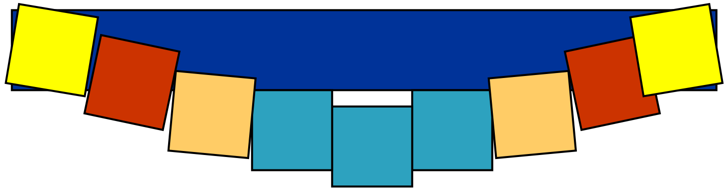


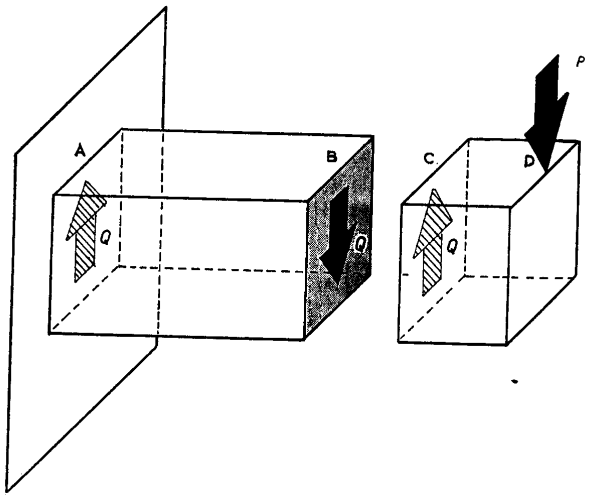


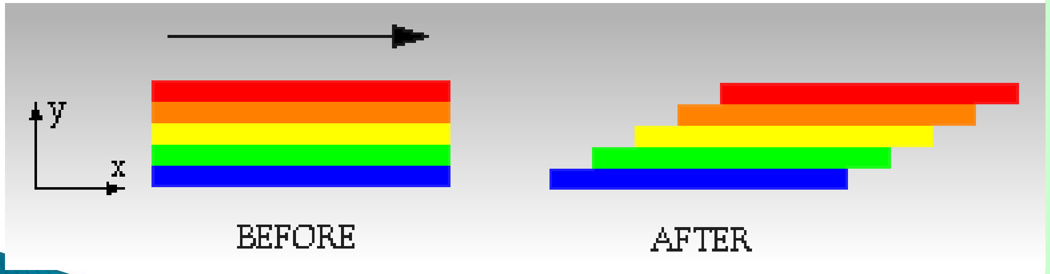


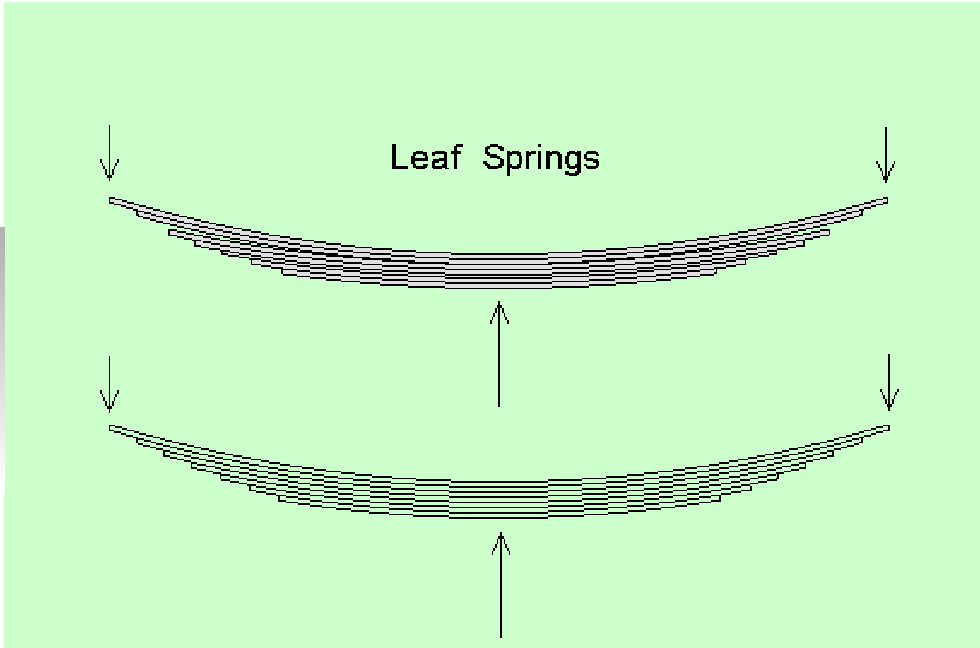
Vertical and Horizontal Shear











How deformation causes all of this

- Forces acting on members (including the reactions from supports) deform these members. As a result of that some of the dimensions of the member change. For example, its overall shape (the beam example), its overall length (the sponge example), the size of its effective cross section (the string example), and the shape of the cross section (the box section example).
- Unless the members are balloons, they resist that, which creates a stress (just like working too much would stress you).
- Each deformation is measured by a parameter called strain.
- The objectives of engineers is to:
  - Find the Stress in each member
  - Make sure the design of each member should assure that it is not overstressed

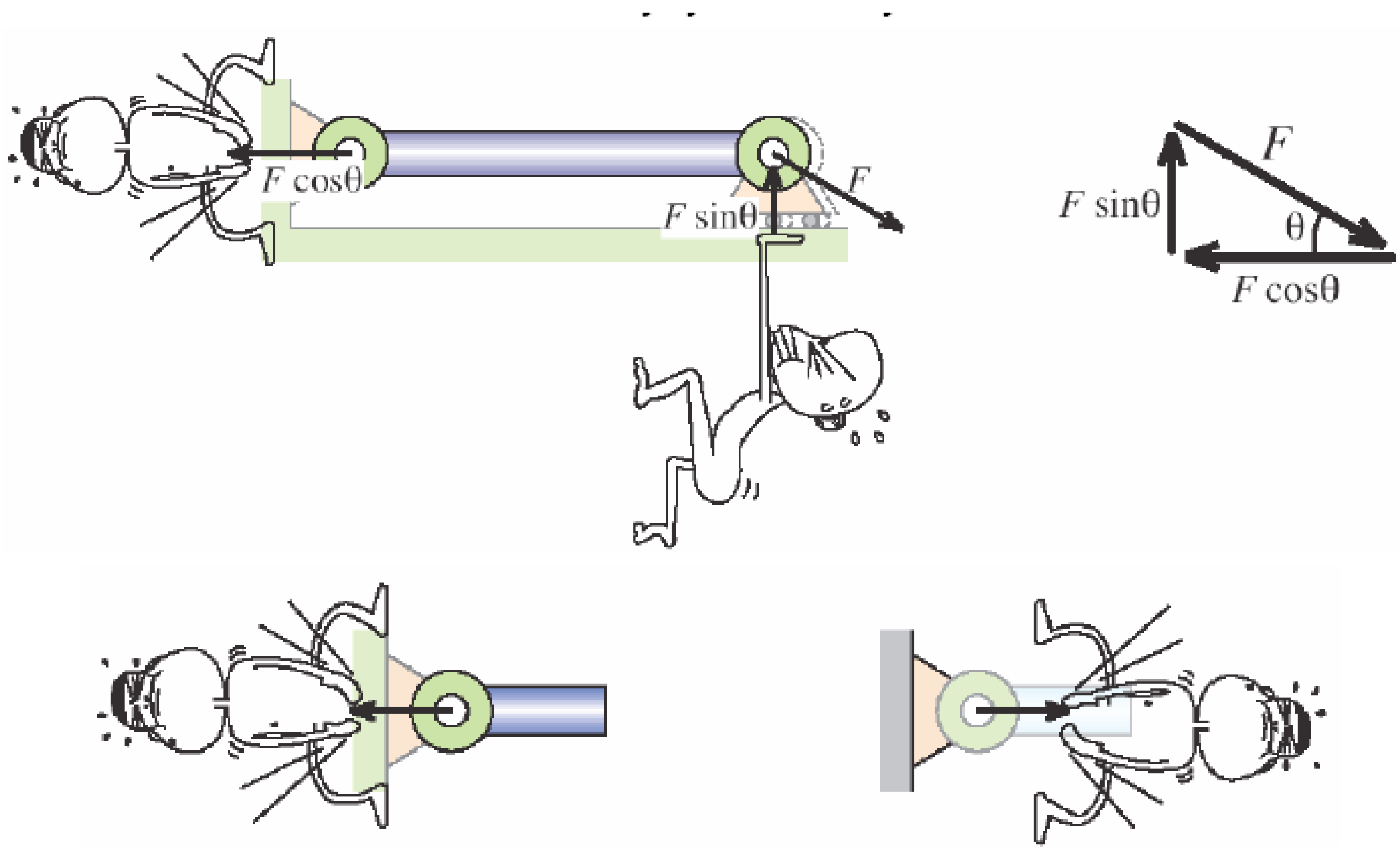
Axial Stress

- Caused by an axial force.
- Can be tension stress or compression stress.
- Symbol used ‘σ’ (sigma)
- Axial stress = Force / Area $\sigma = P/A$
- Units are the same units of pressure (kPa, MPa, etc…)
- This equation tells us that:
  - The greater the applied force the greater the stress.
  - The smaller the area on which the force is applied the greater the stress.

Try to make all stress calculations in MPa

(Forces in N, Area in mm^2)

