Chapter 3 Pulse Modulation

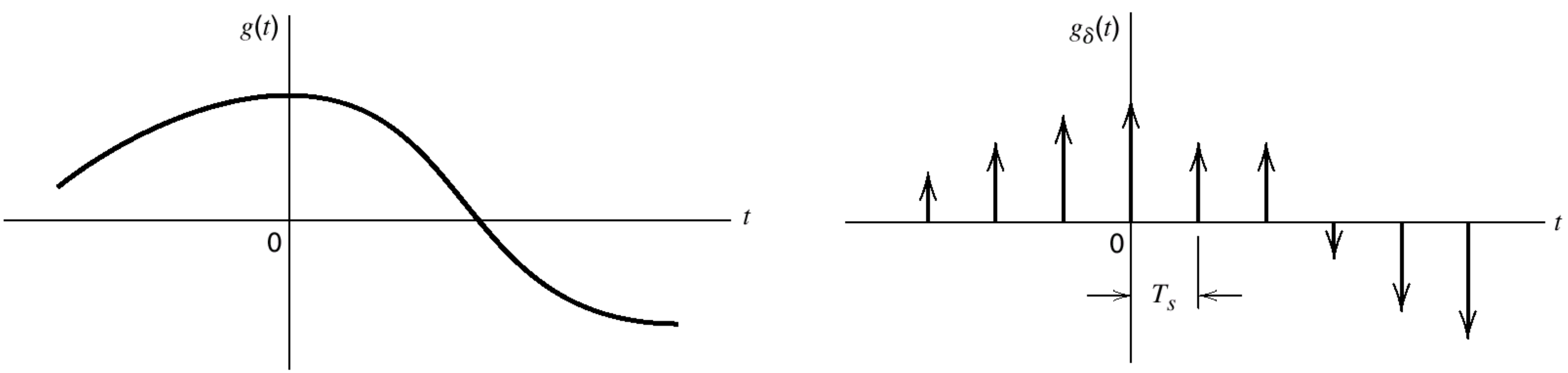
3.1 Introduction

Let $g_\delta(t)$ denote the ideal sampled signal

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\,\delta(t - nT_s) \qquad (3.1)$$

where T_s : sampling period

$f_s = 1/T_s$: sampling rate



(a) (b)

From Table A6.3 we have

$$g(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s) \Leftrightarrow$$

$$G(f) * \frac{1}{T_s}\sum_{m=-\infty}^{\infty}\delta(f-\frac{m}{T_s})$$

$$= \sum_{m=-\infty}^{\infty} f_s G(f-mf_s)$$

$$g_\delta(t) \Leftrightarrow f_s\sum_{m=-\infty}^{\infty} G(f-mf_s) \qquad (3.2)$$

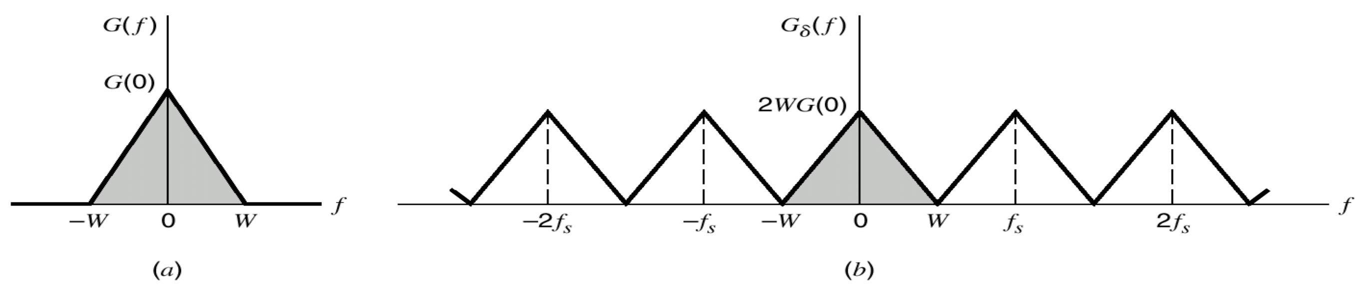
or we may apply Four ier Transform on (3.1) to obtain

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g(nT_s)\exp(-j2\pi nfT_s) \qquad (3.3)$$

or $$G_\delta(f) = f_s G(f) + f_s \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} G(f-mf_s) \qquad (3.5)$$

If $G(f)=0$ for $|f| \geq W$ and $T_s = 1/2W$

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g(\frac{n}{2W})\exp(-\frac{j\pi nf}{W}) \qquad (3.4)$$



With

1. $G(f) = 0$ for $|f| \geq W$

2. $f_s = 2W$

we find from Equation (3.5) that

$$G(f) = \frac{1}{2W} G_\delta(f)\ ,\ -W < f < W \qquad (3.6)$$

Substituting (3.4) into (3.6) we may rewrite $G(f)$ as

$$G(f) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} g(\frac{n}{2W}) \exp(-\frac{j\pi nf}{W})\ , -W < f < W \quad (3.7)$$

$g(t)$ is uniquely determined by $g(\frac{n}{2W})$ for $-\infty < n < \infty$

or $\left\{ g(\frac{n}{2W}) \right\}$ contains all information of $g(t)$

To reconstruct $g(t)$ from $\left\{ g(\frac{n}{2W}) \right\}$, we may have

$$g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi ft)df$$

$$= \int_{-W}^{W} \frac{1}{2W} \sum_{n=-\infty}^{\infty} g(\frac{n}{2W})\exp(-\frac{j\pi n f}{W})\exp(j2\pi f t)df$$

$$= \sum_{n=-\infty}^{\infty} g(\frac{n}{2W})\frac{1}{2W} \int_{-W}^{W} \exp\left[j2\pi f(t-\frac{n}{2W})\right]df \quad (3.8)$$

$$= \sum_{n=-\infty}^{\infty} g(\frac{n}{2W})\frac{\sin(2\pi Wt - n\pi)}{2\pi Wt - n\pi}$$

$$= \sum_{n=-\infty}^{\infty} g(\frac{n}{2W})\sin c(2Wt - n) \ , \ -\infty < t < \infty \qquad (3.9)$$

(3.9) is an interpolation formula of $g(t)$

Sampling Theorem for strictly band - limited signals

1.a signal which is limited to $-W < f < W$, can be completely described by $\left\{ g(\frac{n}{2W}) \right\}$.

2.The signal can be completely recovered from $\left\{ g(\frac{n}{2W}) \right\}$

Nyquist rate $= 2W$

Nyquist interval $= \frac{1}{2W}$

When the signal is not band - limited (under sampling) aliasing occurs .To avoid aliasing, we may limit the signal bandwidth or have higher sampling rate.

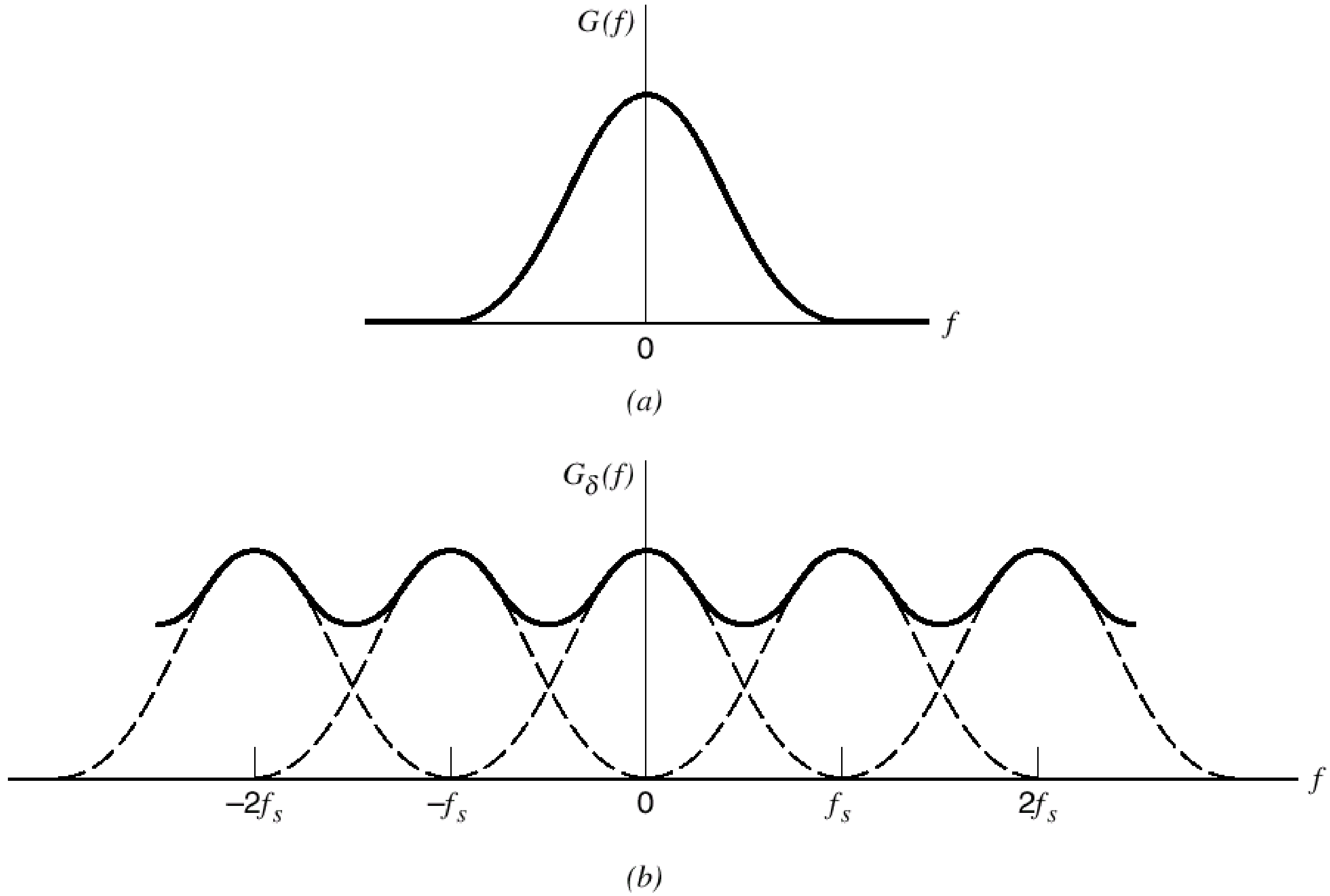


Figure 3.3 (*a*) Spectrum of a signal. (*b*) Spectrum of an undersampled version of the signal exhibiting the aliasing phenomenon.