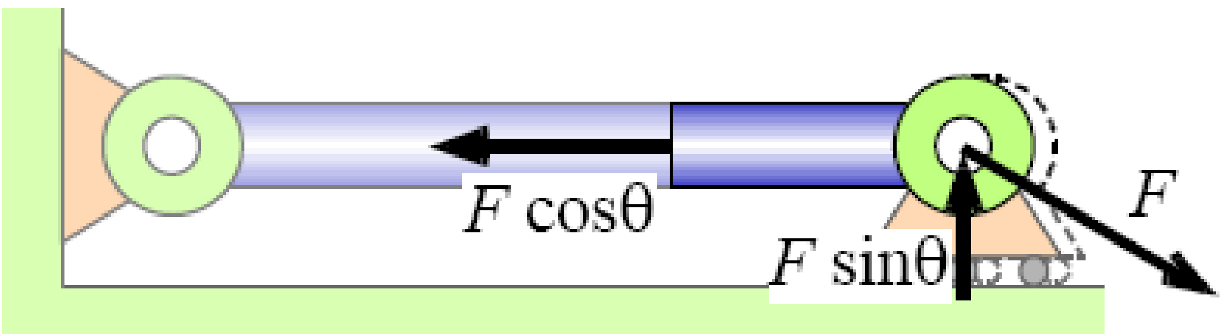
Review: Reactions



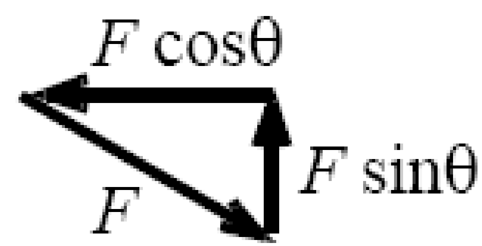
(a) Force from the wall to the rod

(b) Force from the rod to the wall

Cut it loose to see the internal forces

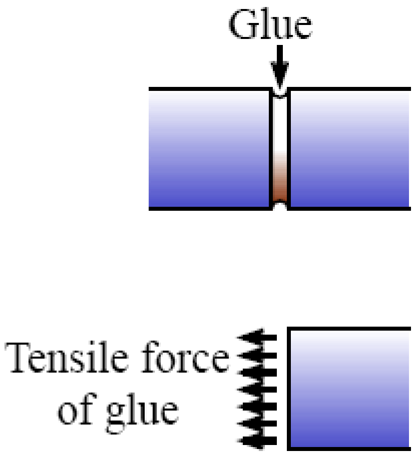


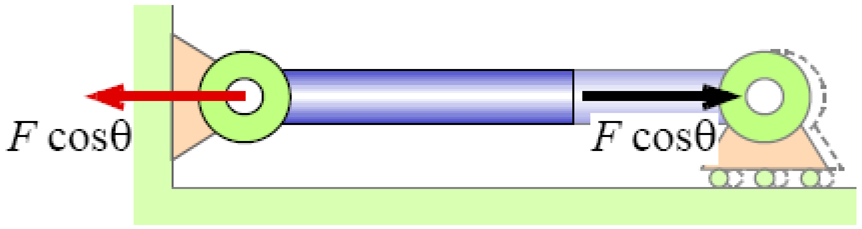
(a) Imaginary cut



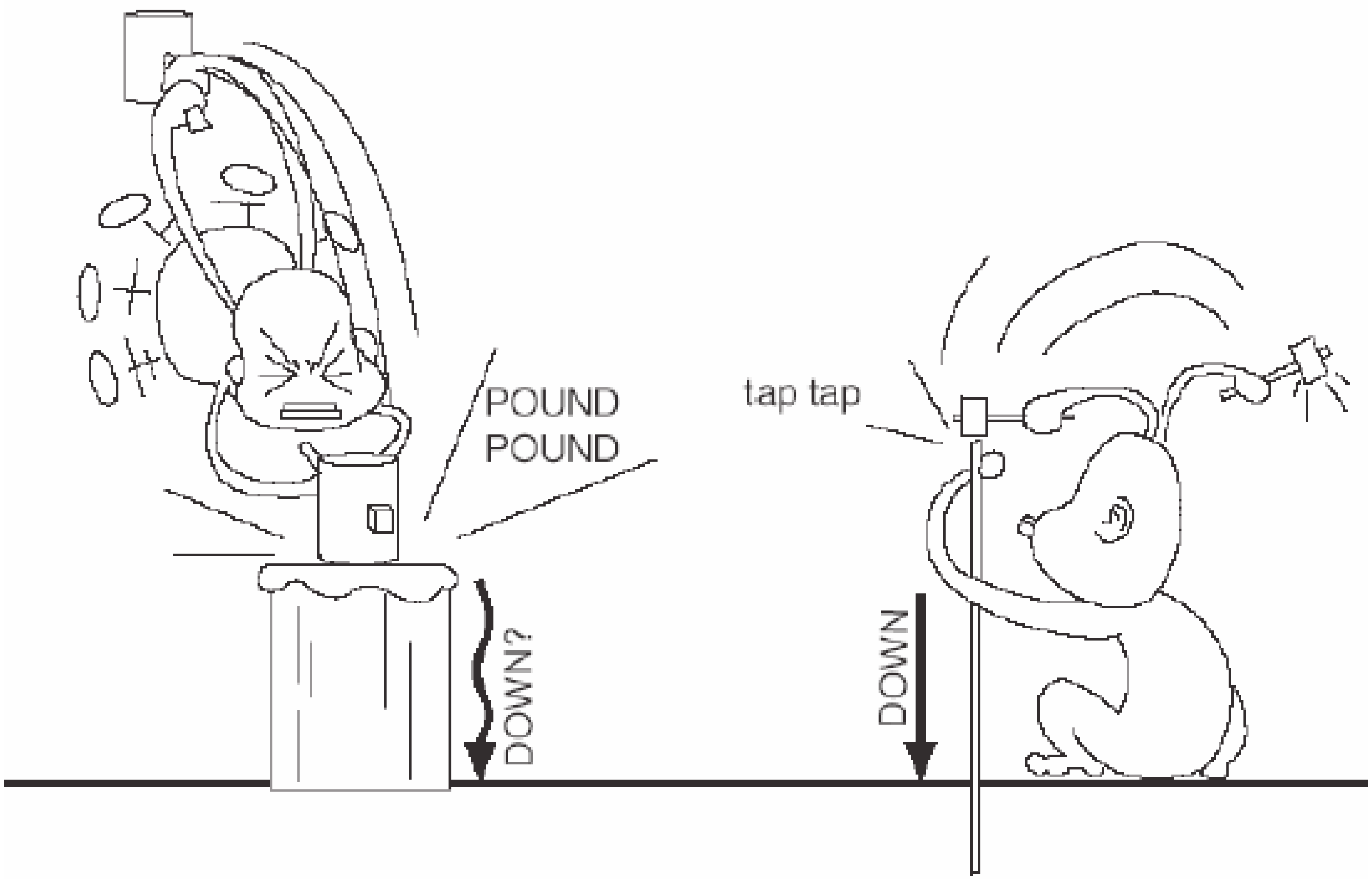
(b) Equilibrium

Holding each other in time of Stress



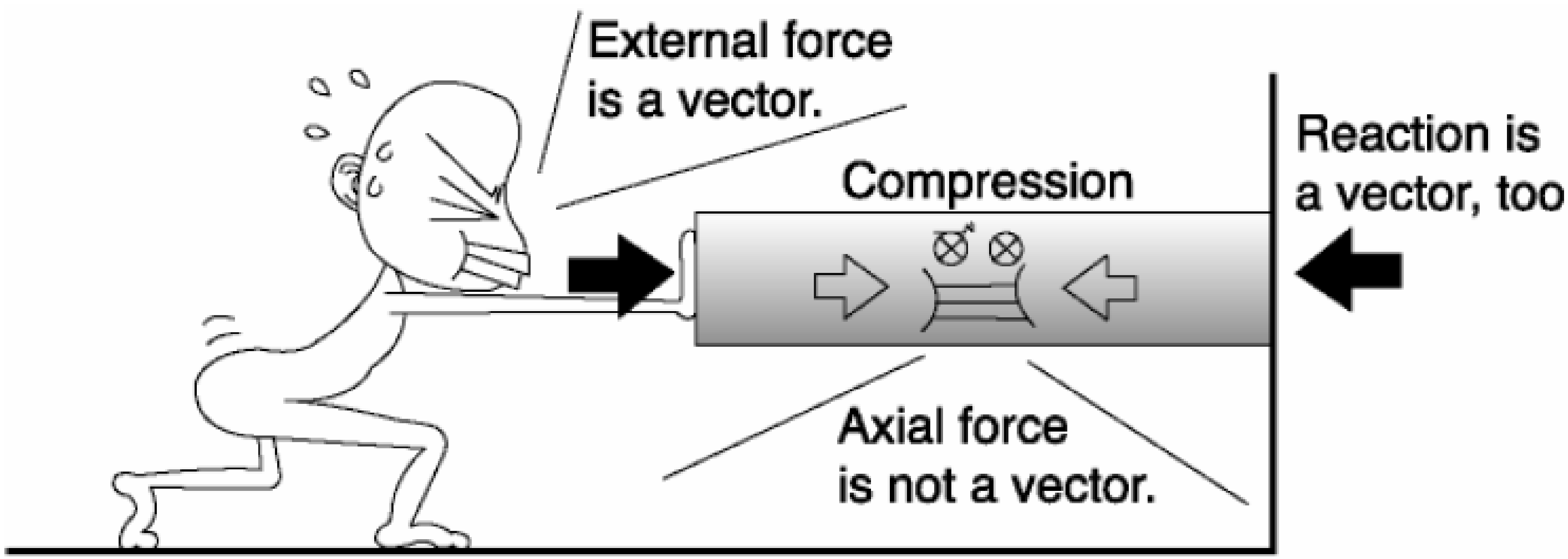


Stress and Area



Stress is small
if cross-section is large

Stress is large
if cross-section is small

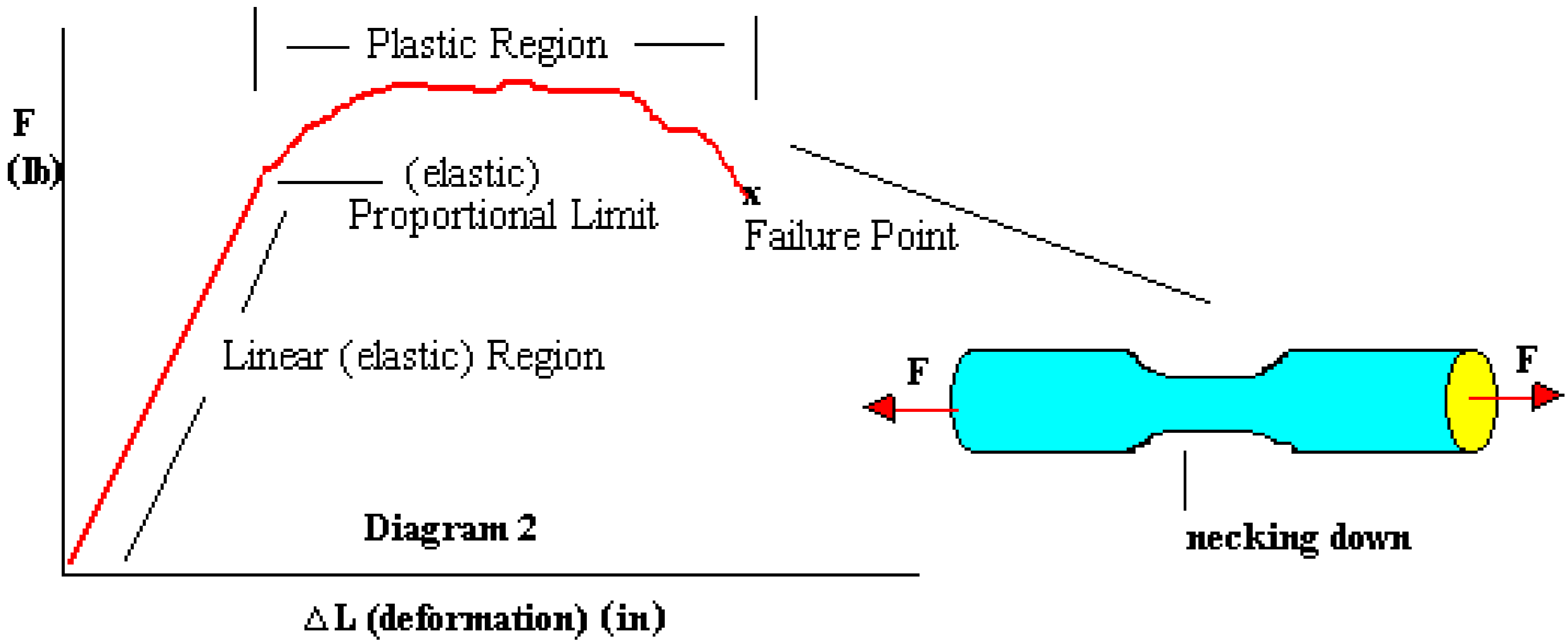

External force
is a vector.
Compression
Reaction is
a vector, too.
Axial force
is not a vector.

Axial Strain

- Stresses cause the material to deform:
  - Tension will cause elongation
  - Compression will cause shrinkage
- Symbol used ‘ε’
- Axial Strain = Change in length / Original length
- $\varepsilon = \Delta L / L$ (no units)

Stress-Strain Relationship

Young's Modulus

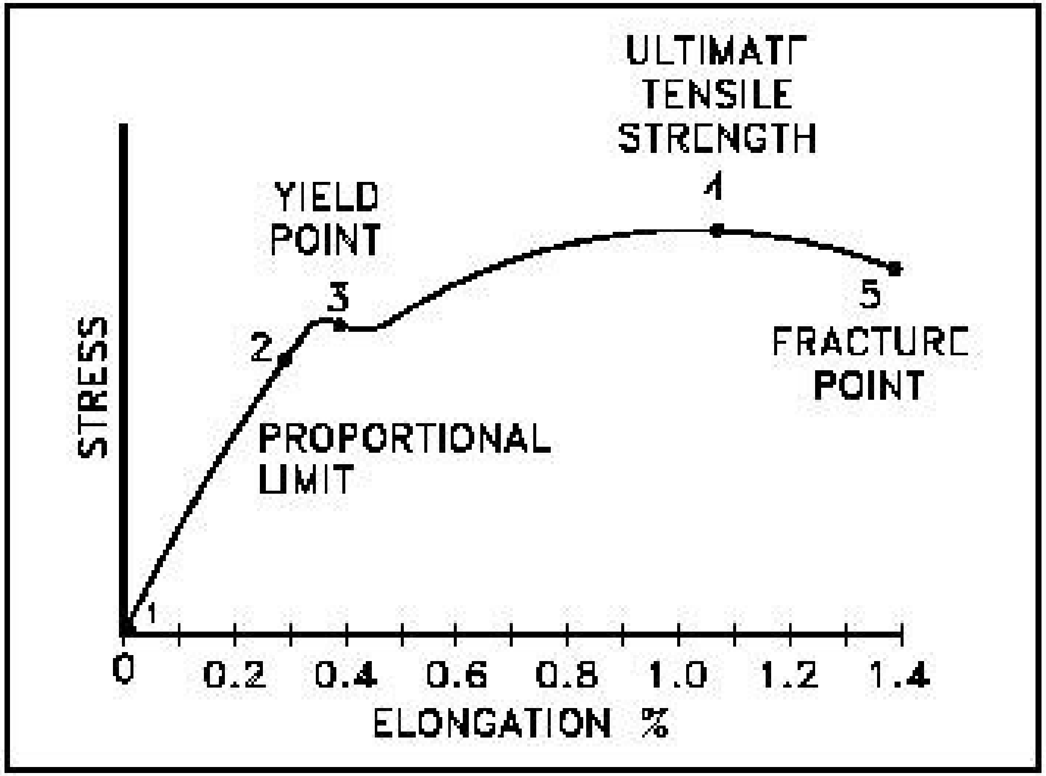


Hooke's Law

- Relates axial stress to axial strain
- It was found that stress is directly to proportional to strain (ie. the more stress applied the greater would be the strain).
- This relation was formalized into:
  - $\sigma = E\,\varepsilon$ E: Modulus of Elasticity
  - The units of E are the same as the units of σ
- The modulus of elasticity is a property of material (each material has a different 'E').
- The modulus of elasticity tells you how stiff a material is. The larger 'E' means a stiffer material.

Stress – Strain Curves

- A plot of stress vs. strain
- Hooke's law is valid from point 1-2
- The slope of line 1-2 is equal to the modulus of elasticity.

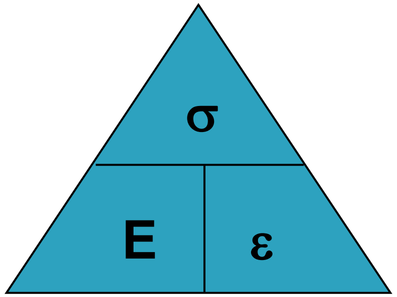


Example on Hooke's law

A Standard channel (C200x21) of 2m length is subject to a tension force of 500kN. Given that the modulus of elasticity for steel E= 200×10^3 MPa

- Calculate the elongation in the steel member

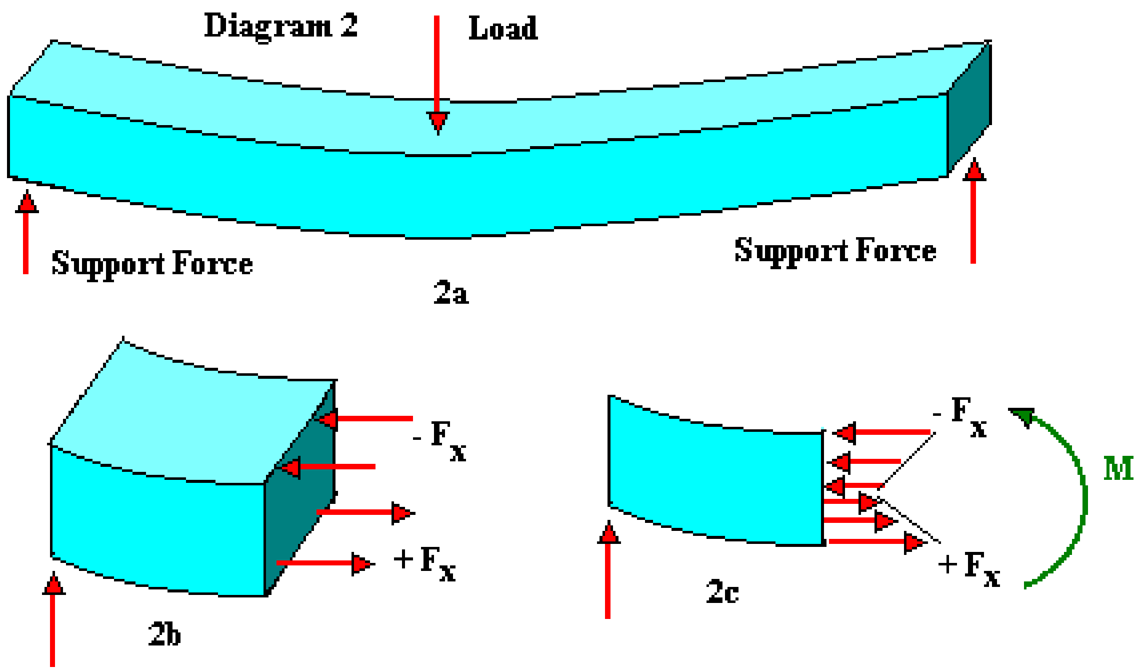
  - From tables: Cross Sectional Area = 2600mm^2
  - $\sigma = (500 \times 1000) / 2600 = 192.31$ MPa (or N/mm^2)
  - $\varepsilon = \sigma / E = 192.31 / 200000 = 9.6154 \times 10^{-4}$
  - $\Delta L = L \times \varepsilon = 2000 \times 9.6154 \times 10-4 = 1.9231$ mm