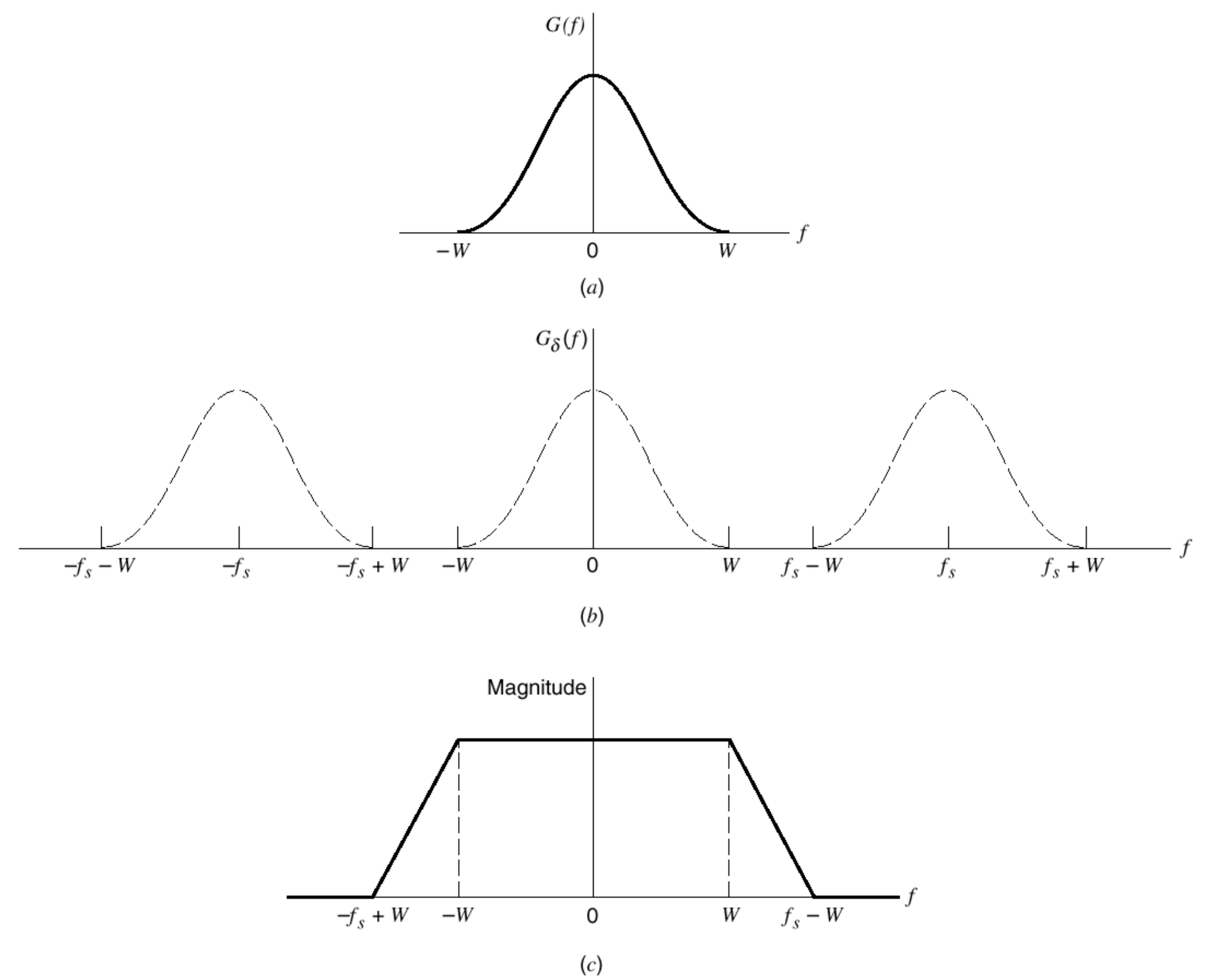
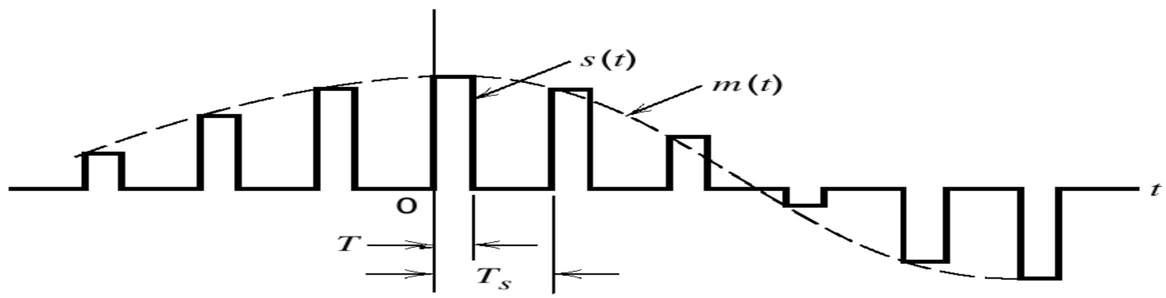


Figure 3.4 (*a*) Anti-alias filtered spectrum of an information-bearing signal. (*b*) Spectrum of instantaneously sampled version of the signal, assuming the use of a sampling rate greater than the Nyquist rate. (*c*) Magnitude response of reconstruction filter.

3.3 Pulse-Amplitude Modulation



Let $s(t)$ denote the sequence of flat-top pulses as

$$s(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\, h(t - nT_s) \qquad (3.10)$$

$$h(t) = \begin{cases} 1, & 0 < t < T \\ 1/2, & t = 0, t = T \\ 0, & \text{otherwise} \end{cases} \qquad (3.11)$$

The instantaneously sampled version of $m(t)$ is

$$m_\delta(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\delta(t - nT_s) \qquad (3.12)$$

$$\begin{aligned} m_\delta(t) * h(t) &= \int_{-\infty}^{\infty} m_\delta(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} m(nT_s)\delta(\tau - nT_s)h(t-\tau)d\tau \\ &= \sum_{n=-\infty}^{\infty} m(nT_s)\int_{-\infty}^{\infty} \delta(\tau - nT_s)h(t-\tau)d\tau \qquad (3.13) \end{aligned}$$

Using the sifting property, we have

$$m_\delta(t) * h(t) = \sum_{n=-\infty}^{\infty} m(nT_s)h(t - nT_s) \qquad (3.14)$$

The PAM signal $s(t)$ is

$$s(t) = m_\delta(t) * h(t) \qquad (3.15)$$

$$\Leftrightarrow S(f) = M_\delta(f)H(f) \qquad (3.16)$$

Recall (3.2) $g_\delta(t) \Leftrightarrow f_s \sum_{m=-\infty}^{\infty} G(f - mf_s)$ (3.2)

$$\mathrm{M}_\delta(f) = f_s \sum_{k=-\infty}^{\infty} M(f - k f_s) \qquad (3.17)$$

$$S(f) = f_s \sum_{k=-\infty}^{\infty} M(f - k f_s)H(f) \qquad (3.18)$$

Pulse Amplitude Modulation – Natural and Flat-Top Sampling

- The circuit of Figure 11-3 is used to illustrate pulse amplitude modulation (PAM). The FET is the switch used as a sampling gate.
- When the FET is on, the analog voltage is shorted to ground; when off, the FET is essentially open, so that the analog signal sample appears at the output.
- Op-amp 1 is a noninverting amplifier that isolates the analog input channel from the switching function.

Pulse Amplitude Modulation – Natural and Flat-Top Sampling

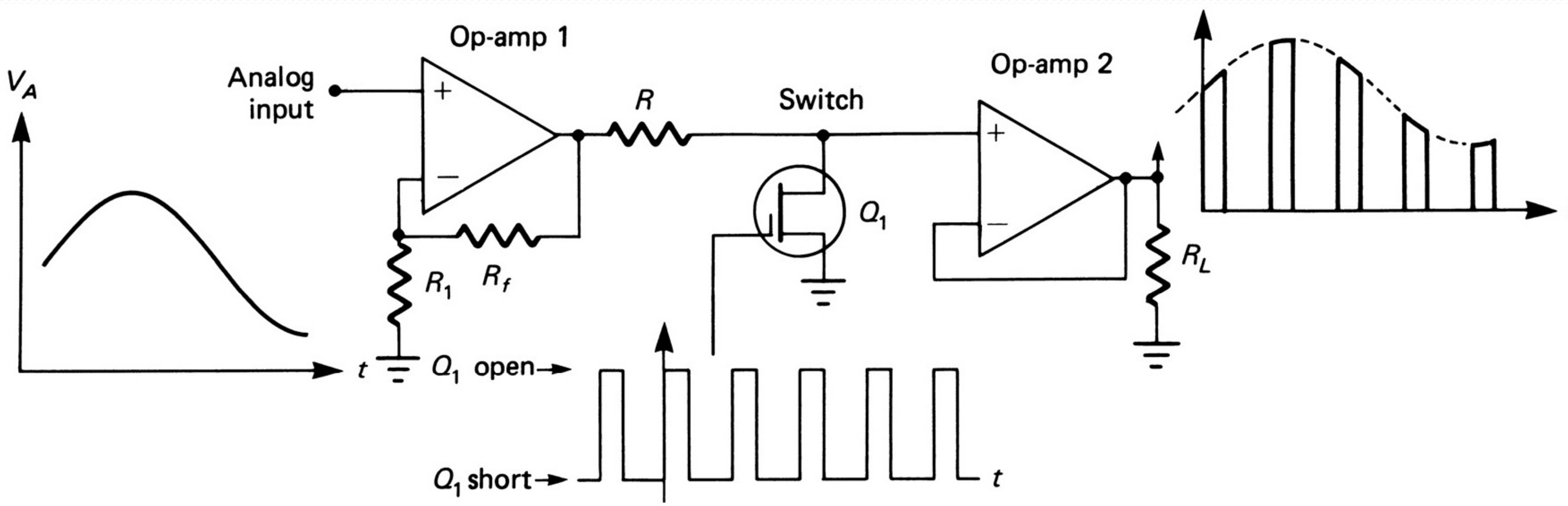


圖 11-3　脈衝幅度調變器，自然取樣。

Figure 11-3. Pulse amplitude modulator, natural sampling.

Pulse Amplitude Modulation – Natural and Flat-Top Sampling

- **Op-amp 2 is a high input-impedance voltage follower capable of driving low-impedance loads (high “fanout”).**

- **The resistor *R* is used to limit the output current of op-amp 1 when the FET is “on” and provides a voltage division with r_d of the FET. (r_d, the drain-to-source resistance, is low but not zero)**

Pulse Amplitude Modulation – Natural and Flat-Top Sampling

- **The most common technique for sampling voice in PCM systems is to a sample-and-hold circuit.**
- **As seen in Figure 11-4, the instantaneous amplitude of the analog (voice) signal is held as a constant charge on a capacitor for the duration of the sampling period T_s.**
- **This technique is useful for holding the sample constant while other processing is taking place, but it alters the frequency spectrum and introduces an error, called aperture error, resulting in an inability to recover exactly the original analog signal.**

Pulse Amplitude Modulation – Natural and Flat-Top Sampling

- **The amount of error depends on how mach the analog changes during the holding time, called aperture time.**
- **To estimate the maximum voltage error possible, determine the maximum slope of the analog signal and multiply it by the aperture time ΔT in Figure 11-4.**

Pulse Amplitude Modulation – Natural and Flat-Top Sampling

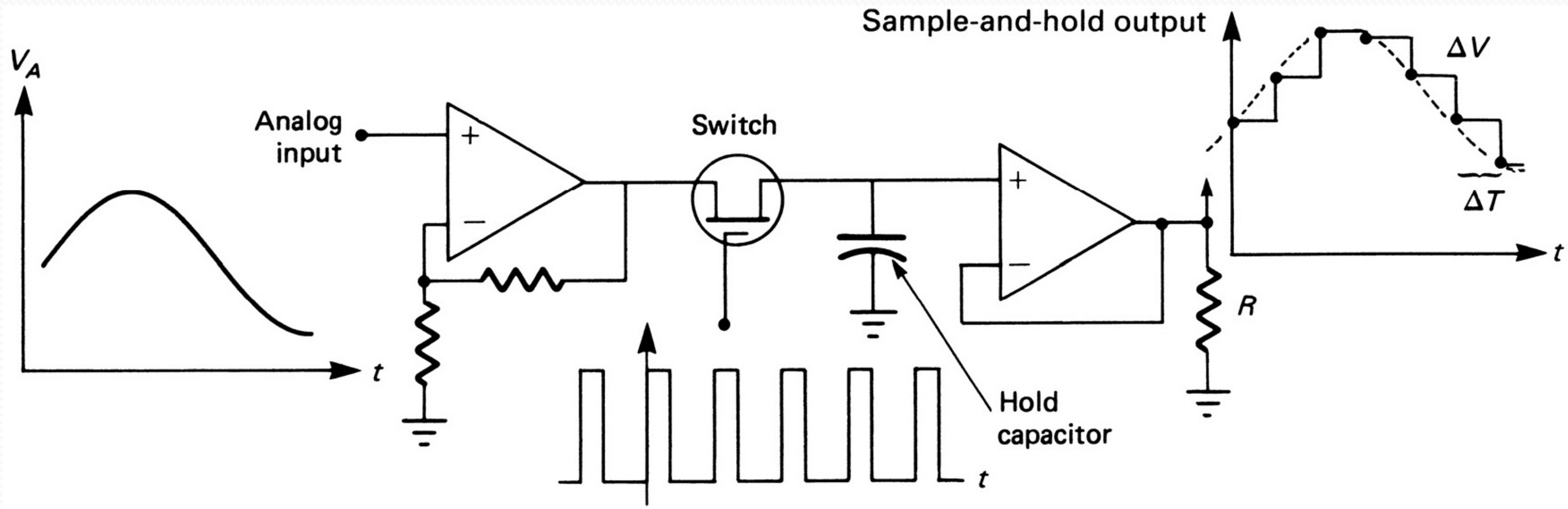


圖 11-4　樣本－和－持保電路和平頂取樣。

Figure 11-4. Sample-and-hold circuit and flat-top sampling.