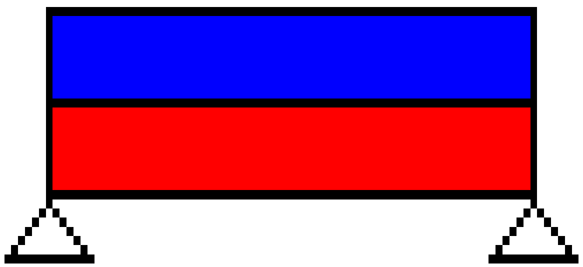


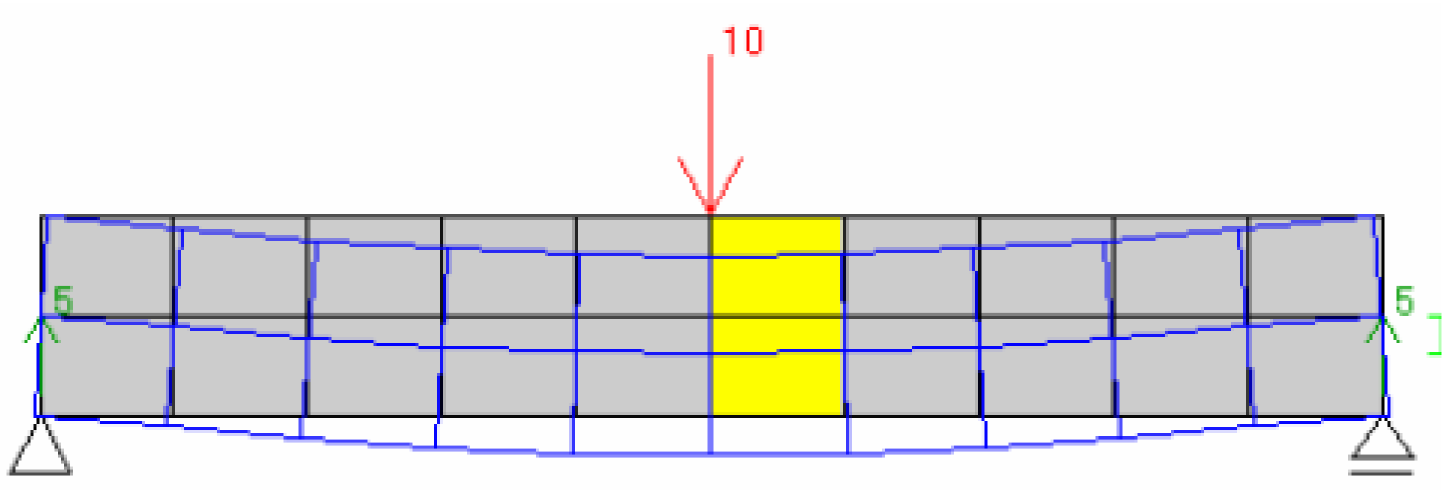
Axial compression stresses

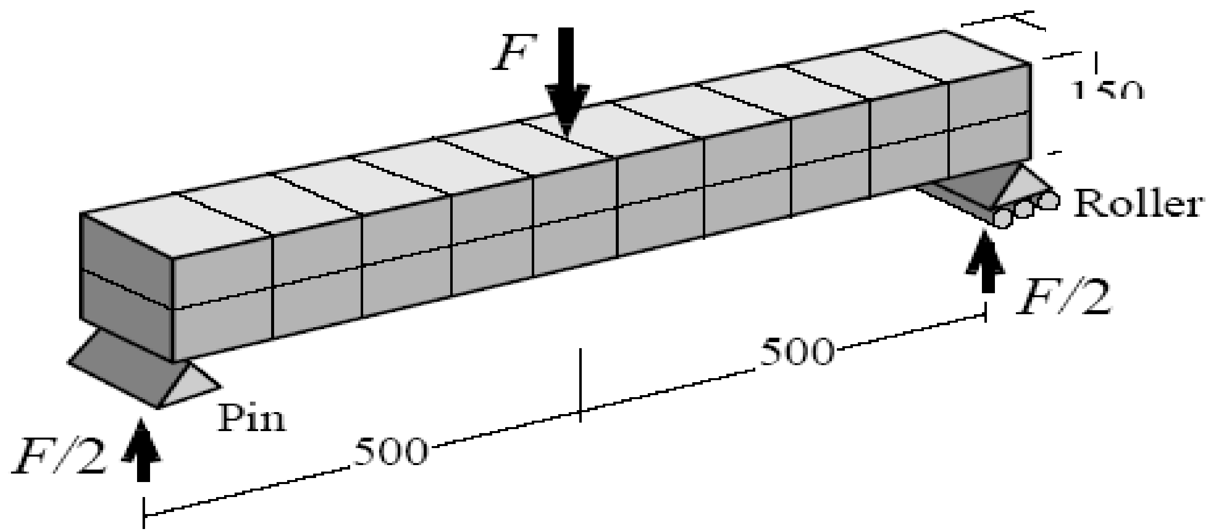
- When a member is in compression, failure may occur due to:
  - Yielding: Compressive stresses exceed the material's compressive strength
  - Buckling: Out-of-plane bending of the member that causes excessive deformations and subsequent failure