Pulse Amplitude Modulation – Natural and Flat-Top Sampling

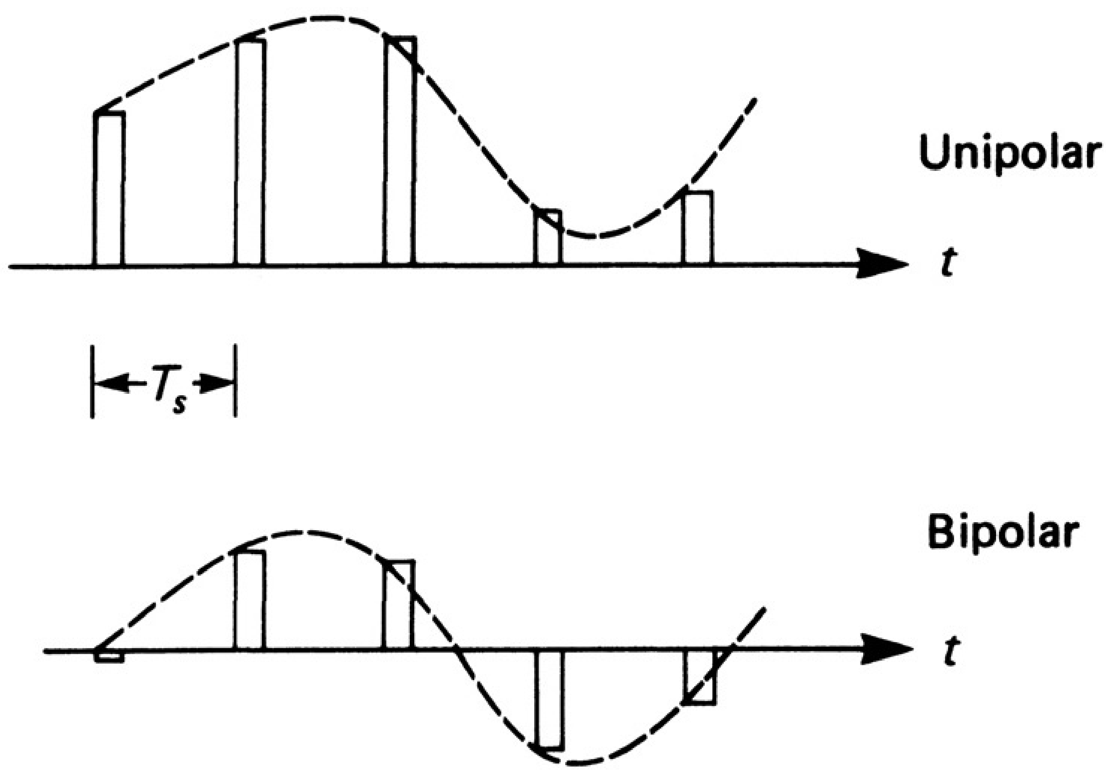
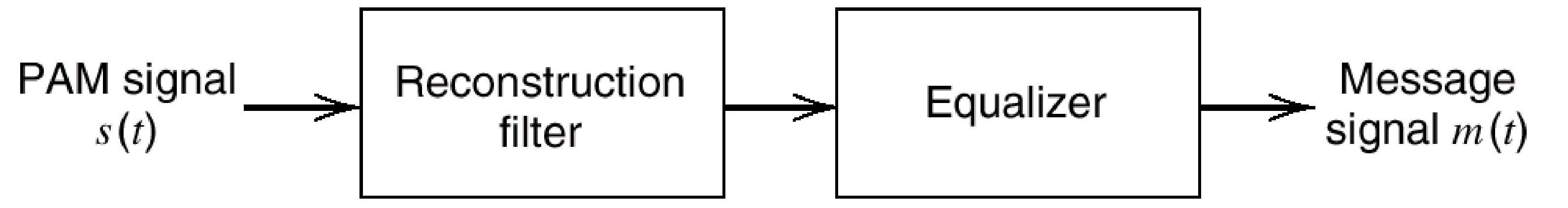


圖 11-5 平頂 PAM 訊號。

Figure 11-5. Flat-top PAM signals.

Recovering the original message signal m(t) from PAM signal



Where the filter bandwidth is W

The filter output is $f_s M(f)H(f)$. Note that the Fourier transform of $h(t)$ is given by

$$H(f) = T\,\mathrm{sinc}(fT)\exp(-j\pi f T) \qquad (3.19)$$

amplitude distortion   delay $= T/2$

$\Rightarrow$ aparture effect

Let the equalizer response is

$$\frac{1}{H(f)} = \frac{1}{T\,\mathrm{sinc}(fT)} = \frac{\pi f}{\sin(\pi f T)} \qquad (3.20)$$

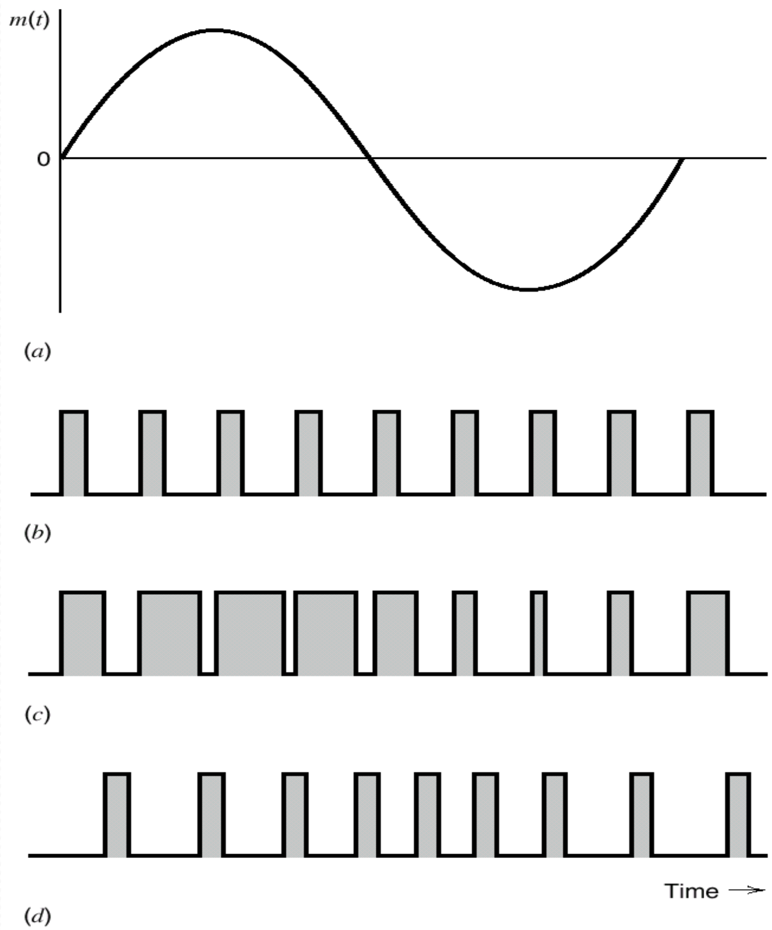
Ideally the original signal $m(t)$ can be recovered completely.

3.4 Other Forms of Pulse Modulation

a. Pulse-duration modulation (PDM)

b. Pulse-position modulation (PPM)

PPM has a similar noise performance as FM.



Pulse Width and Pulse Position Modulation

- **In pulse width modulation (PWM), the width of each pulse is made directly proportional to the amplitude of the information signal.**
- **In pulse position modulation, constant-width pulses are used, and the position or time of occurrence of each pulse from some reference time is made directly proportional to the amplitude of the information signal.**
- **PWM and PPM are compared and contrasted to PAM in Figure 11-11.**

Pulse Width and Pulse Position Modulation

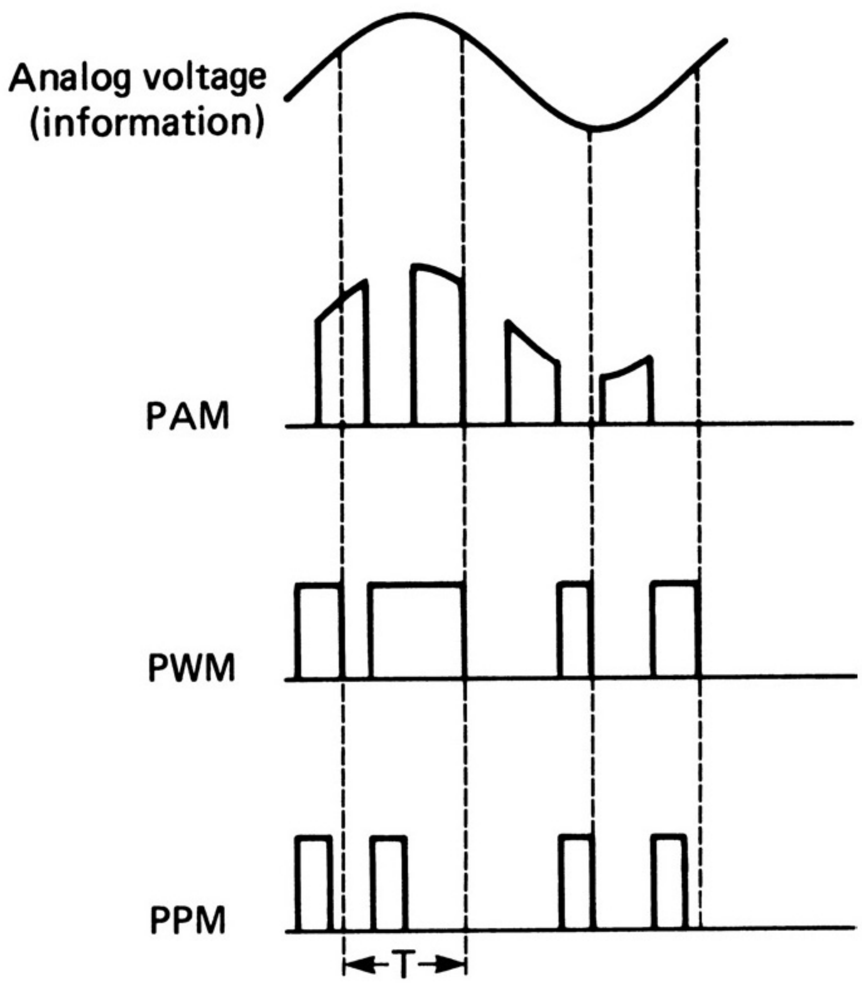


圖 11-11　類比脈衝調變訊號。

Figure 11-11. Analog/pulse modulation signals.