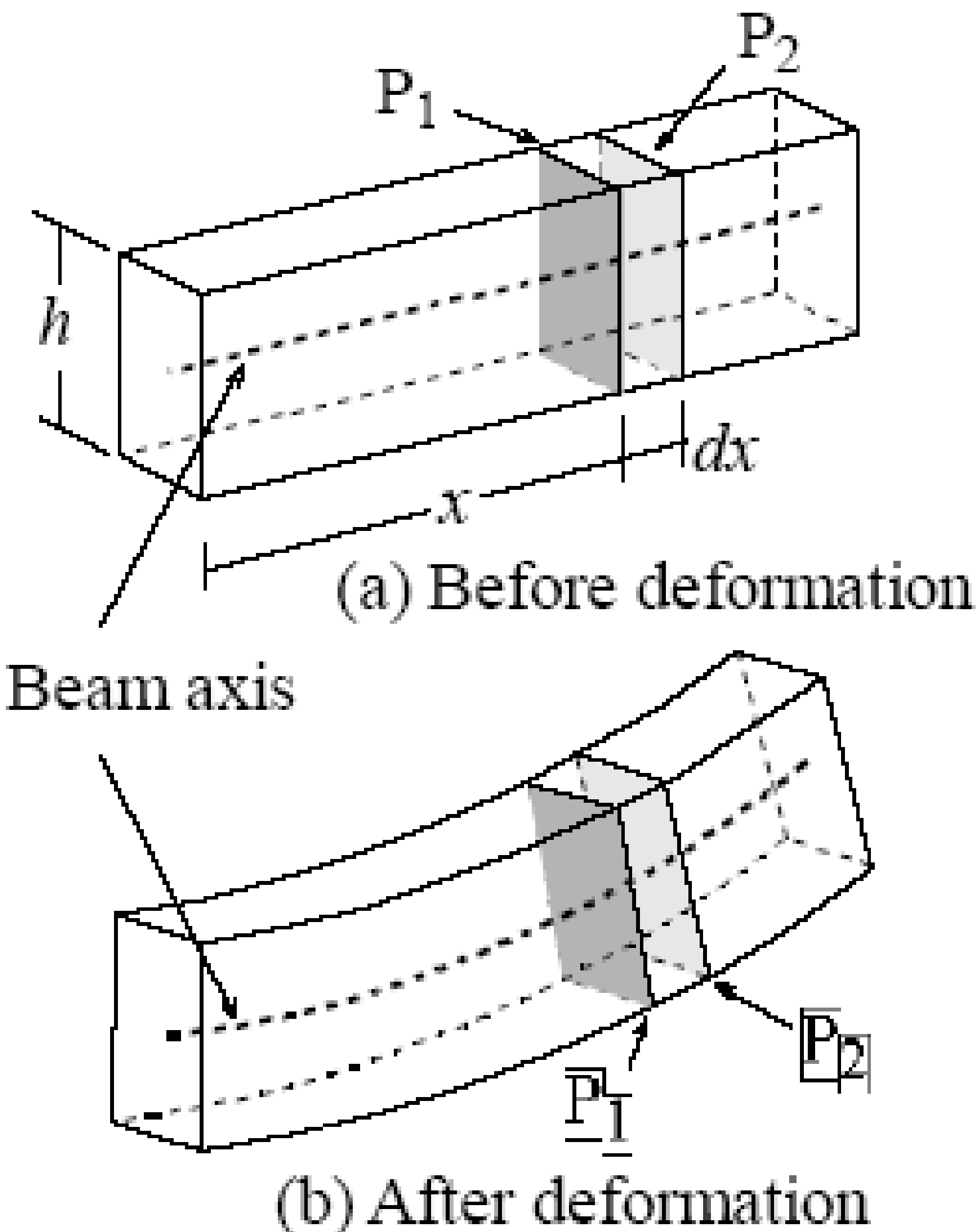
Bending Has it all



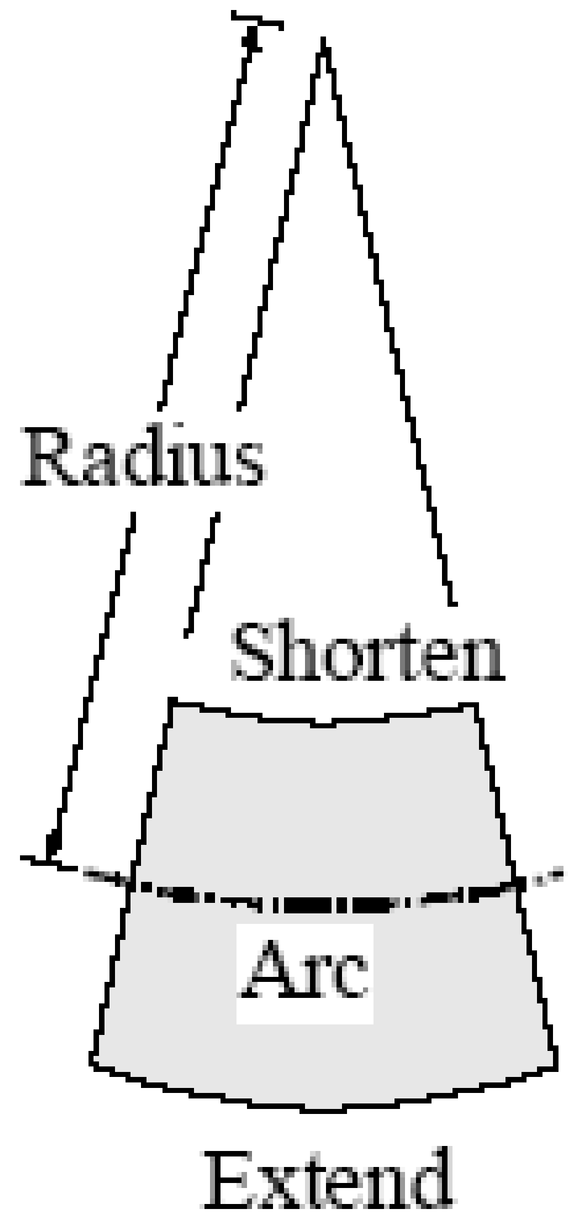



10
5
5

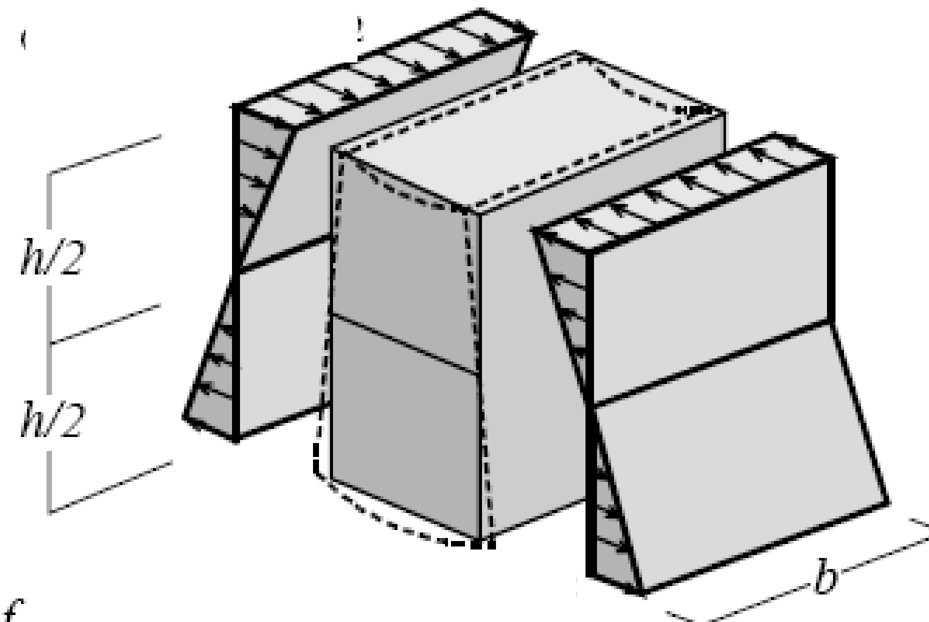

F
Roller
F/2
500
Pin
F/2
500

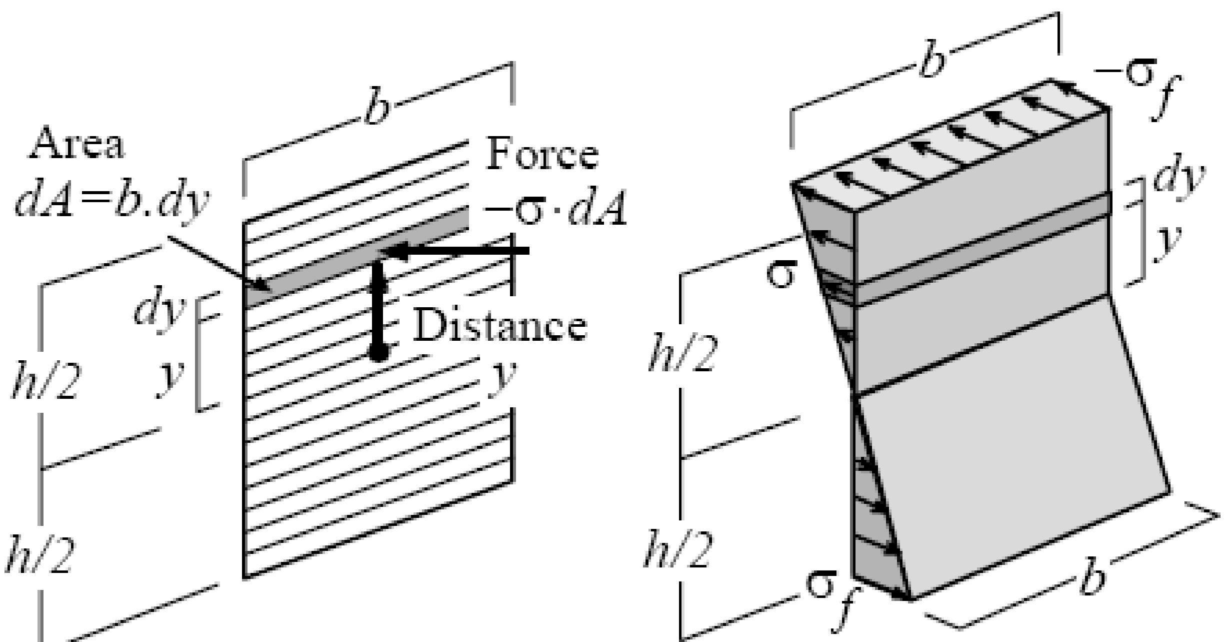


P1
P2
h
dx
x
(a) Before deformation
Beam axis
P1
P2
(b) After deformation



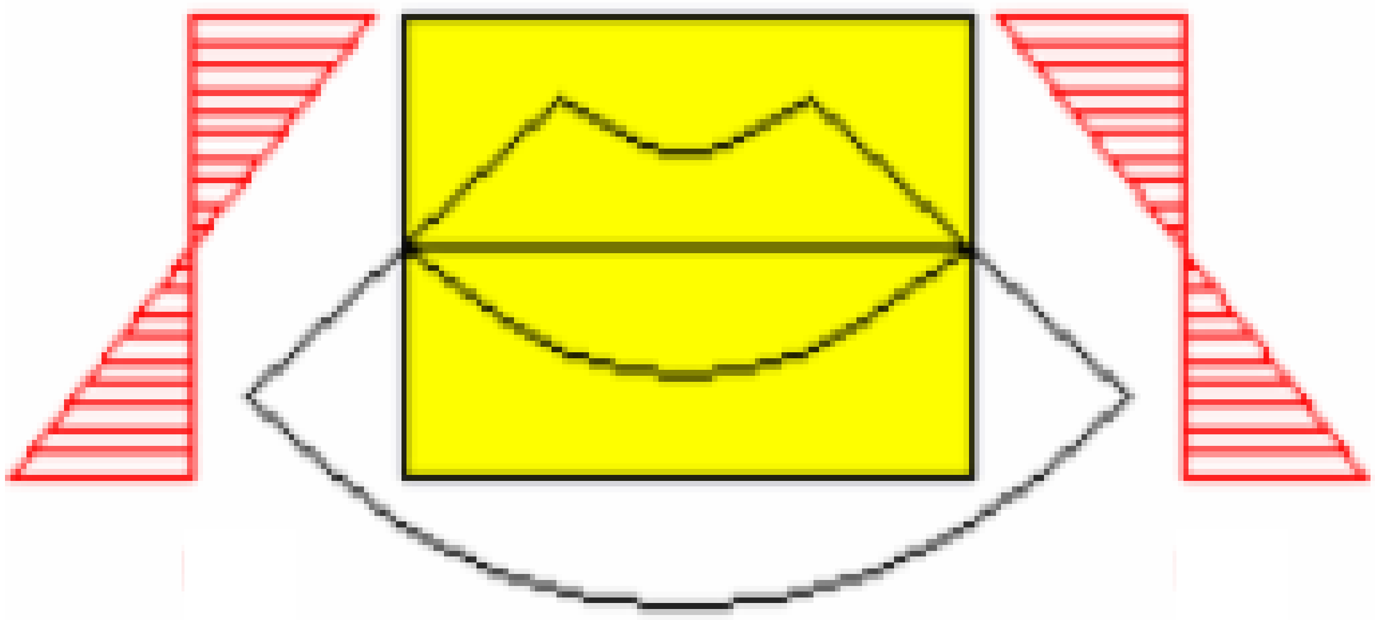
Radius
Shorten
Arc
Extend





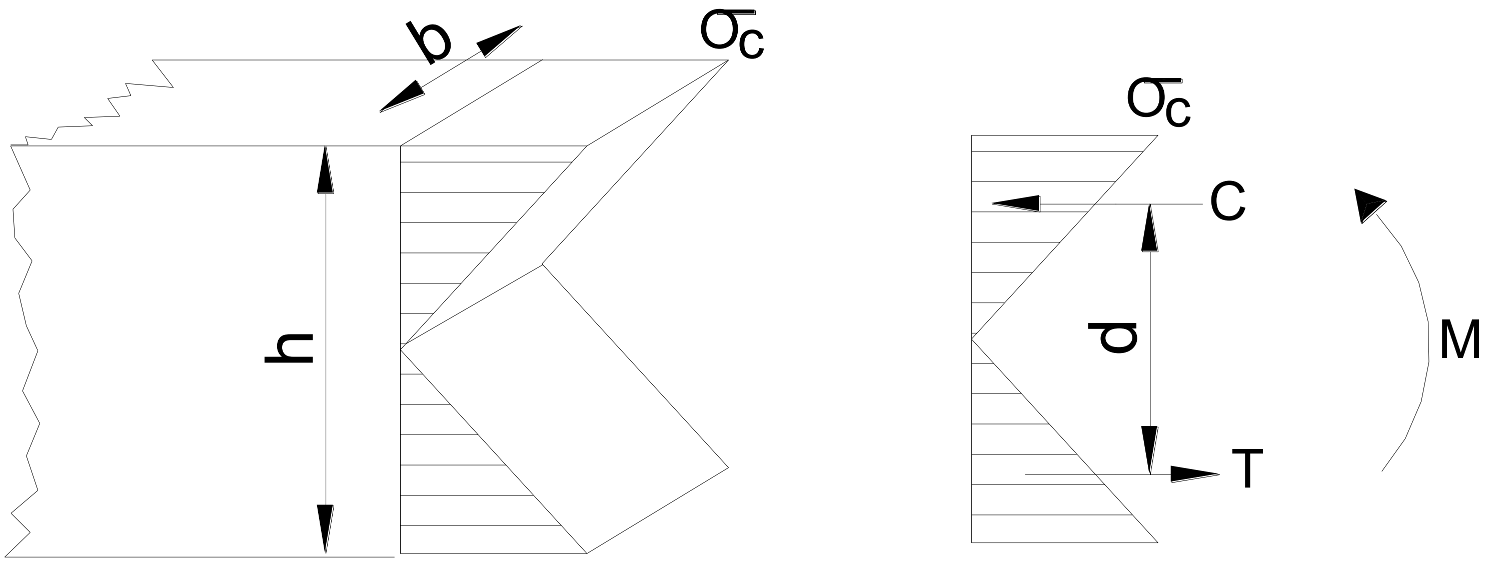
(a) Partitioned section (b) Stress distribution

Just For Fun

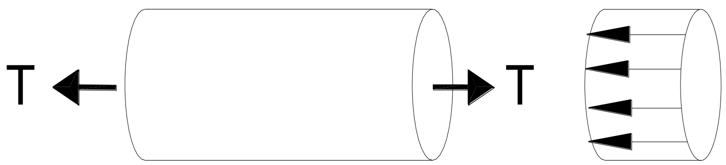


Bending stresses

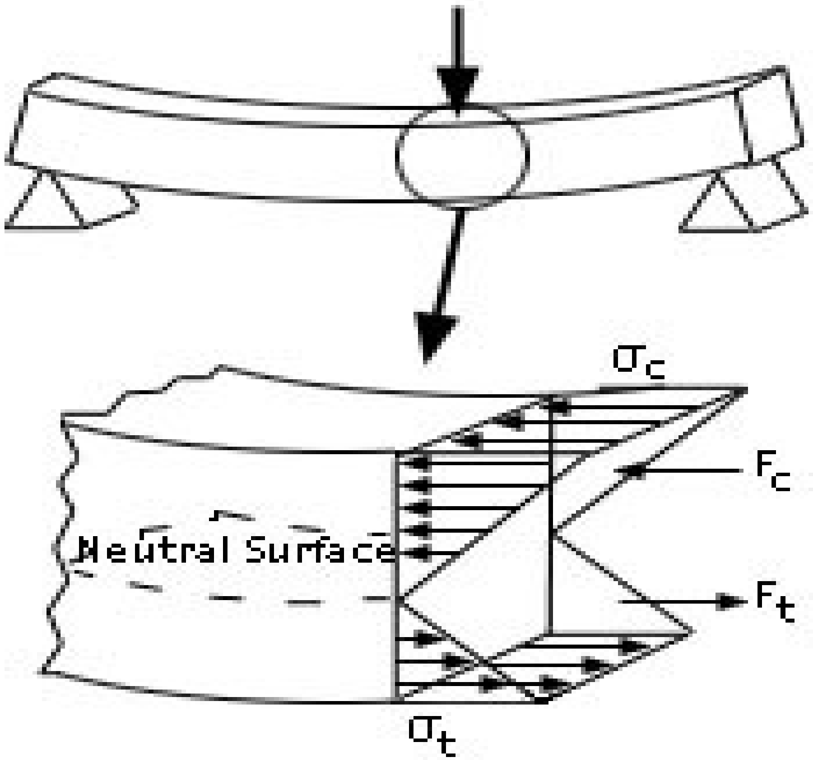
- The stresses due to bending are distributed over the cross section as shown.
- The centroidal axis of the cross section has zero bending stress
- Stresses increase uniformly. The maximum stress value is found in the top or bottom fibers of the section



Bending stresses



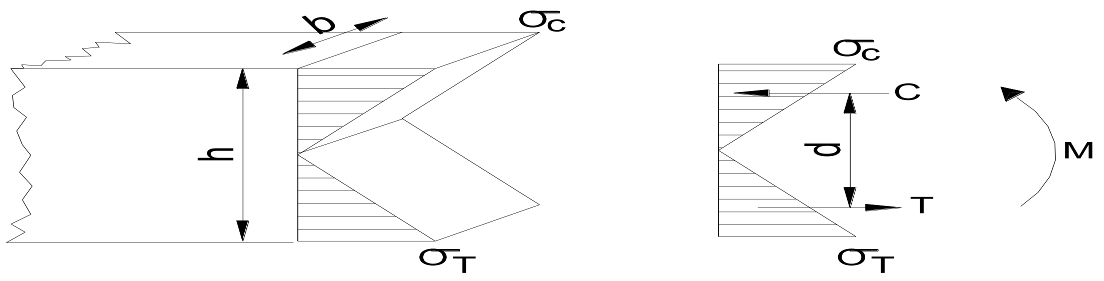
Axial Stress: All points in the cross section have the same value of stress



Bending Stress: Point in the cross section have a different value of stress

Bending stresses

- The value of the compression force is equal to the volume of the compression stress block
  - $C = (\sigma_c)\ (b)\ (h/2)$
- The value of the tension force is equal to the volume of the tension stress block
  - $T = (\sigma_T)\ (b)\ (h/2)$
- Always T = C
- The internal moment causing these stresses on this section M = (C) (d)



The Theory

$$M = \sum_{y=0}^{y\max} F_x \cdot y$$

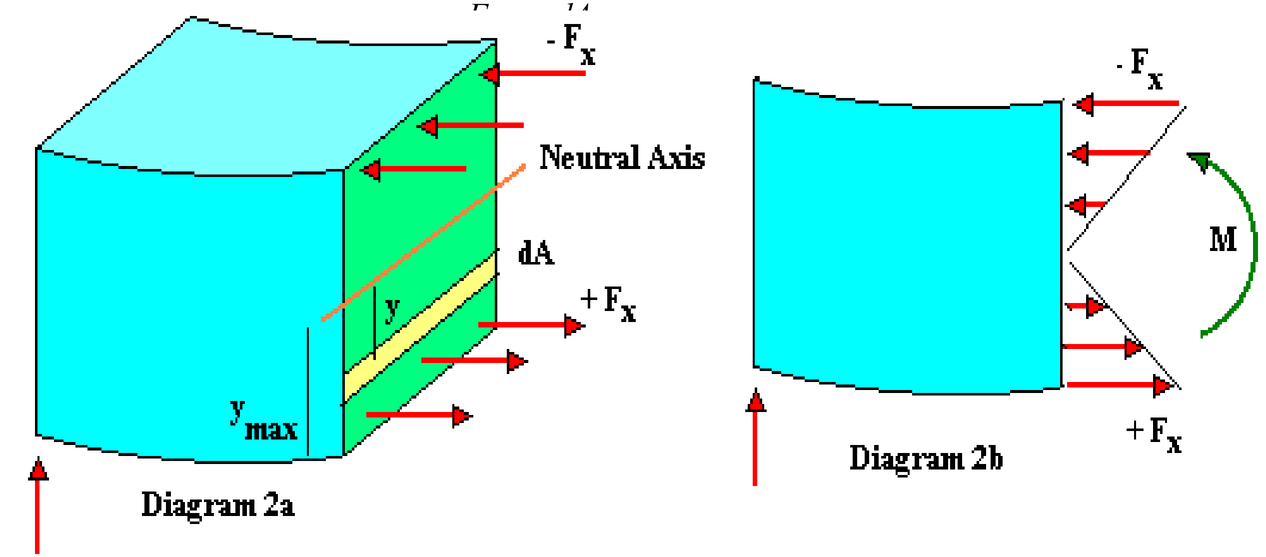


Diagram 2a

Diagram 2b

Bending stresses

- The value of stress at any point ‘a’ on the cross section
  - $\sigma_a = (M)\,(y_a) / I$
  - Where M: Internal moment at cross section
    y_a: Distance from centroidal axis to point ‘a’
    I: Moment of inertia of section about axis which bending occurs

  In this case $I = bh^3 / 12$

When using this Eqn. take care of your units

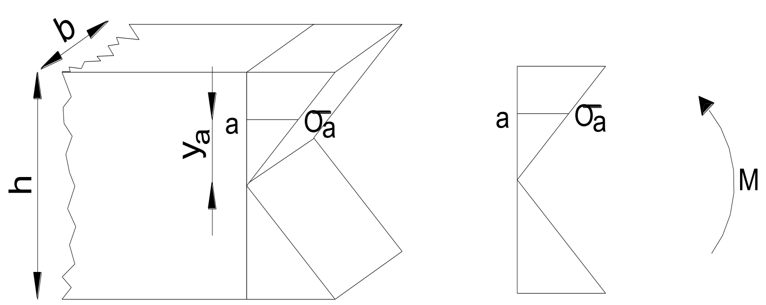
<u>Best use:</u>

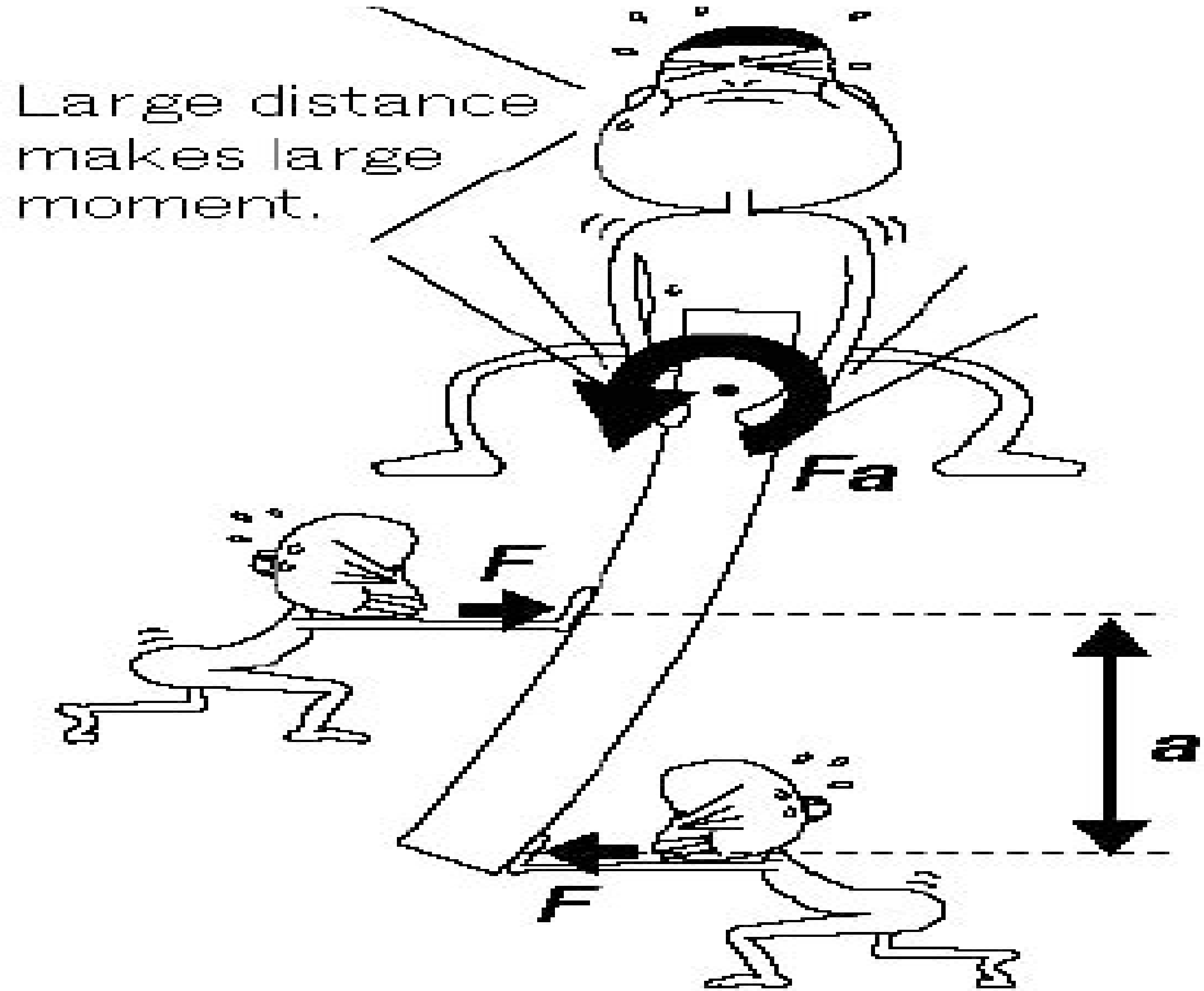
M: N.mm

y: mm

I: mm4

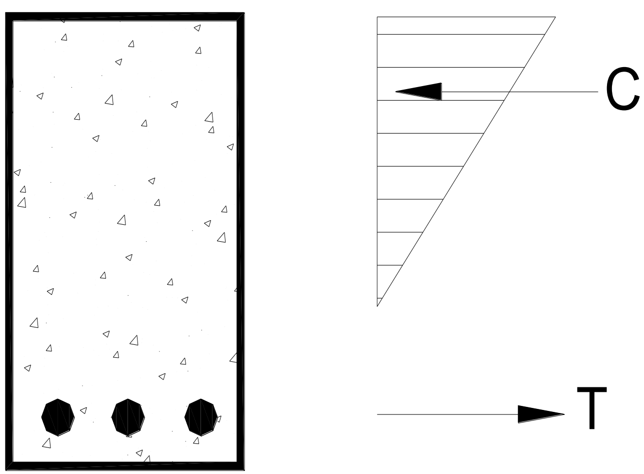
σ: MPa

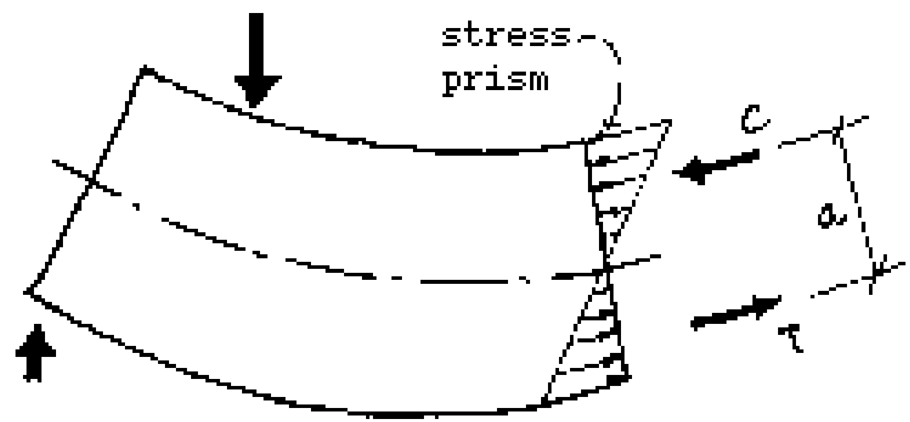



Large distance
makes large
moment.
Fa
F
a
F

Bending stresses: Concrete Beams

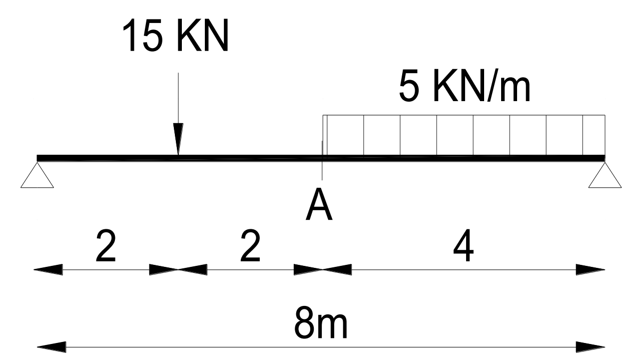
- Concrete beams usually have steel reinforcement where the fibers are subject to tension.
  - Concrete is very strong in compression, but very weak in tension
  - Steel is strong in both tension and compression

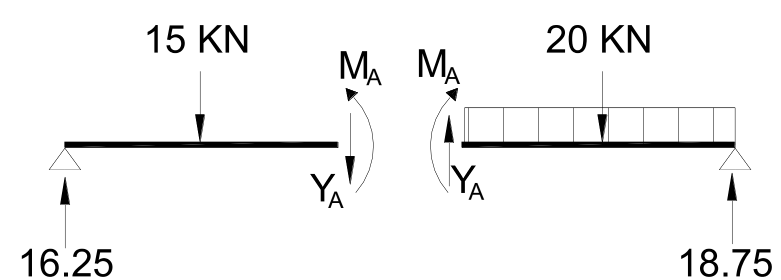




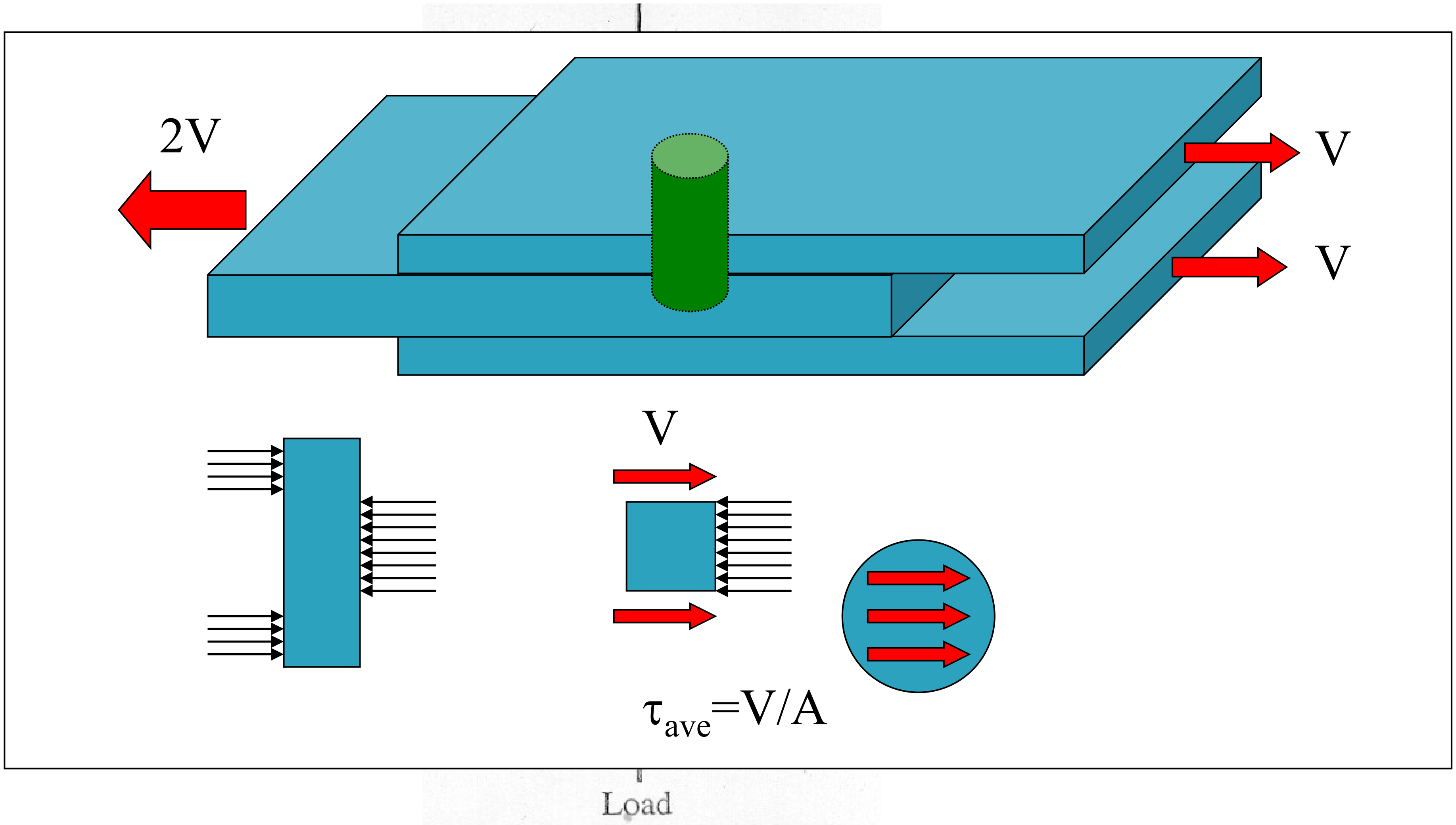
Internal forces in beams

- Internal forces at a specific point in a beam can be calculated by cutting the beam at that point
- Cutting the beam at point A will expose the internal forces at A.
- Generally there are 3 internal actions (2Forces + 1 Moment).
- In beams where all forces are vertical we only have 2 internal actions.
  - Vertical force → Shear
  - Moment → Bending moment





Seeing shear



What are Shear & Moment Diagrams?

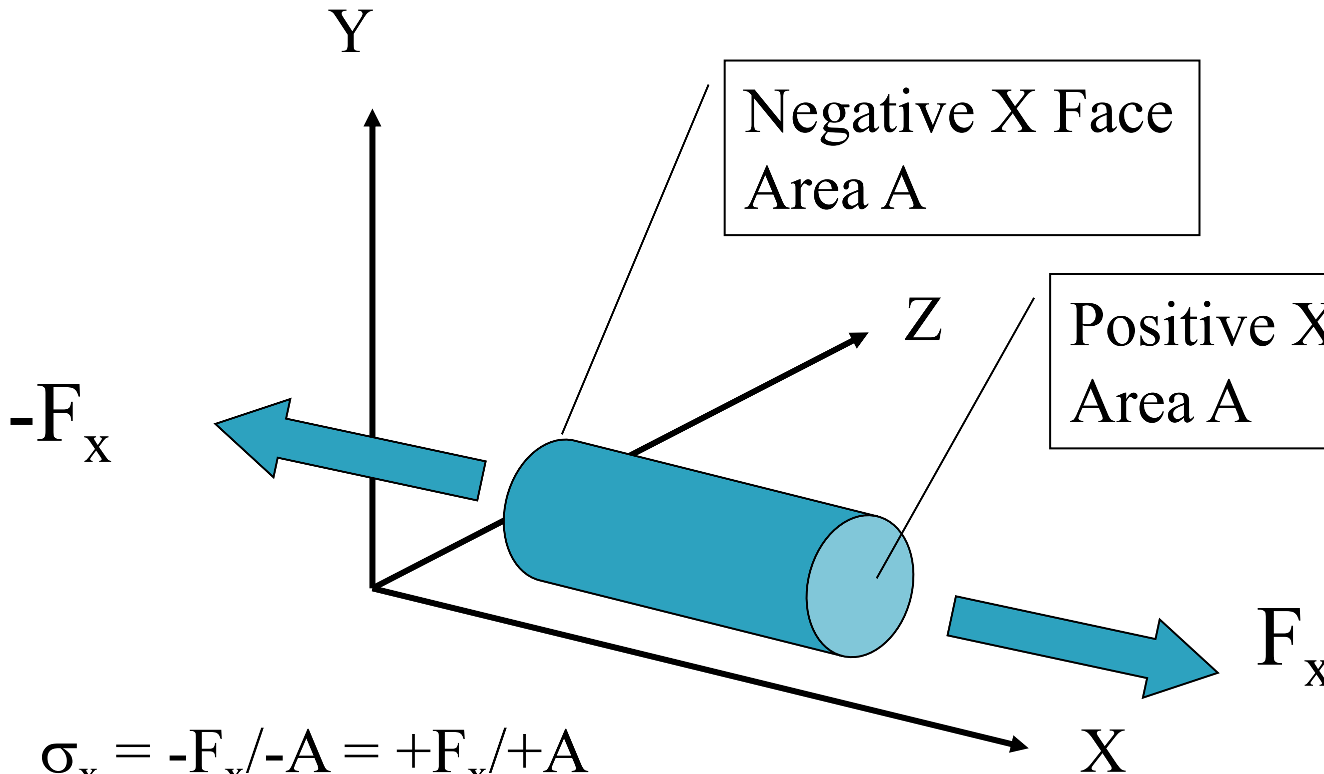
- They are diagrams that illustrate the value of the internal forces (shear / moment) that occur along a structure.

- The shear and moment diagrams are very important- In addition to showing V and M:
  - The shear diagram tells you if you have calculated the forces on the structure correctly so that: $\boxed{\sum \mathbf{F_y} = 0}$
  - The moment diagram verifies that the calculated forces satisfy: $\boxed{\sum \mathbf{M} = 0}$
  - If the diagrams do not close, i.e. start and end at zero, then equilibrium is not satisfied and you know there is an error