Pulse Width and Pulse Position Modulation

- **Figure 11-12 shows a PWM modulator. This circuit is simply a high-gain comparator that is switched on and off by the sawtooth waveform derived from a very stable-frequency oscillator.**
- **Notice that the output will go to $+V_{cc}$ the instant the analog signal exceeds the sawtooth voltage.**
- **The output will go to $-V_{cc}$ the instant the analog signal is less than the sawtooth voltage. With this circuit the average value of both inputs should be nearly the same.**
- **This is easily achieved with equal value resistors to ground. Also the $+V$ and $–V$ values should not exceed V_{cc}.**

Pulse Width and Pulse Position Modulation

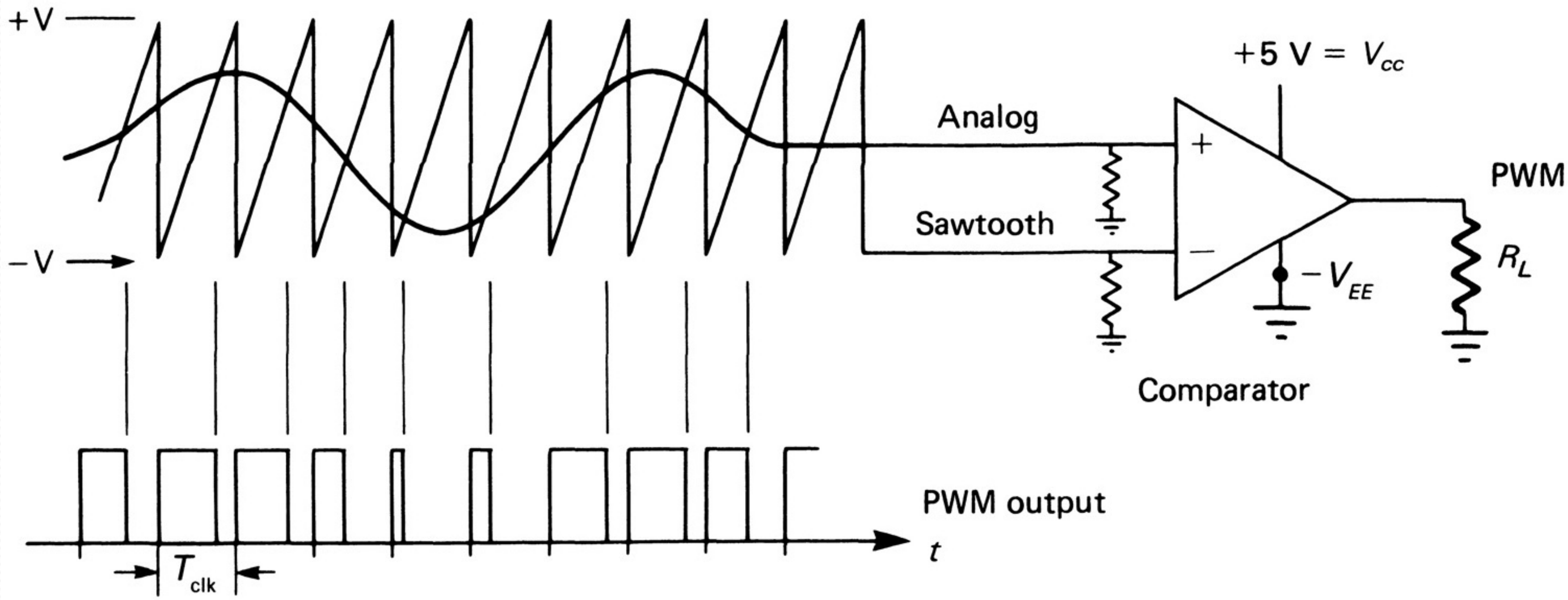


圖 11-12　脈衝寬度調變器。

Figure 11-12. Pulse width modulator.

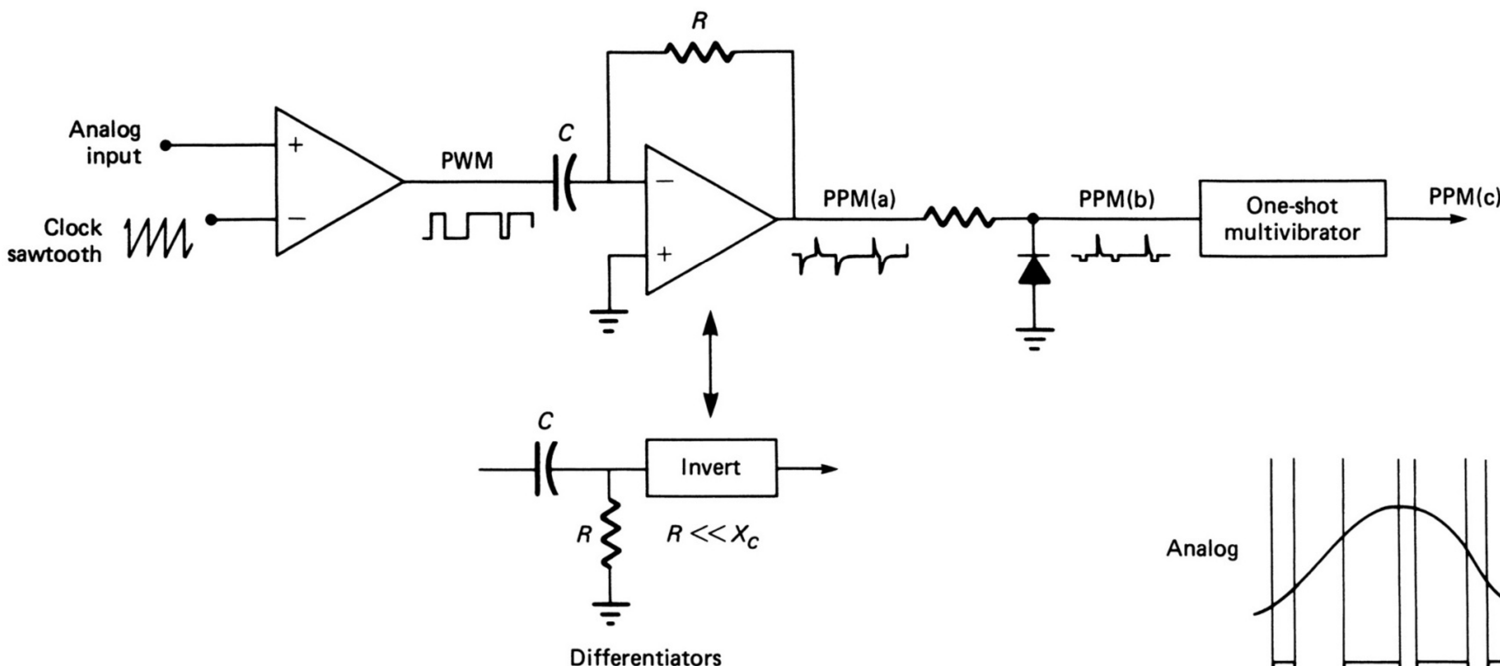
Pulse Width and Pulse Position Modulation

- A 710-type IC comparator can be used for positive-only output pulses that are also TTL compatible. PWM can also be produced by modulation of various voltage-controllable multivibrators.
- One example is the popular 555 timer IC. Other (pulse output) VCOs, like the 566 and that of the 565 phase-locked loop IC, will produce PWM.
- This points out the similarity of PWM to continuous analog FM. Indeed, PWM has the advantages of FM---constant amplitude and good noise immunity---and also its disadvantage---large bandwidth.

Demodulation

- Since the width of each pulse in the PWM signal shown in Figure 11-13 is directly proportional to the amplitude of the modulating voltage.
- The signal can be differentiated as shown in Figure 11-13 (to PPM in part a), then the positive pulses are used to start a ramp, and the negative clock pulses stop and reset the ramp.
- This produces frequency-to-amplitude conversion (or equivalently, pulse width-to-amplitude conversion).
- The variable-amplitude ramp pulses are then time-averaged (integrated) to recover the analog signal.

Pulse Width and Pulse Position Modulation



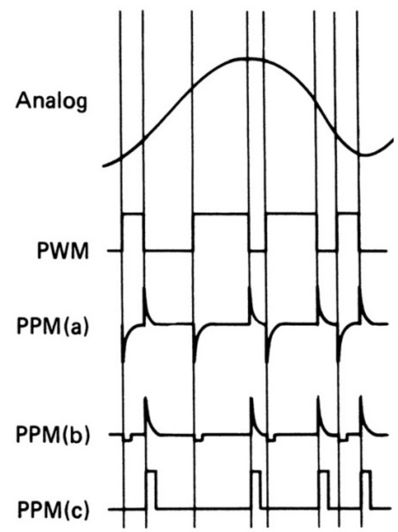


圖 11-13　脈波位置調變器。

Figure 11-13. Pulse position modulator.

Demodulation

- As illustrated in Figure 11-14, a narrow clock pulse sets an RS flip-flop output high, and the next PPM pulses resets the output to zero.
- The resulting signal, PWM, has an average voltage proportional to the time difference between the PPM pulses and the reference clock pulses.
- Time-averaging (integration) of the output produces the analog variations.
- PPM has the same disadvantage as continuous analog phase modulation: a coherent clock reference signal is necessary for demodulation.
- The reference pulses can be transmitted along with the PPM signal.

Demodulation

- **This is achieved by full-wave rectifying the PPM pulses of Figure 11-13a, which has the effect of reversing the polarity of the negative (clock-rate) pulses.**
- **Then an edge-triggered flipflop (J-K or D-type) can be used to accomplish the same function as the RS flip-flop of Figure 11-14, using the clock input.**
- **The penalty is: more pulses/second will require greater bandwidth, and the pulse width limit the pulse deviations for a given pulse period.**

Demodulation

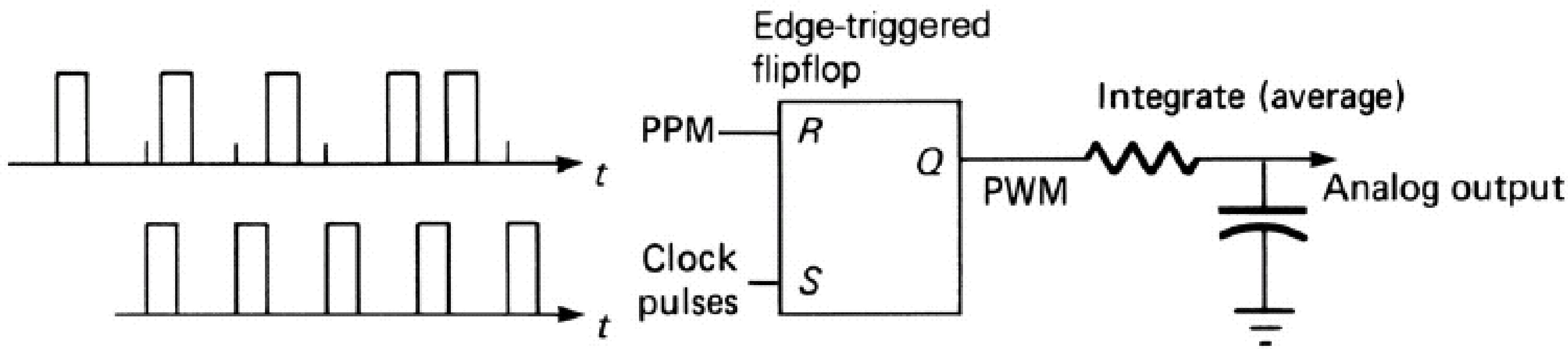


圖 11-14　PPM 解調器。

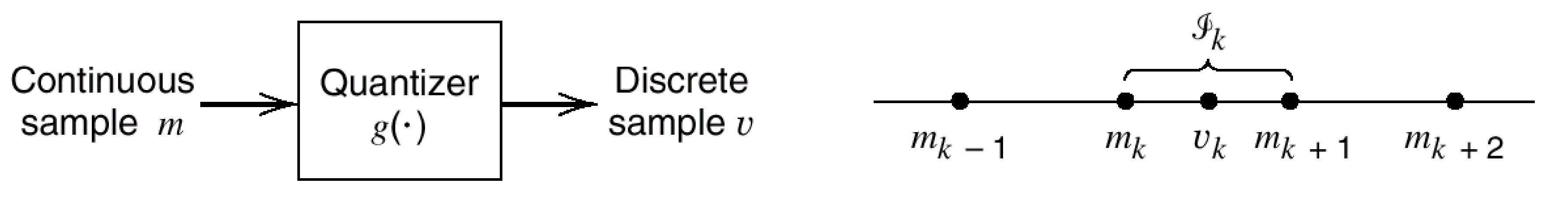
Figure 11-14. PPM demodulator.

Pulse Code Modulation (PCM)

- Pulse code modulation (PCM) is produced by analog-to-digital conversion process.
- As in the case of other pulse modulation techniques, the rate at which samples are taken and encoded must conform to the Nyquist sampling rate.
- The sampling rate must be greater than, twice the highest frequency in the analog signal,

$$f_s > 2f_A(\text{max})$$

3.6 Quantization Process



(*a*) (*b*)

Define partition cell

$$J_k : \{m_k < m \le m_{k+1}\} , k = 1,2,\cdots, L \qquad (3.21)$$

Where m_k is the decision level or the decision threshold.

Amplitude quantization : The process of transforming the sample amplitude $m(nT_s)$ into a discrete amplitude $v(nT_s)$ as shown in Fig 3.9

If $m(t) \in J_k$ then the quantizer output is v_k where v_k , $k = 1,2,\cdots,$ L are the representation or reconstruction levels , $m_{k+1} - m_k$ is the step size.

The mapping $v = \mathrm{g}(m) \qquad (3.22)$

is called the quantizer characteristic, which is a staircase function.

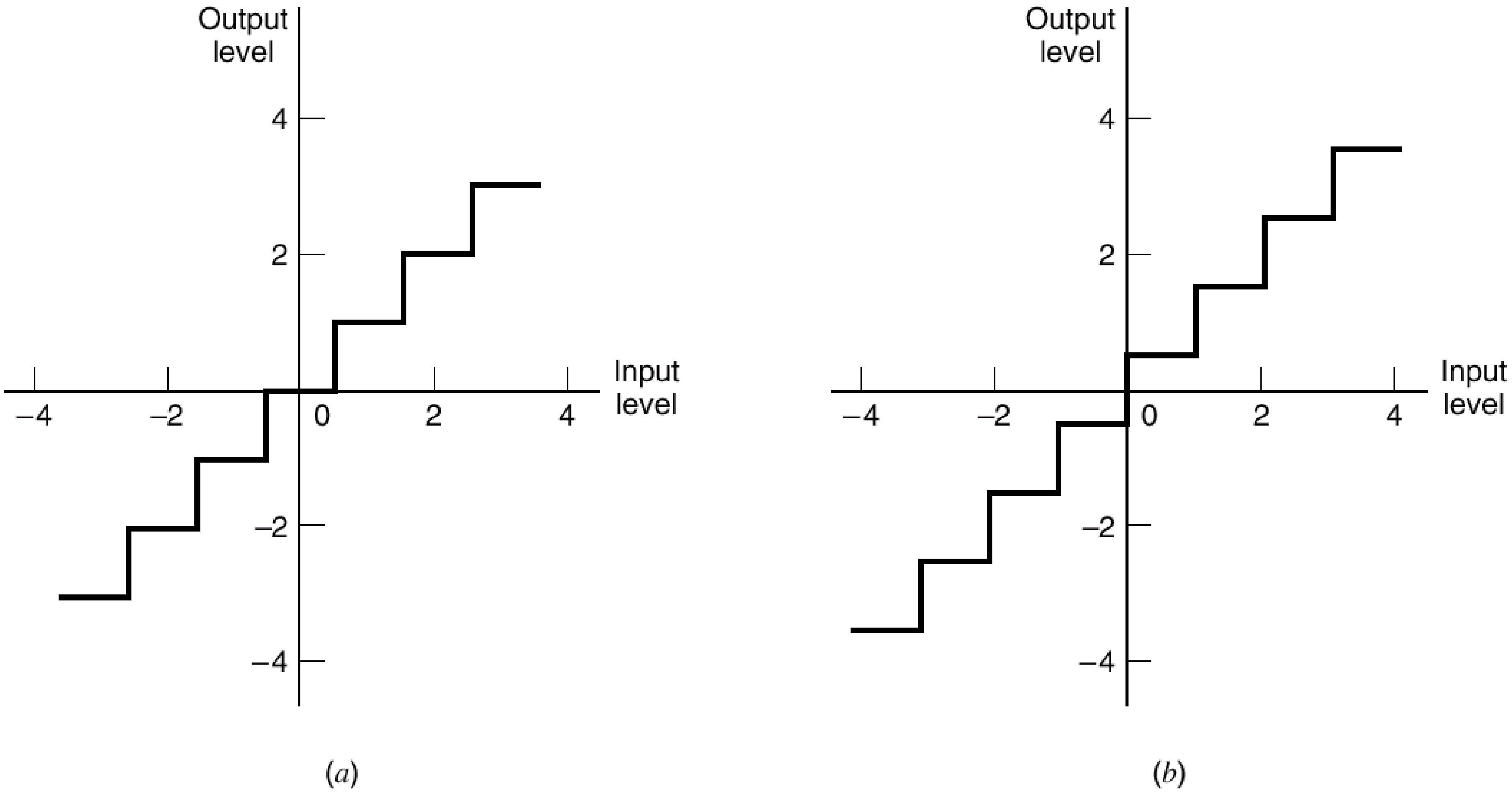


Figure 3.10 Two types of quantization: (*a*) midtread and (*b*) midrise.

Quantization Noise

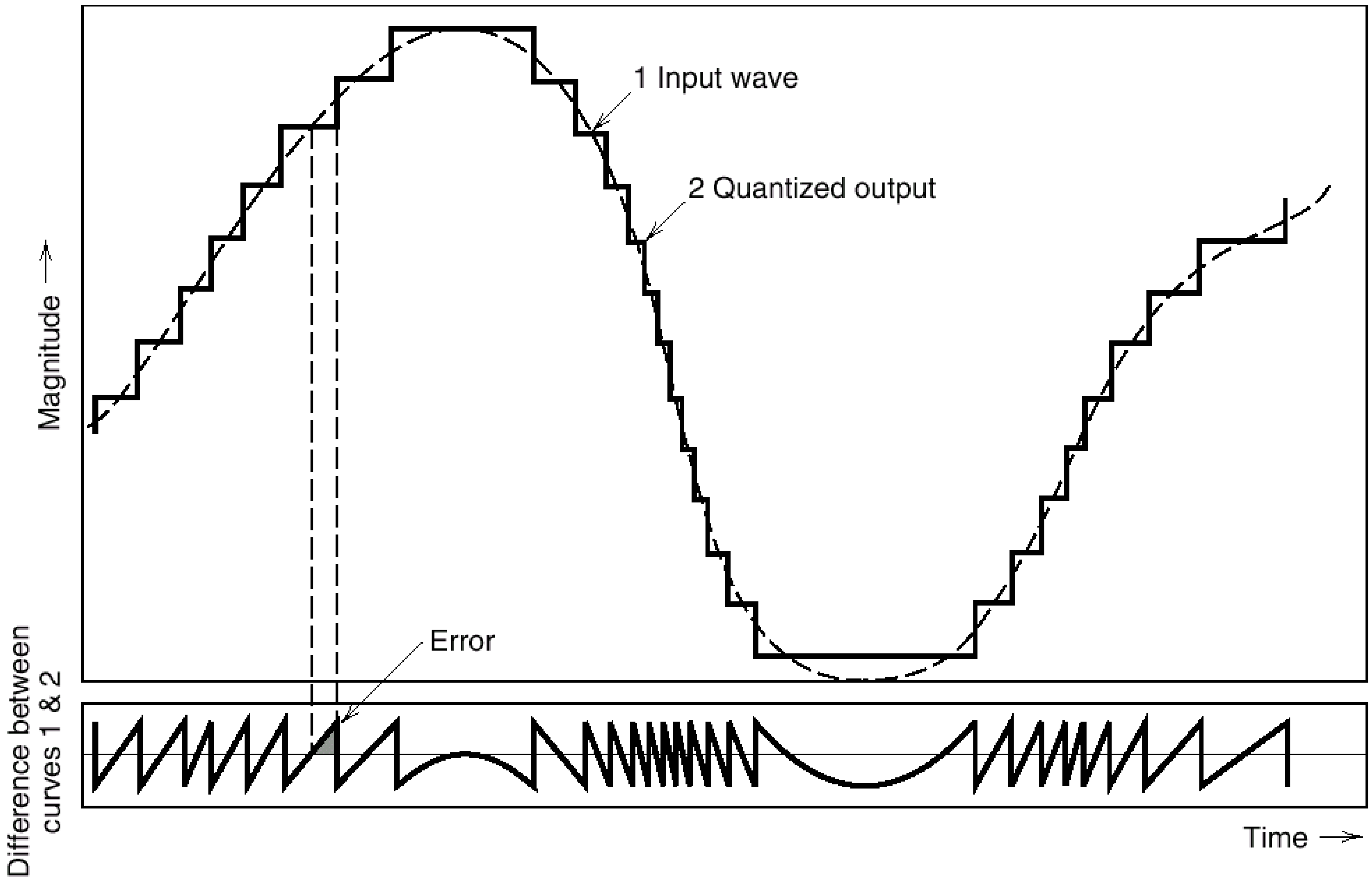


Figure 3.11 Illustration of the quantization process. (Adapted from Bennett, 1948, with permission of AT&T.)

Let the quantization error be denoted by the random variable Q of sample value q

$$q = m - v \qquad (3.23)$$

$$Q = M - V, \ (E[M] = 0) \qquad (3.24)$$

Assuming a uniform quantizer of the midrise type

the step-size is $\Delta = \dfrac{2m_{\max}}{L}$ (3.25)

$-m_{\max} < m < m_{\max}$, L : total number of levels

$$f_Q(q) = \begin{cases} \dfrac{1}{\Delta}, & -\dfrac{\Delta}{2} < q \le \dfrac{\Delta}{2} \\ 0, & \text{otherwise} \end{cases} \qquad (3.26)$$

$$\sigma_Q^2 = E[Q^2] = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} q^2 f_Q(q)dq = \frac{1}{\Delta}\int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} q^2 dq$$

$$= \frac{\Delta^2}{12} \qquad (3.28)$$

When the quatized sample is expressed in binary form,

$$L = 2^R \qquad (3.29)$$

where R is the number of bits per sample

$$R = \log_2 L \qquad (3.30)$$

$$\Delta = \frac{2m_{\max}}{2^R} \qquad (3.31)$$

$$\sigma_Q^2 = \frac{1}{3} m_{\max}^2 2^{-2R} \qquad (3.32)$$

Let P denote the average power of $m(t)$

$$\Rightarrow (SNR)_{\mathrm{o}} = \frac{P}{\sigma_Q^2}$$

$$= \left(\frac{3P}{m_{\max}^2}\right) 2^{2R} \qquad (3.33)$$

$(\mathrm{SNR})_{\mathrm{o}}$ increases exponentially with increasing R (bandwidth).

▶ EXAMPLE 3.1 Sinusoidal Modulating Signal

Consider the special case of a full-load sinusoidal modulating signal of amplitude A_m, which utilizes all the representation levels provided. The average signal power is (assuming a load of 1 ohm)

$$P = \frac{A_m^2}{2}$$

The total range of the quantizer input is $2A_m$, because the modulating signal swings between $-A_m$ and A_m. We may therefore set $m_{\max} = A_m$, in which case the use of Equation (3.32) yields the average power (variance) of the quantization noise as

$$\sigma_Q^2 = \tfrac{1}{3}A_m^2 2^{-2R}$$

Thus the output signal-to-noise ratio of a uniform quantizer, for a full-load test tone, is

$$(\mathrm{SNR})_O = \frac{A_m^2/2}{A_m^2 2^{-2R}/3} = \frac{3}{2}(2^{2R}) \tag{3.34}$$

Expressing the signal-to-noise ratio in decibels, we get

$$10 \log_{10}(\mathrm{SNR})_O = 1.8 + 6R \tag{3.35}$$

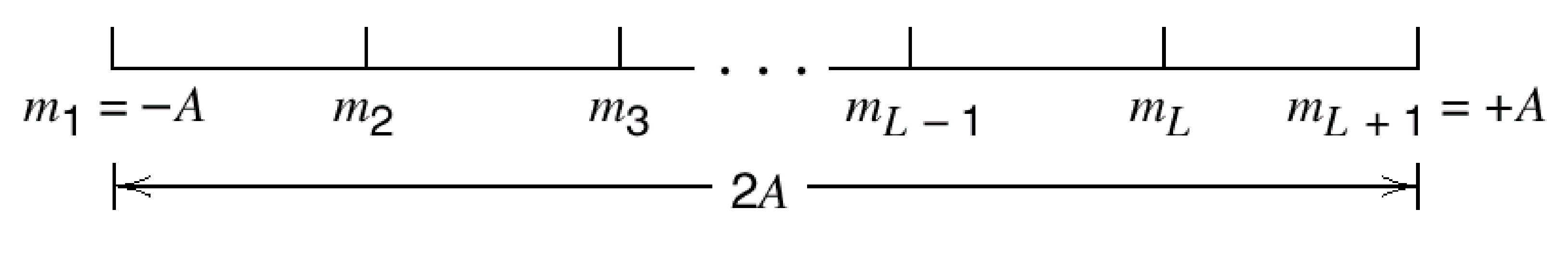
TABLE 3.1 ***Signal-to-(quantization) noise ratio for varying number of representation levels for sinusoidal modulation***

Number of Representation Levels, L	*Number of Bits per Sample,* R	*Signal-to-Noise Ratio (dB)*
32	5	31.8
64	6	37.8
128	7	43.8
256	8	49.8

Conditions for Optimality of scalar Quantizers

Let $m(t)$ be a message signal drawn from a stationary process $M(t)$

$$-A \le m \le A$$



$$m_1 = -A$$
$$m_{L+1} = A$$

$$m_k \le m_{k+1} \quad \text{for } k=1,2,\ldots, L$$

The kth partition cell is defined as

$$J_k: m_k < m \le m_{k+1} \text{ for } k=1,2,\ldots, L$$

$d(m,v_k)$: distortion measure for using v_k to represent values inside J_k.