How to draw a Shear diagram

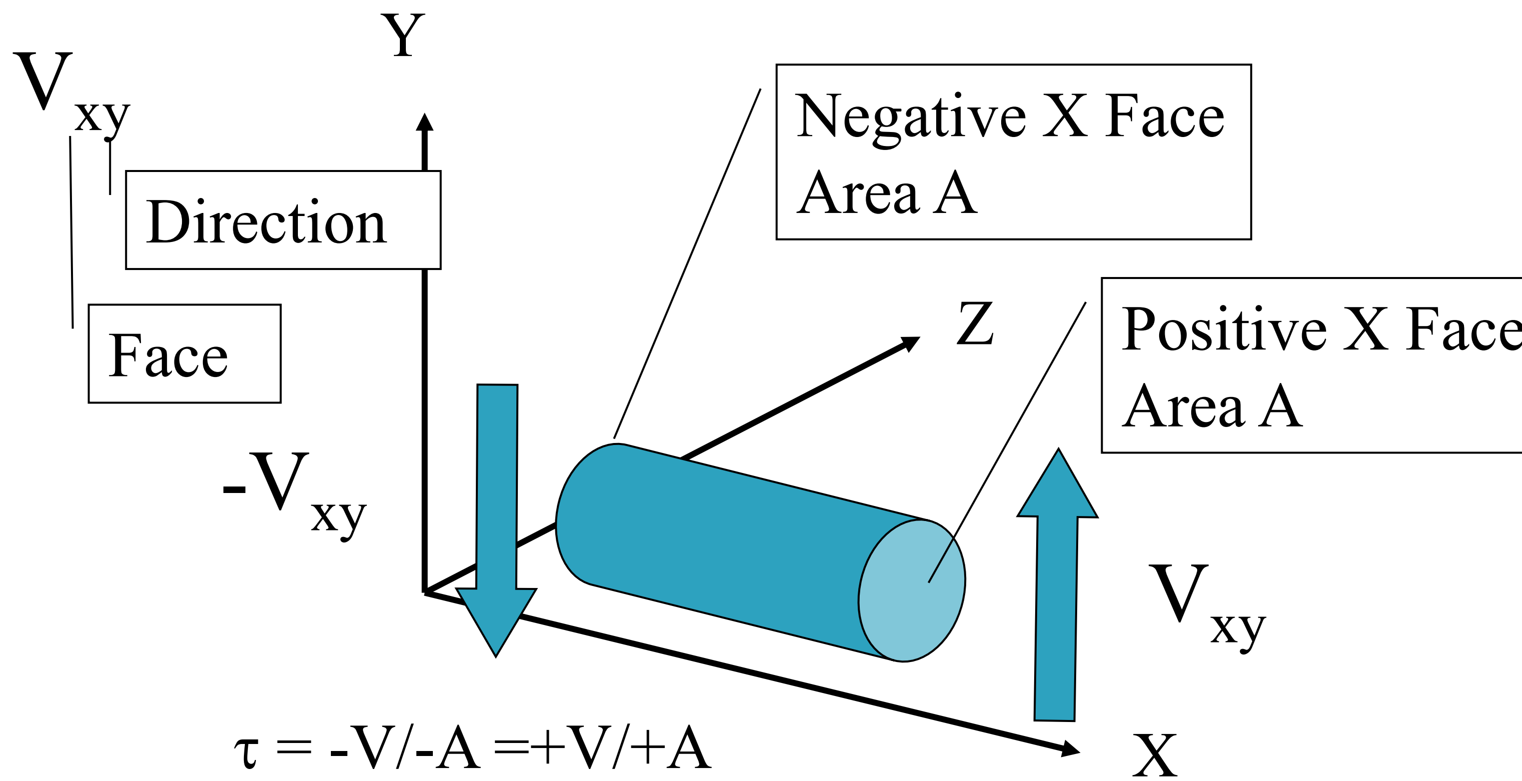
- The shear diagram is the graphic representation of the shear force at successive points along the beam. Upward acting forces are assumed positive and downward forces negative.
- The shear force (V) at any point is equal to the algebraic sum of the external loads and reactions to the left of that point.
- Since the entire beam must be in equilibrium (the sum of V = 0), the shear diagram **must** close to zero at the right end.
- Abrupt changes in loading cause abrupt changes in the slope of the shear curve. Concentrated loads produce vertical lines (a jump) in the shear curve

Axial Stress Signs (A new Trick)



$\sigma_x = -F_x/-A = +F_x/+A$

The same Trick applied to Shear



$\tau = -V/-A = +V/+A$

How to draw a Shear diagram

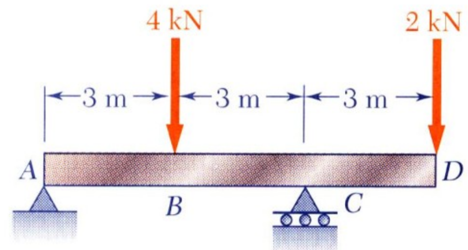
- Consider the type of loading along the beam's length in order to determine the shape of the curve:
  - If there is no change in the load along the length under consideration, the shear curve is a straight horizontal line (or a curve of zero slope). The slope at any point is defined as the tangent to the curve at that point.
  - If a load exists, but does not change in magnitude over the length under consideration (uniformly distributed), the slope of the shear curve is constant and non-horizontal.
  - If a load exists, and increases in magnitude over successive increments, the slope of the shear curve is positive (approaches the vertical); if the magnitude decreases, the slope of the shear curve is negative (approaches the horizontal).

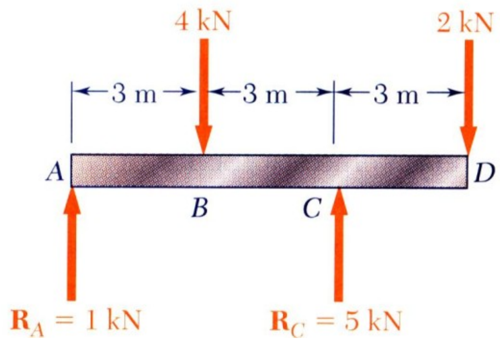
How to draw a Bending Moment diagram

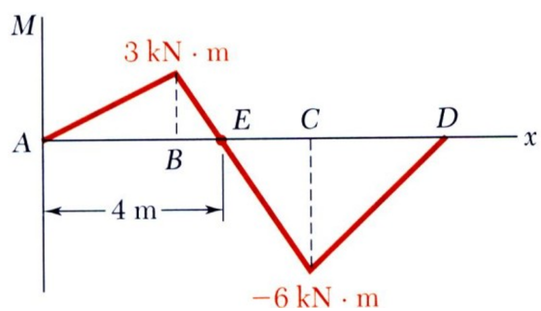
- The moment diagram is the graphical representation of the magnitude of the bending moment at successive points along the beam.
- The bending moment for the moment diagram (M) at any point equals the sum of moments of the forces to the left about that point.
- Since the entire beam is in equilibrium (Sum of M=0), the bending moment diagram must close to zero at the right side.

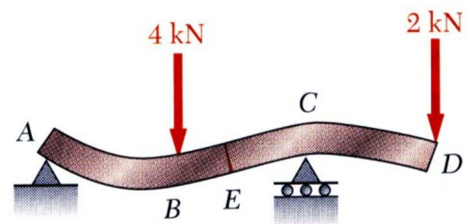
How to draw a Bending Moment diagram

- Consider the values of the SFD:
  - If the magnitudes of successive shear ordinates are constant, the moment curve has a constant slope at that increment.
  - If the magnitudes of successive shear ordinates increase, the slope of the moment curve is positive (it approaches the vertical).
  - If the magnitudes of successive shear ordinates decreases, the slope of the moment curve is negative (it approaches the horizontal).
  - Abrupt changes in the shear diagram will produce changes in the shape of the moment curve. Concentrated moments produce vertical lines in the moment curve.