Find the two sets $\{v_k\}_{k=1}^{L}$ and $\{J_k\}_{k=1}^{L}$, that minimize the average distortion

$$D = \sum_{k=1}^{L} \int_{m \in J_k} d(m, v_k) f_M(m) dm \qquad (3.37)$$

where $f_M(m)$ is the pdf

The mean-square distortion is used commonly

$$d(m, v_k) = (m - v_k)^2 \qquad (3.38)$$

The optimization is a nonlinear problem which may not have closed form solution. However the quantizer consists of two components: an encoder characterized by J_k, and a decoder characterized by v_k

Condition 1. Optimality of the encoder for a given decoder

Given the set $\{v_k\}_{k=1}^{L}$, find the set $\{J_k\}_{k=1}^{L}$ that minimizes D.

That is to find the encoder defined by the nonlinear mapping

$$\mathrm{g}(m) = v_k, \quad k = 1, 2, \ldots, L \tag{3.40}$$

such that we have

$$D = \int_{-A}^{A} d(m, g(m))\, f_M(m)\, dm \geq \sum_{k=1}^{L} \int_{\mathrm{m} \in J_k} \left[\min d(m, v_k)\right] f_M(m)\, dm \tag{3.41}$$

To attain the lower bound, if

$$d(m, v_k) \leq d(m, v_j) \quad \text{holds for all } j \neq k \tag{3.42}$$

This is called nearest neighbor condition.

Condition 2. Optimality of the decoder for a given encoder

Given the set $\{J_k\}_{k=1}^{L}$, find the set $\{v_k\}_{k=1}^{L}$ that minimized D.

For mean-square distortion

$$D = \sum_{\mathrm{k}=1}^{L} \int_{m \in J_k} (m - v_k)^2 f_M(m)\,dm,\ f_M(m) \qquad (3.43)$$

$$\frac{\partial D}{\partial v_k} = -2 \sum_{\mathrm{k}=1}^{L} \int_{m \in J_k} (m - v_k)^2 f_M(m)\,dm = 0 \qquad (3.44)$$

$$v_{k,\,\mathrm{opt}} = \frac{\int_{m \in J_k} m f_M(m)\,dm}{\int_{m \in J_k} f_M(m)\,dm} \qquad (3.45)$$

Probability $\mathrm{P_k}$ (given)

$$= E\left[M \middle| m_k < m \le m_{k+1}\right] \qquad (3.47)$$

Using iteration, until D reaches a minimum

Pulse Code Modulation

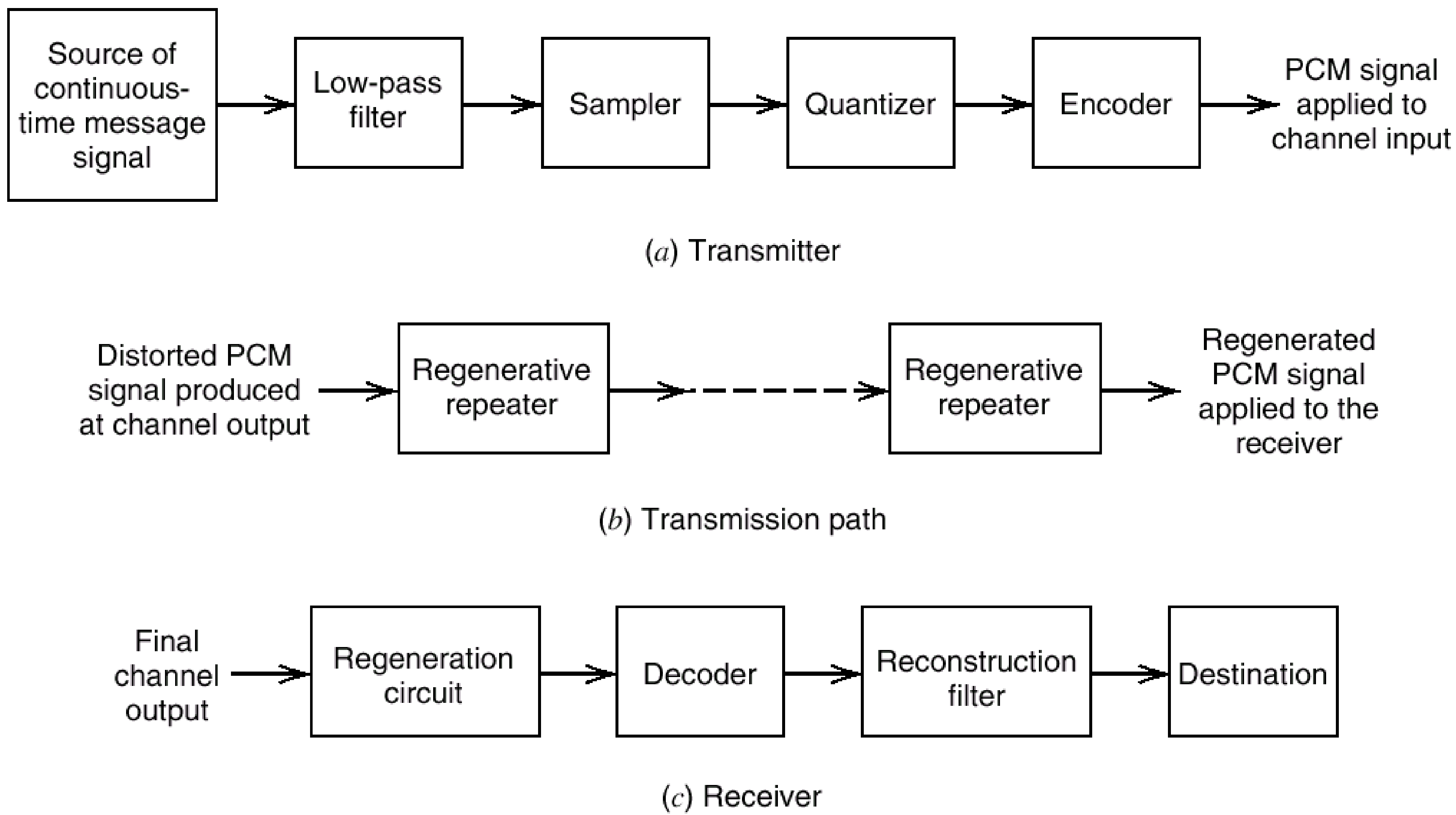


Figure 3.13 The basic elements of a PCM system.

Quantization (nonuniform quantizer)

μ - law

$$|v| = \frac{\log(1+\mu|m|)}{\log(1+\mu)} \qquad (3.48)$$

$$\frac{d|m|}{d|v|} = \frac{\log(1+\mu)}{\mu}(1+\mu|m|) \qquad (3.49)$$

A - law

$$|v| = \begin{cases} \dfrac{A(m)}{1+\log A} & 0 \le |m| \le \dfrac{1}{A} \\ \dfrac{1+\log(A|m|)}{1+\log A} & \dfrac{1}{A} \le |m| \le 1 \end{cases} \qquad (3.50)$$

$$\frac{d|m|}{d|v|} = \begin{cases} \dfrac{1+\log A}{A} & 0 \le |m| \le \dfrac{1}{A} \\ (1+A)|m| & \dfrac{1}{A} \le |m| \le 1 \end{cases} \qquad (3.51)$$

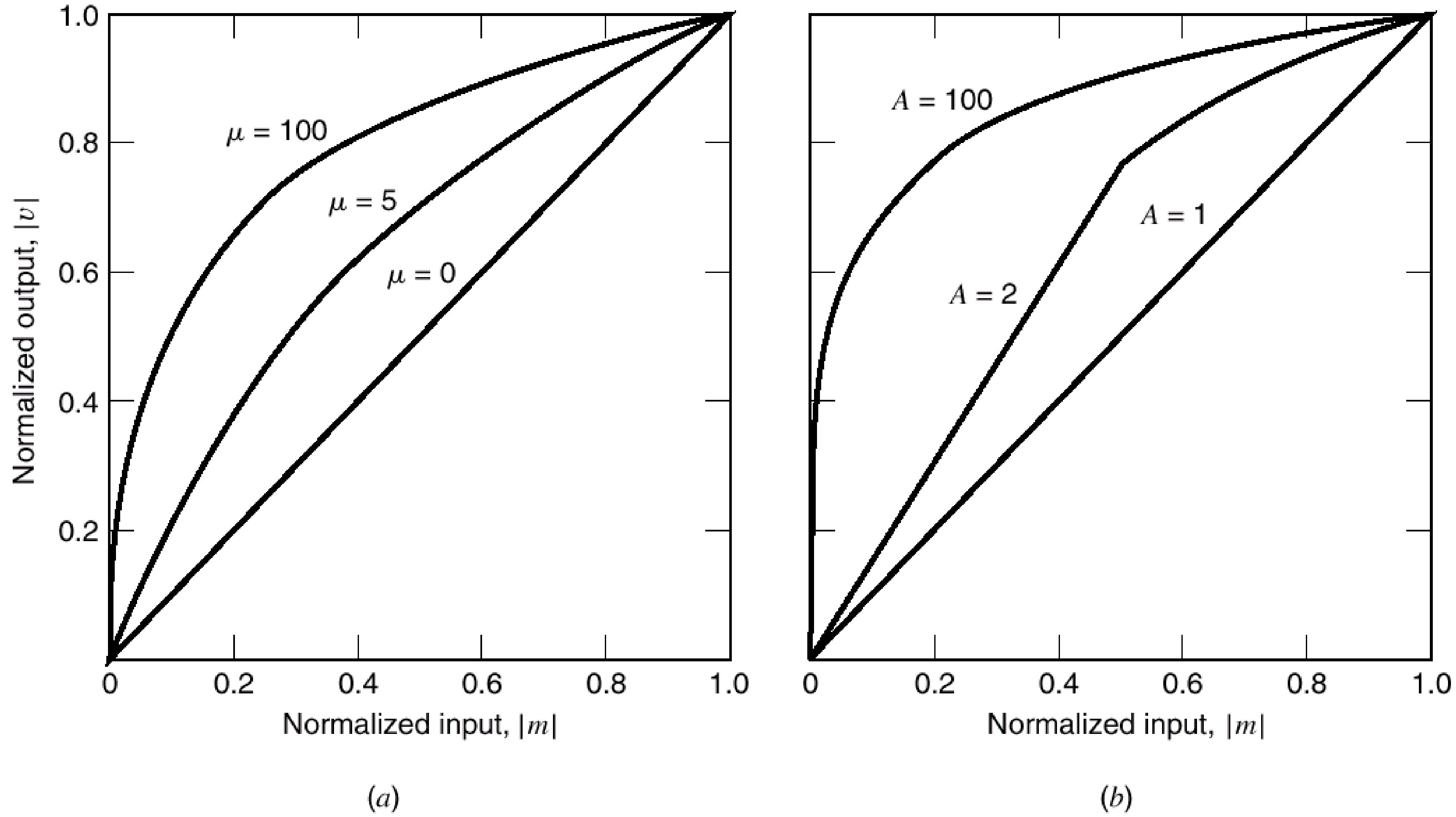


Figure 3.14 Compression laws. (*a*) μ -law. (*b*) A-law.

Encoding

TABLE 3.2 *Binary number system for R = 4 bits/sample*

Ordinal Number of Representation Level	Level Number Expressed as Sum of Powers of 2	Binary Number
0		0000
1	2^0	0001
2	2^1	0010
3	$2^1 + 2^0$	0011
4	2^2	0100
5	$2^2 + 2^0$	0101
6	$2^2 + 2^1$	0110
7	$2^2 + 2^1 + 2^0$	0111
8	2^3	1000
9	$2^3 + 2^0$	1001
10	$2^3 + 2^1$	1010
11	$2^3 + 2^1 + 2^0$	1011
12	$2^3 + 2^2$	1100
13	$2^3 + 2^2 + 2^0$	1101
14	$2^3 + 2^2 + 2^1$	1110
15	$2^3 + 2^2 + 2^1 + 2^0$	1111

Line codes:

1. Unipolar nonreturn-to-zero (NRZ) Signaling
2. Polar nonreturn-to-zero(NRZ) Signaling
3. Unipor nonreturn-to-zero (RZ) Signaling
4. Bipolar nonreturn-to-zero (BRZ) Signaling
5. Split-phase (Manchester code)

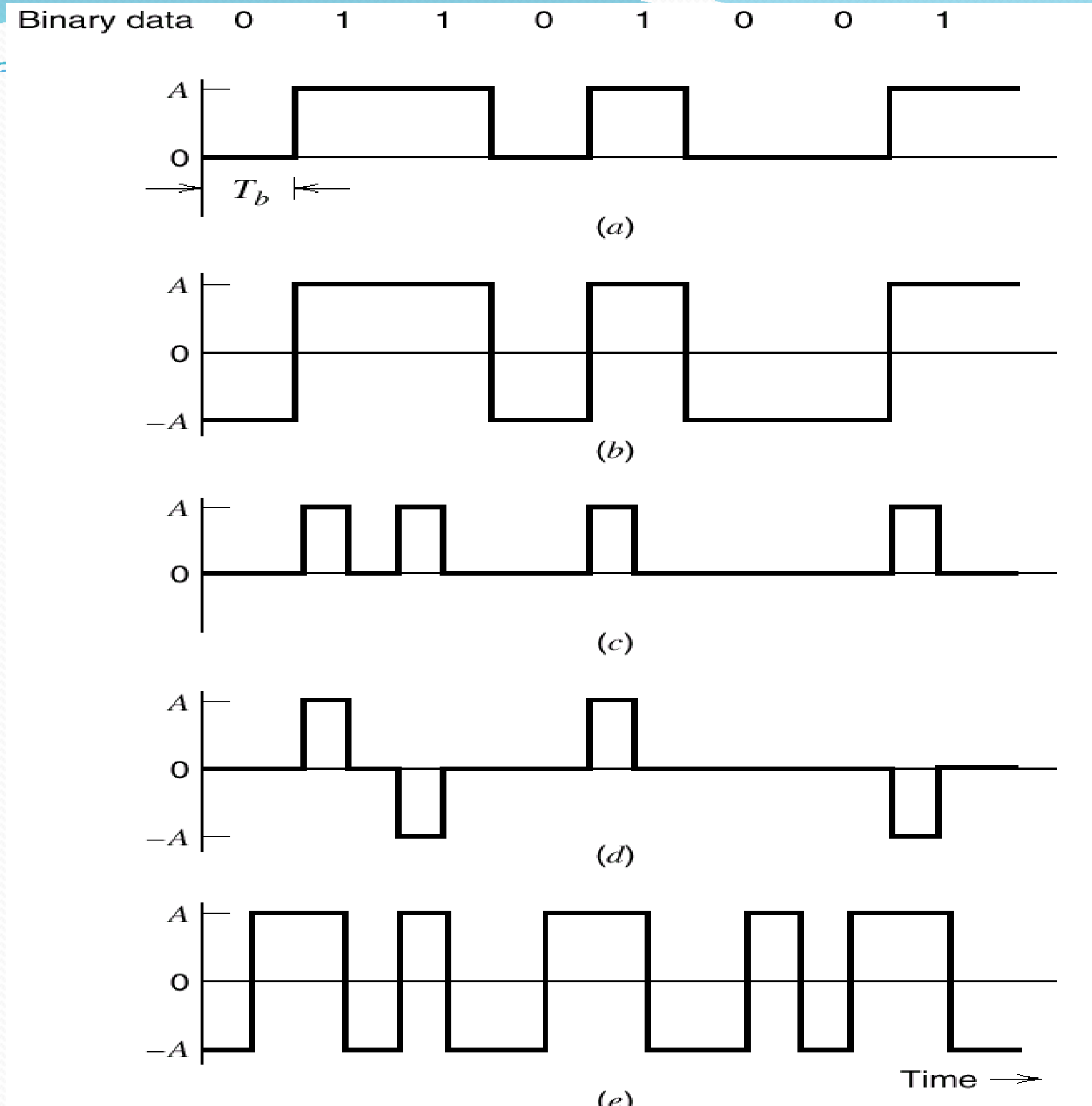


Figure 3.15 Line codes for the electrical representations of binary data.
(*a*) Unipolar NRZ signaling. (*b*) Polar NRZ signaling.
(c) Unipolar RZ signaling. (*d*) Bipolar RZ signaling.
(*e*) Split-phase or Manchester code.

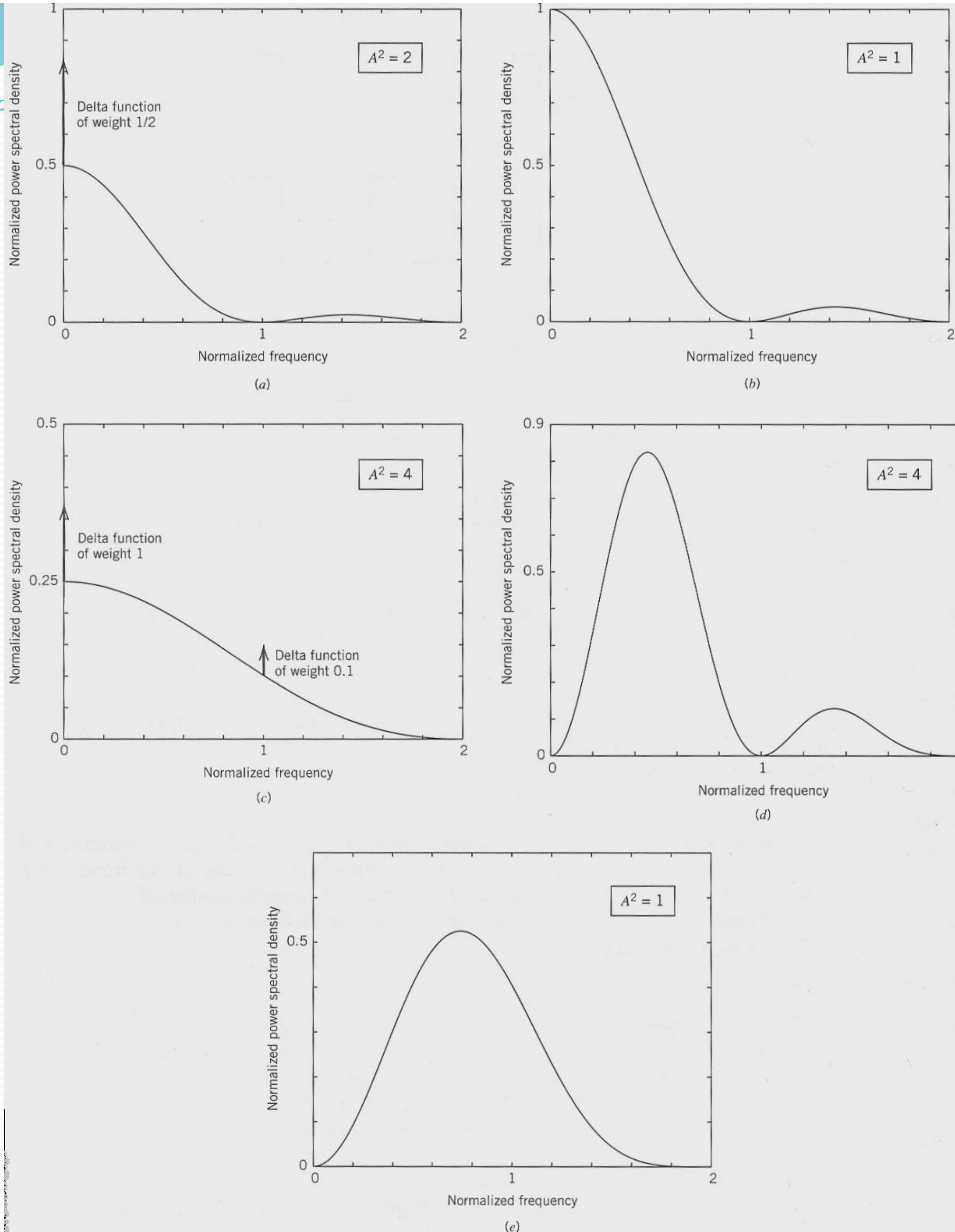
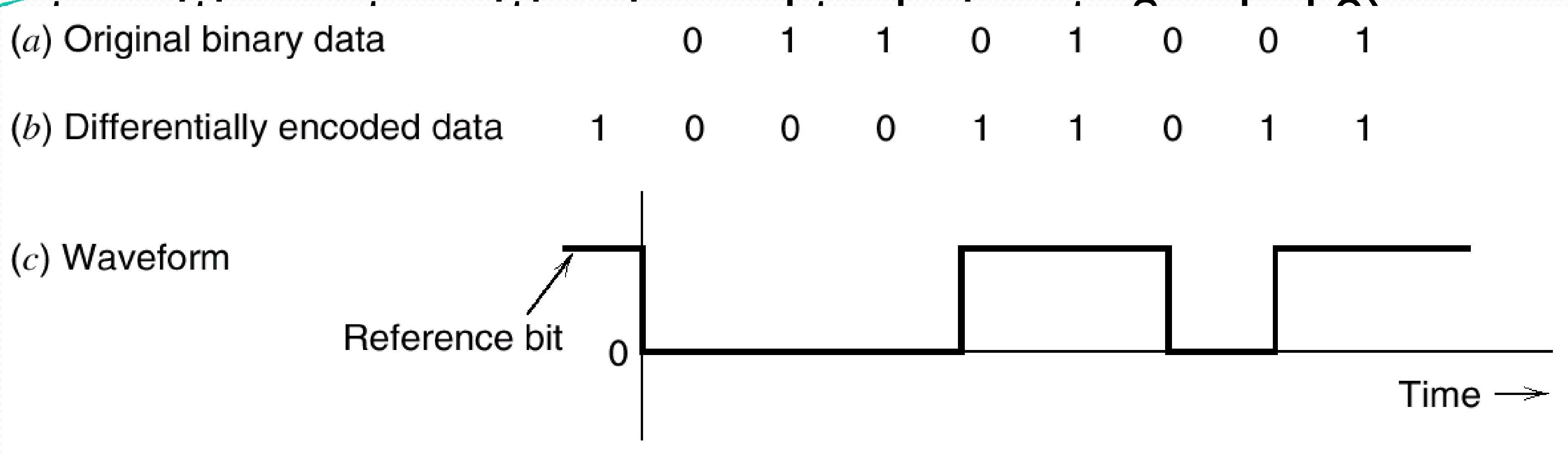
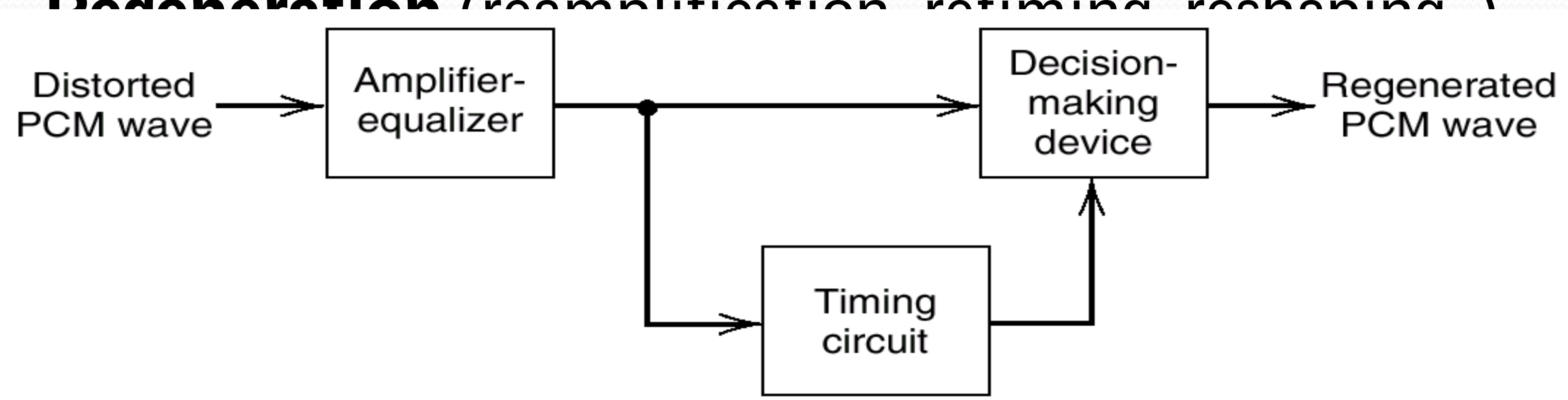


FIGURE 3.16 Power spectra of line codes: (*a*) Unipolar NRZ signal. (*b*) Polar NRZ signal. (*c*) Unipolar RZ signal. (*d*) Bipolar RZ signal. (*e*) Manchester-encoded signal. The frequency is normalized with respect to the bit rate $1/T_b$, and the average power is normalized to unity.