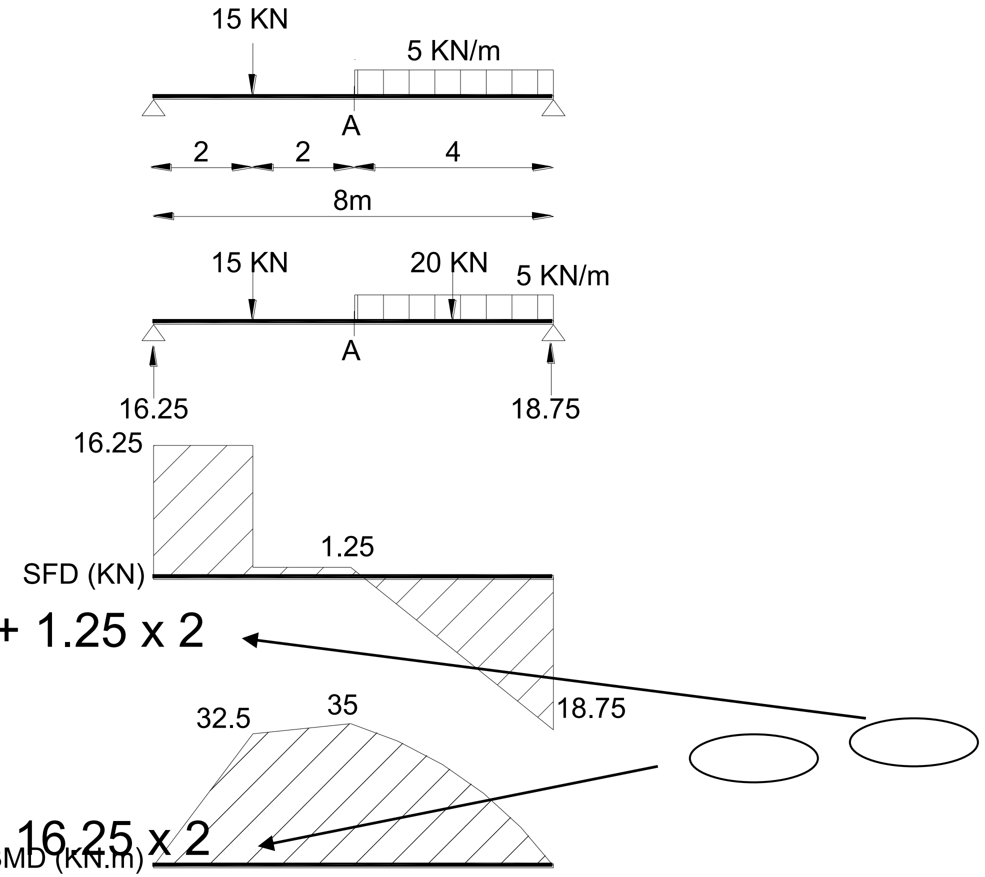






Sample Diagrams

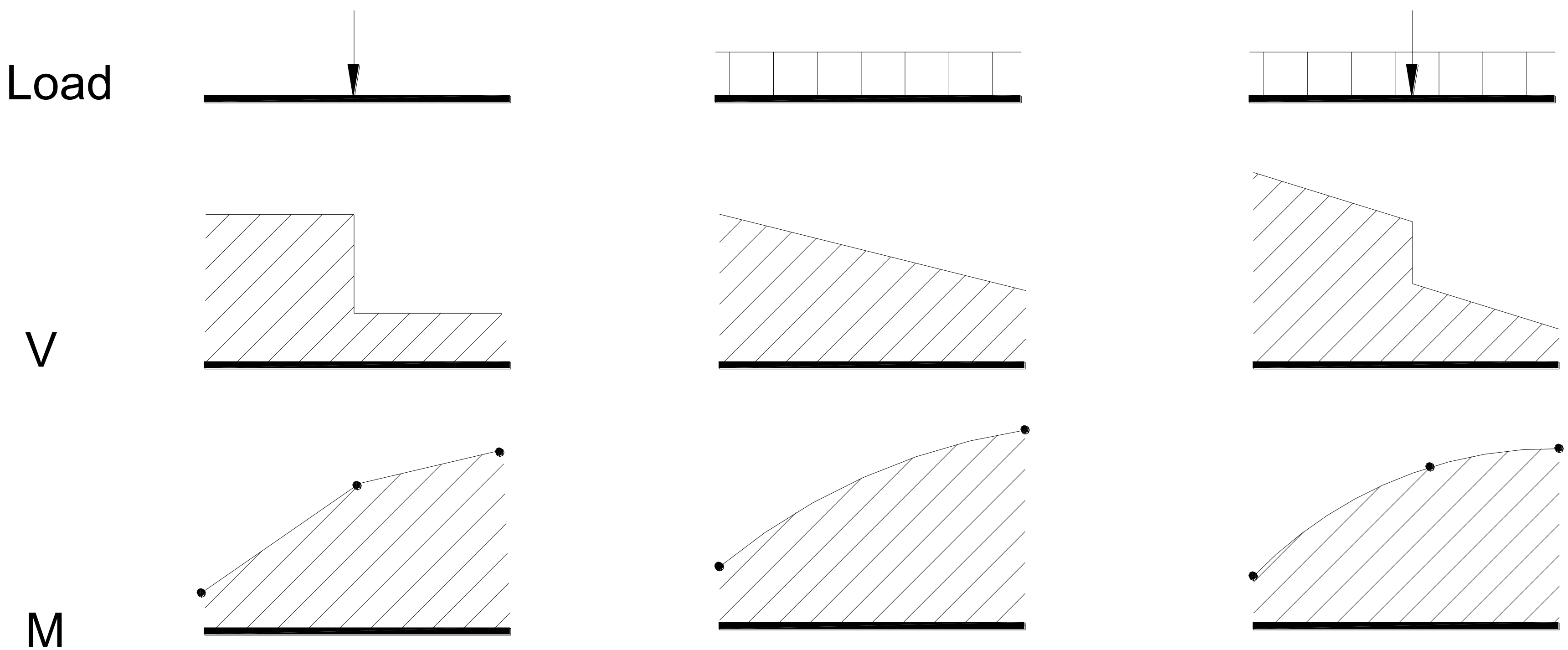
Example

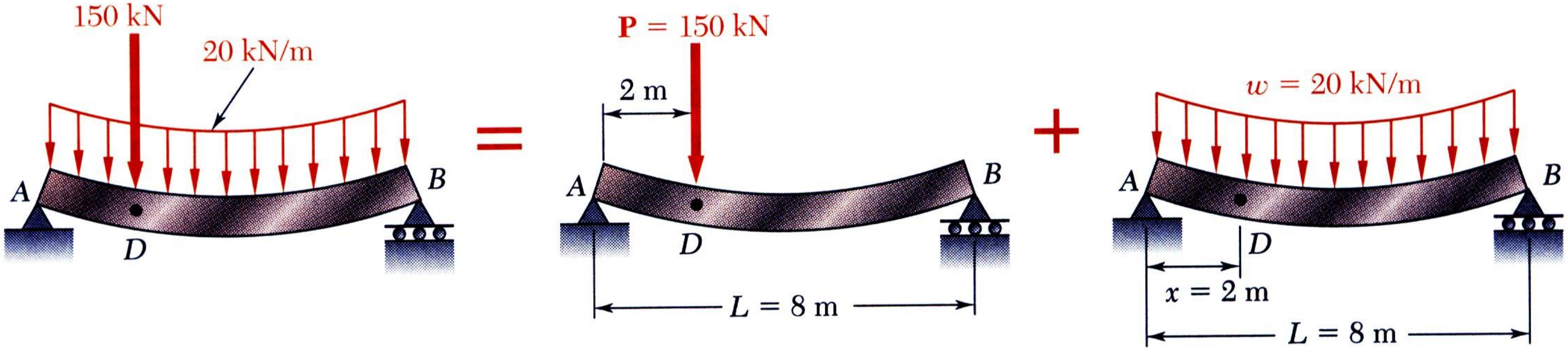


= 32.5 + 1.25 x 2

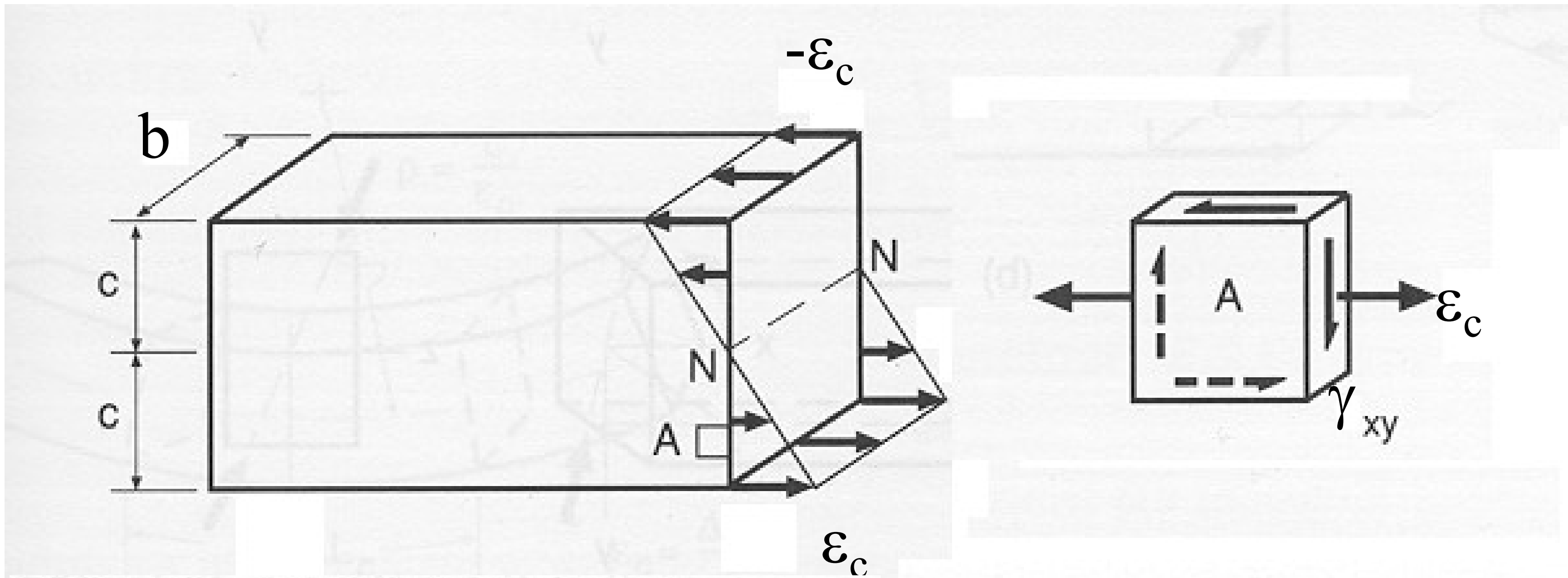
= 16.25 x 2

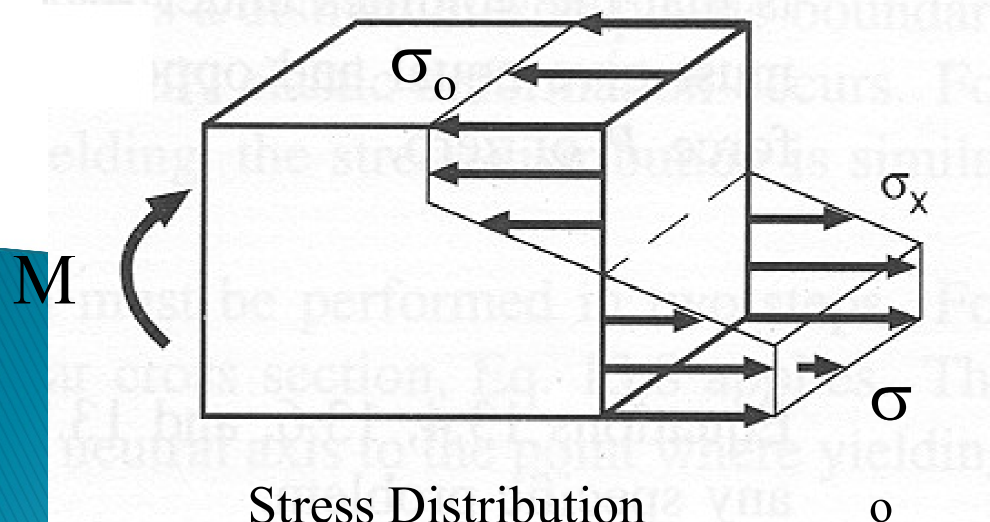
SFD and BMD for common loading

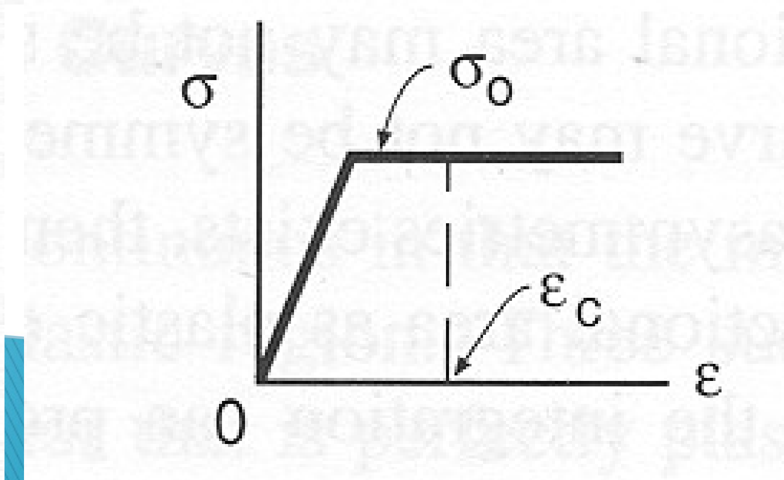




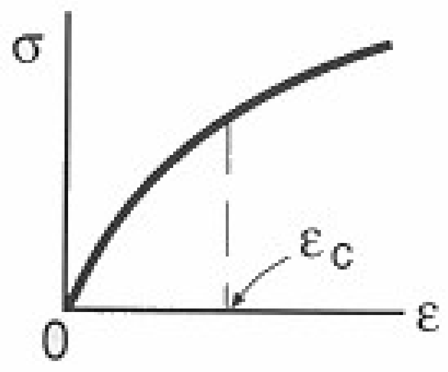
150 kN
20 kN/m
A
B
D
P = 150 kN
2 m
A
B
D
L = 8 m
w = 20 kN/m
A
B
D
x = 2 m
L = 8 m


-εc
b
c
c
N
N
A
εc
A
εc
γ xy

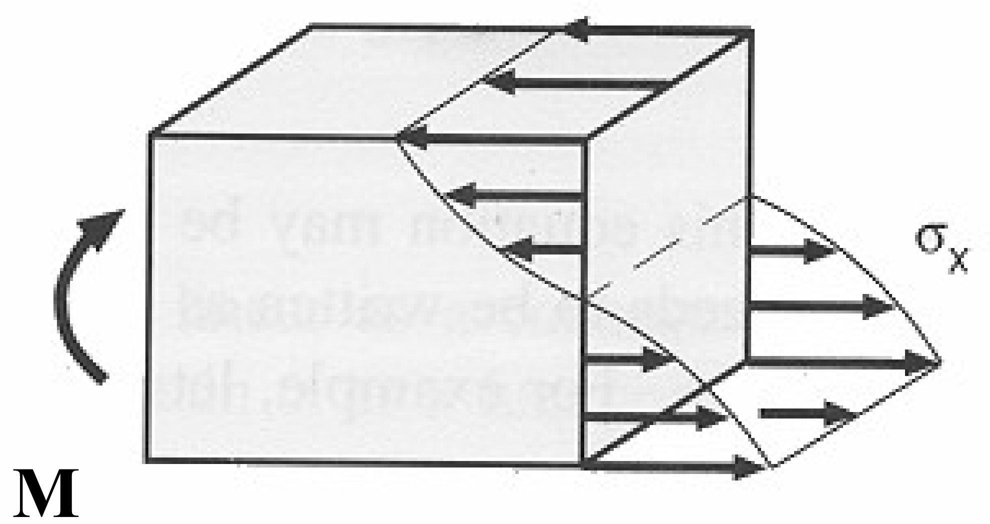

σo
σx
M
σ
Stress Distribution
o


σ
σo
εc
ε
0

Elastic-plastic Material

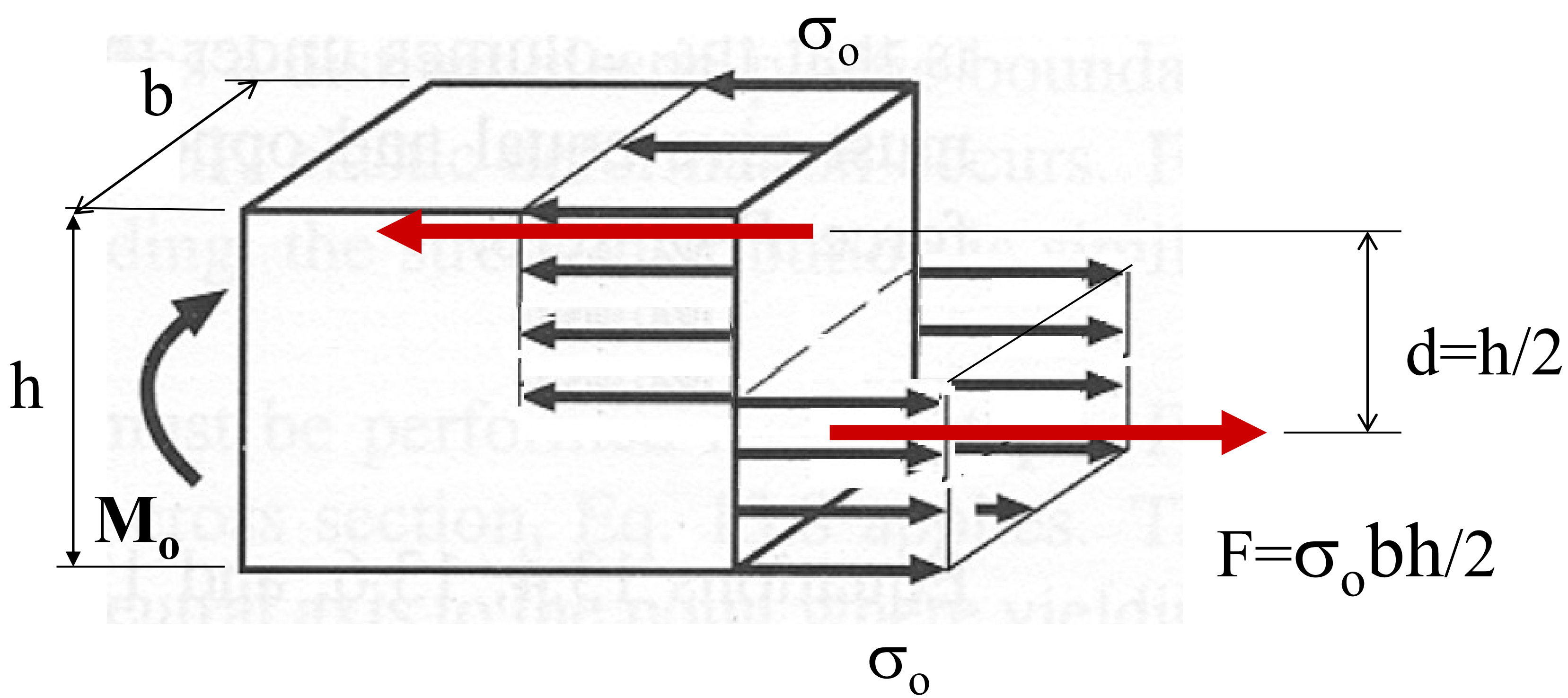


Stress-Strain Relationship



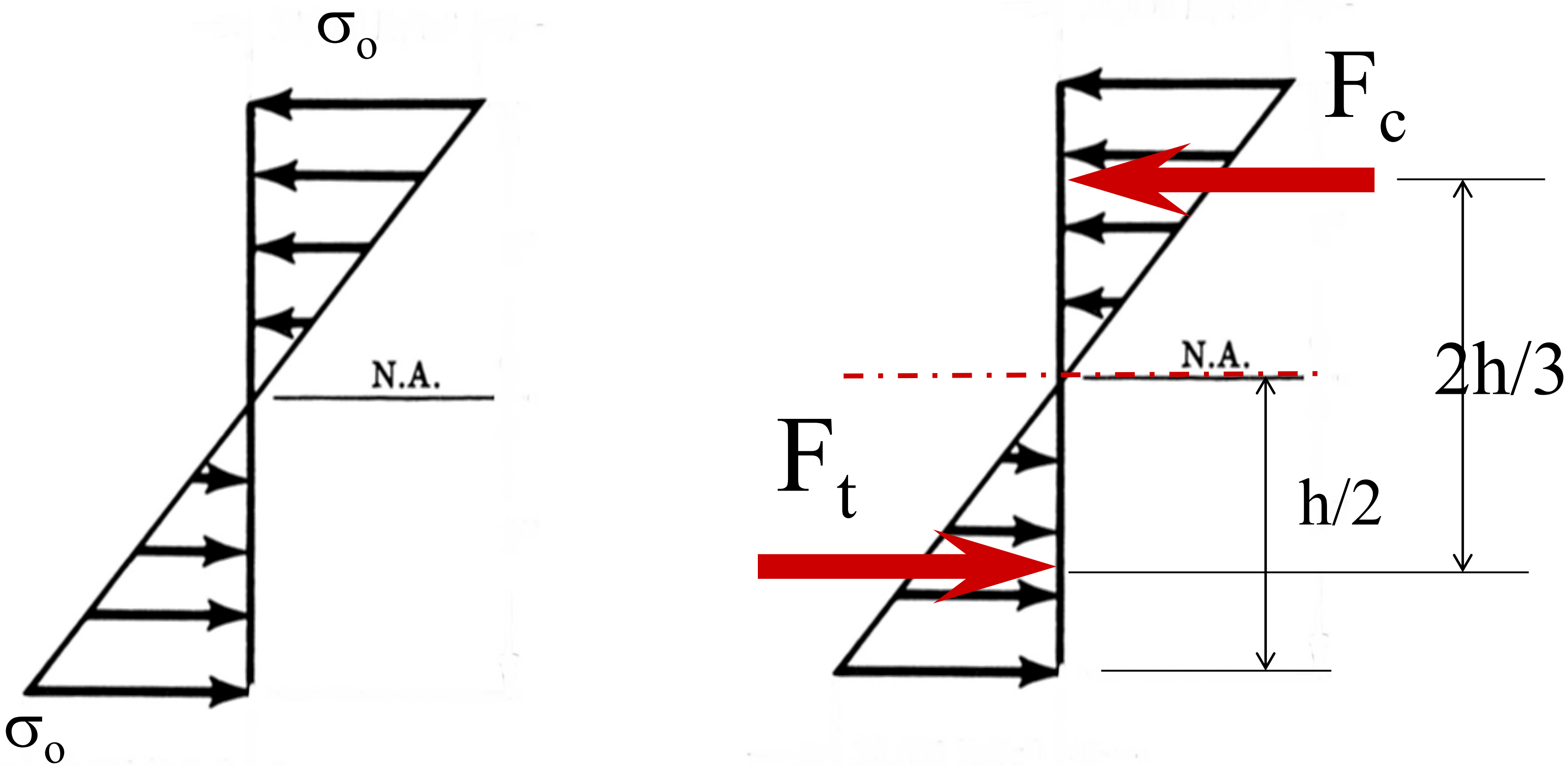
Stress Distribution

Fully-Plastic Moment



$$M_o = (\sigma_o bh/2)(h/2)$$
$$= \sigma_o bh^2/4$$

Maximum Elastic Moment

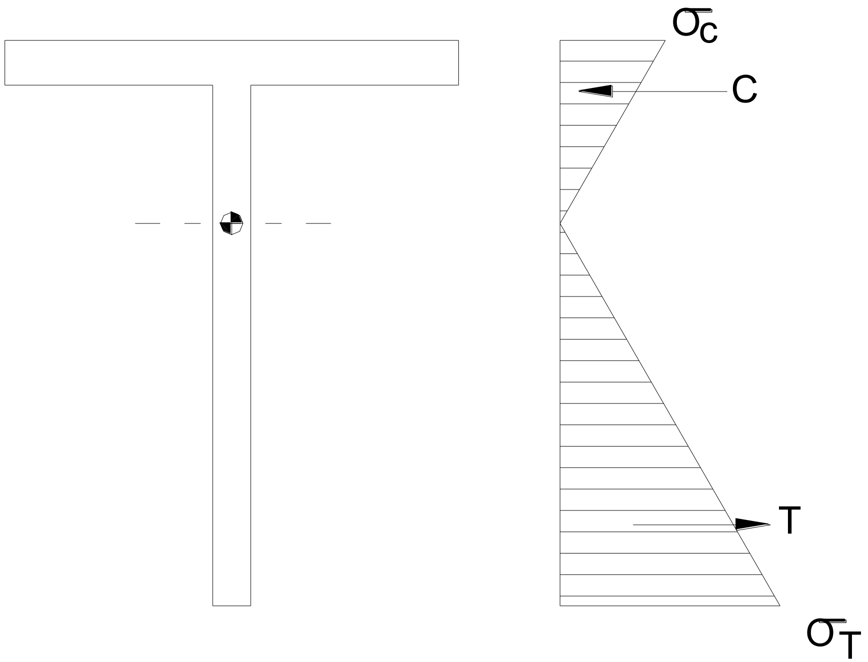


$$F_c = F_t = (\sigma_o/2)bh/2$$

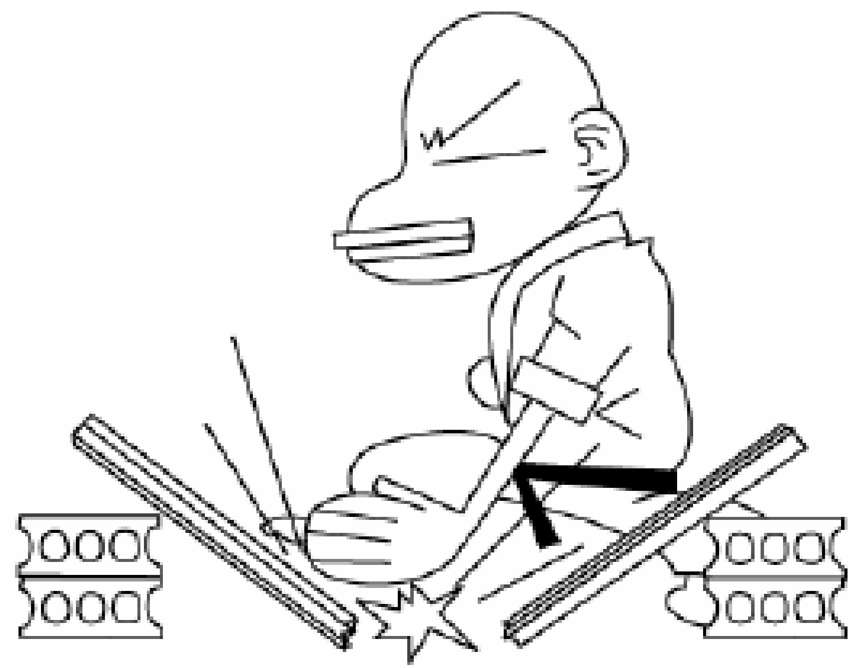
$$M = F_t d = (\sigma_o bh/4)(2h/3) = \sigma_o bh^2/6$$

Bending stresses

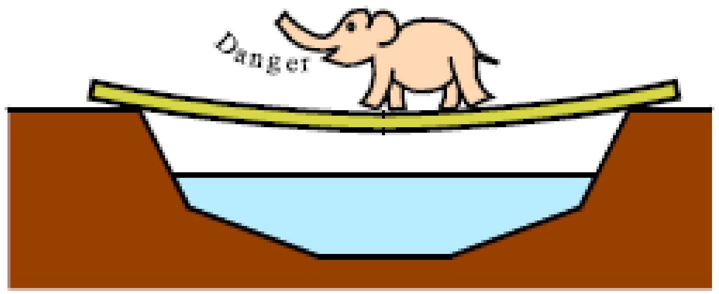
- Stress blocks in unsymmetrical shapes:

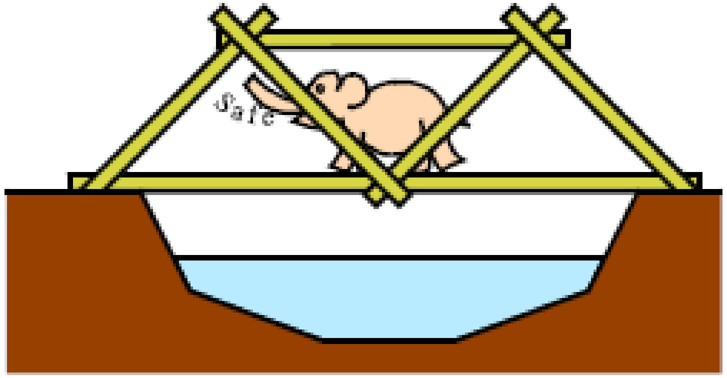


Bending Vs. Axial Stress

Beam Vs Truss





Axial compression stresses

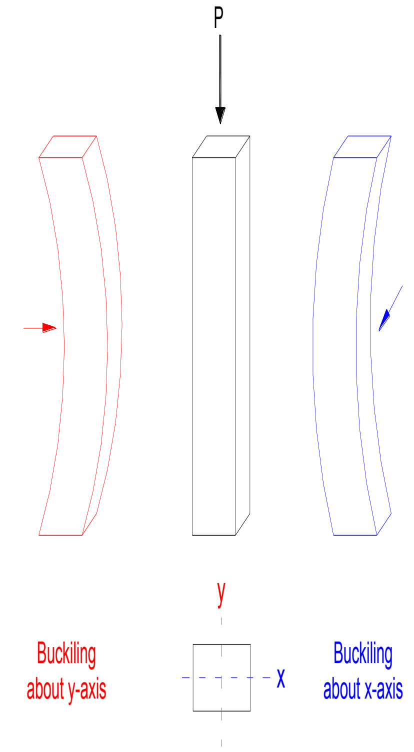
- What effects the tendency of a member to buckle?
  - Length: The longer the member the easier it will buckle.
  - Modulus of Elasticity: The stiffer the member's material (higher 'E'), the more difficult it will be able to buckle
  - Moment of Inertia: The greater the cross-section's moment of inertia, the more difficult it will be able to buckle

Axial compression stresses

- The previous relations are expressed by:
  - $P_{Euler} = (\Pi^2 EI) / L^2$
    - P_{Euler}: Load at which buckling will occur
    - E: Modulus of elasticity of the member’s material
    - I: Moment of Inertia of the member’s cross section
    - L: Unsupported length of member

Axial compression stresses

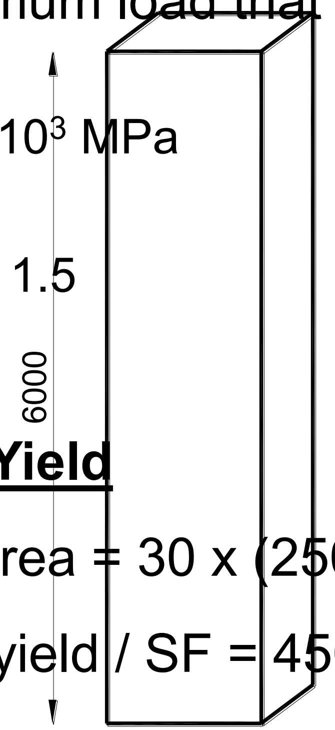
- Buckling may occur about either the x-axis or the y-axis
- If the supported length in each direction is the same, buckling will occur around the weaker axis (the axis having the smaller moment of inertia)
- In this case, Ix < Iy. Thus buckling will occur around the x-axis



Confused? Remember the ruler!

Example on Buckling

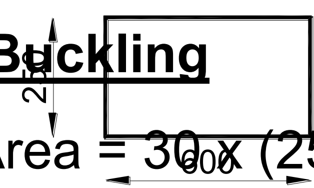
- What is the maximum load that can be held by the shown concrete column can?
  - $E_{concrete} = 20 \times 10^3$ MPa
  - $\sigma_{yield} = 30$ MPa
  - Safety Factor= 1.5



Step1: Check Yield

$P_{yield} = \sigma_{yield} \times Area = 30 \times (250 \times 600) = 4500$ kN

Pallowable = Pyield / SF = 4500 / 1.5 = 3000 kN



Step1: Check Buckling

$P_{Euler} = \sigma_{yield} \times Area = 30 \times (250 \times 600) = 4500$ kN

Pallowable = Pyield / SF = 4500 / 1.5 = 3000 kN

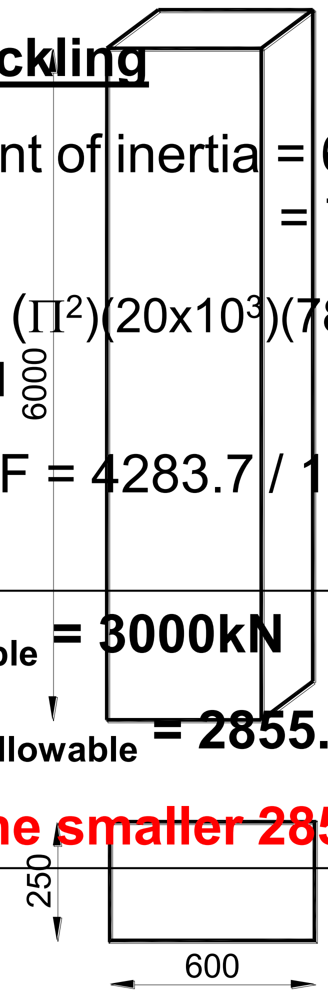
Example on Buckling

<u>Step 2: Check Buckling</u>

The smaller moment of inertia $= 600 \times (250)^3 / 12$
$= 781.25 \times 10^6\ mm^4$

$P_{Euler} = (\Pi^2 EI) / L^2 = (\Pi^2)(20\times10^3)(781.25\times10^6) / (6000^2)$
$= 4283.7\ kN$

$P_{allowable} = P_{Euler} / SF = 4283.7 / 1.5 = 2855.8\ kN$



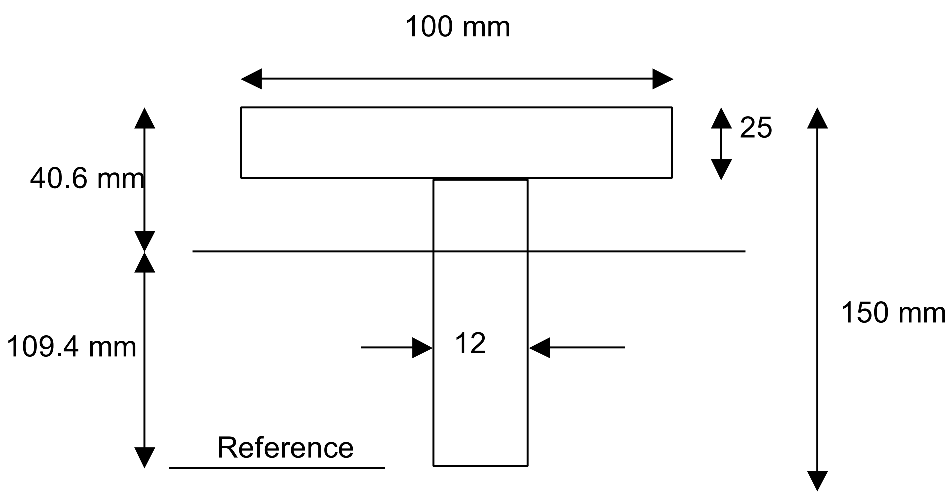
<u>Yield:</u> $P_{allowable}$ = 3000kN

<u>Buckling:</u> $P_{allowable}$ = 2855.8kN

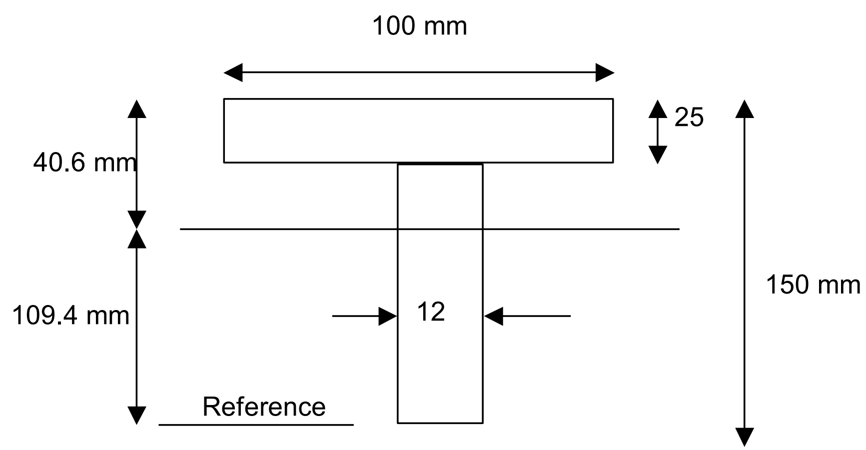
→ Choose the smaller 2855.8kN

Example

Example: A uniform T-section beam is 100 mm wide and 150 mm deep with a flange thickness of 25 mm and a web thickness of 12 mm. If the limiting bending stresses for the material of the beam are 80 MN/m^2 in compression and 160 MN/m^2 , find the maximum uniformly distributed load (u.d.l) that the beam can carry over a simply supported span of 5 m.



Solution



Solution: The second moment of area, I used in the simple bending theory is about the neutral axis, thus in order to determine the I value of the T-section shown, it is necessary first to determine the position of the centroid and hence the neutral axis.

Find the Neutral Axis:

$A \; y = A_1 y_1 + A_2 \; y_2$

$(100 \times 25) + (125 \times 12)\; y = (100 \times 25) \times 137.5 + (12 \times 125 \times 62.5)$

$(2500 + 1500)\; y = 343750 + 93750 = 437500$

$y = 109.4$ mm (see in figure)

Thus the N.A. is positioned, as shown, a distance of 109.4 mm above the base.

Solution Contd.

The second moment of area, I can now be found by dividing the section into convenient rectangles with their edges in the neutral axis.

$$I_{xx} = \frac{12\ x\ 125^3}{12} + (12\ x\ 125)\ x\ 46.9^2 + \frac{100\ x\ 25^3}{12} + (100\ x\ 25)\ x\ 28.1^2$$

= 1953125 + 3299415 + 130208.3 + 1974025

= 7356773.3 mm^4 = 7.36 x 10^{-6} m^4

Maximum compressive stress will occur at the upper surface, where y = 40.6 mm and using the limiting compressive stress value quoted:

$$M = \frac{\sigma\ I}{y} = \frac{80\ x\ 10^6\ x\ 7.36\ x\ 10^{-6}}{40.6\ x\ 10^{-3}} = 14.5\ kNm$$

Solution Concluded

This suggests a maximum allowable bending moment of 14.5 kN m. It is now necessary, however, to check the tensile stress criterion which must apply on the lower surface,

$$M = \frac{\sigma\, I}{y} = \frac{160\ x\ 10^{6}\ x\ 7.36\ x\ 10^{-6}}{109.4\ x\ 10^{-3}} = 10.76\ kN\ m$$

The greatest moment that can therefore be applied to retain stresses within both conditions quoted is therefore M = 10.76 kN m

But for a simply supported beam with u.d.l, $M_{max} = w\,L^2/8$

$W = 8\,M/L^2 = (8 \times 10.76 \times 10^3) / 5^2 =$ **3.4 kN/m**

The u.d.l must be limited to 3.4 kN/m.