Differential Encoding (encode information in terms of signal



Regeneration (reamplification, retiming, reshaping .)



Two measure factors: bit error rate (BER) and jitter.

Decoding and Filtering

3.8 Noise consideration in PCM systems

(Channel noise, quantization noise)

(will be discussed in Chapter 4)

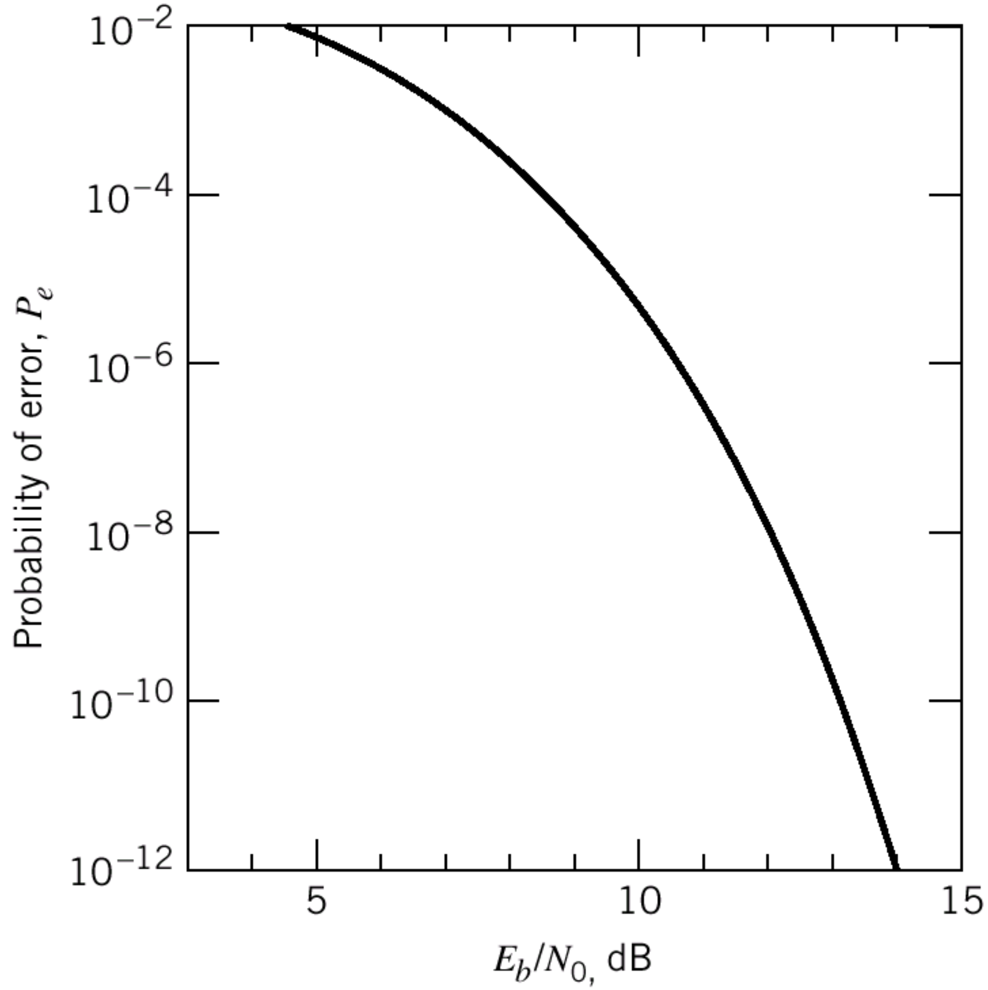


TABLE 3.3 *Influence of E_b/N_0 on the probability of error*

E_b/N_0	*Probability of Error P_e*	*For a Bit Rate of 10^5 b/s, This Is About One Error Every*
4.3 dB	10^{-2}	10^{-3} second
8.4	10^{-4}	10^{-1} second
10.6	10^{-6}	10 seconds
12.0	10^{-8}	20 minutes
13.0	10^{-10}	1 day
14.0	10^{-12}	3 months

Time-Division Multiplexing

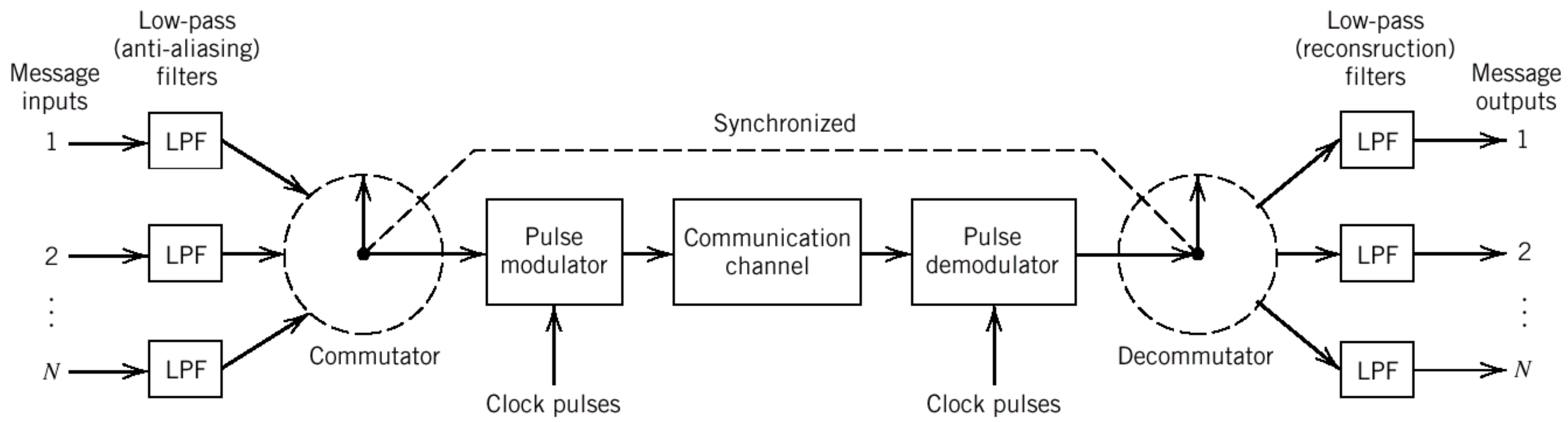


Figure 3.19 Block diagram of TDM system.

Synchronization

Example 2.2 The T1 System

TABLE 3.4 *The 15-segment companding characteristic (μ = 255)*

Linear Segment Number	*Step-Size*	*Projections of Segment End Points onto the Horizontal Axis*
0	2	±31
1a, 1b	4	±95
2a, 2b	8	±223
3a, 3b	16	±479
4a, 4b	32	±991
5a, 5b	64	±2015
6a, 6b	128	±4063
7a, 7b	256	±8159

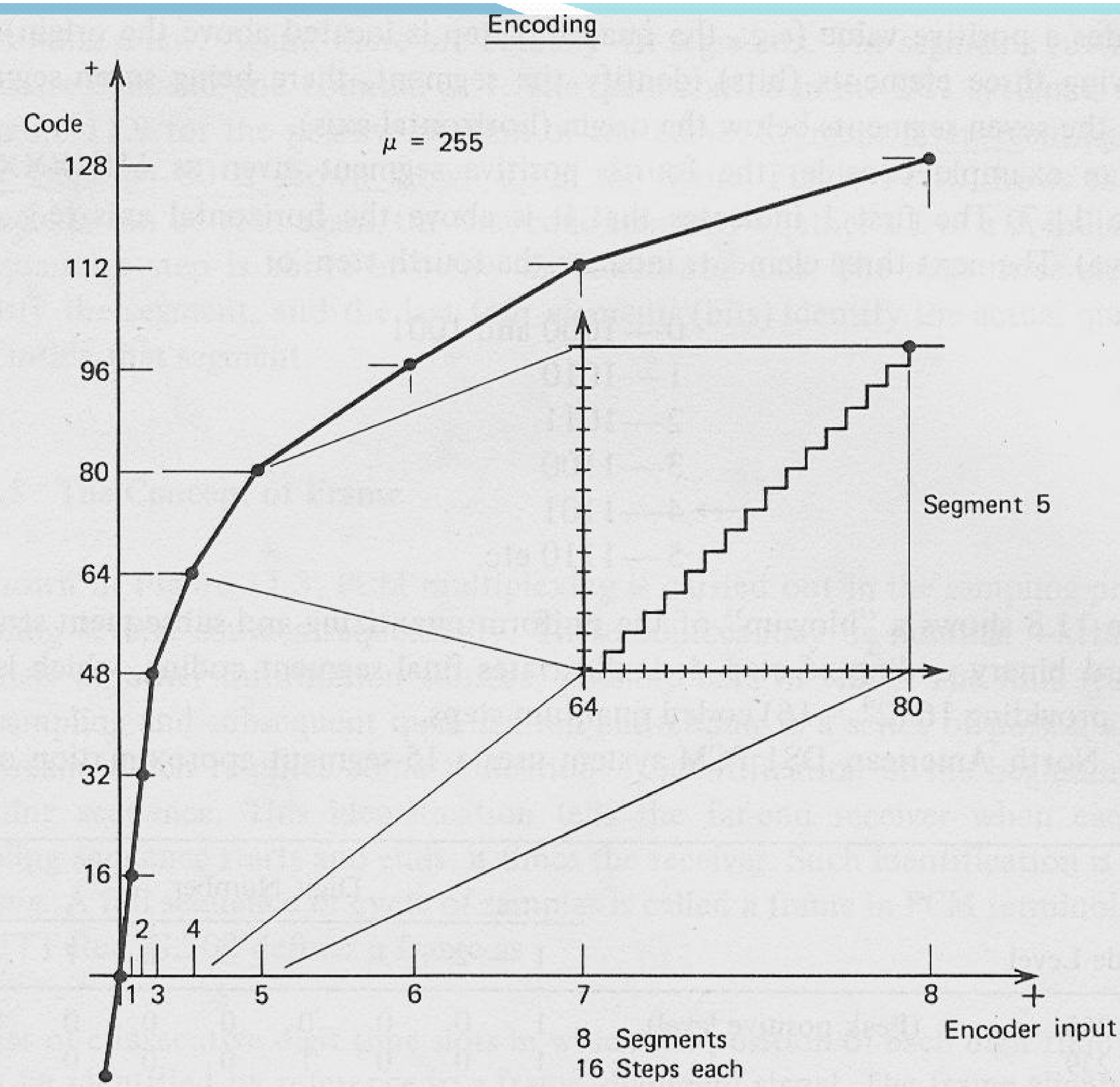
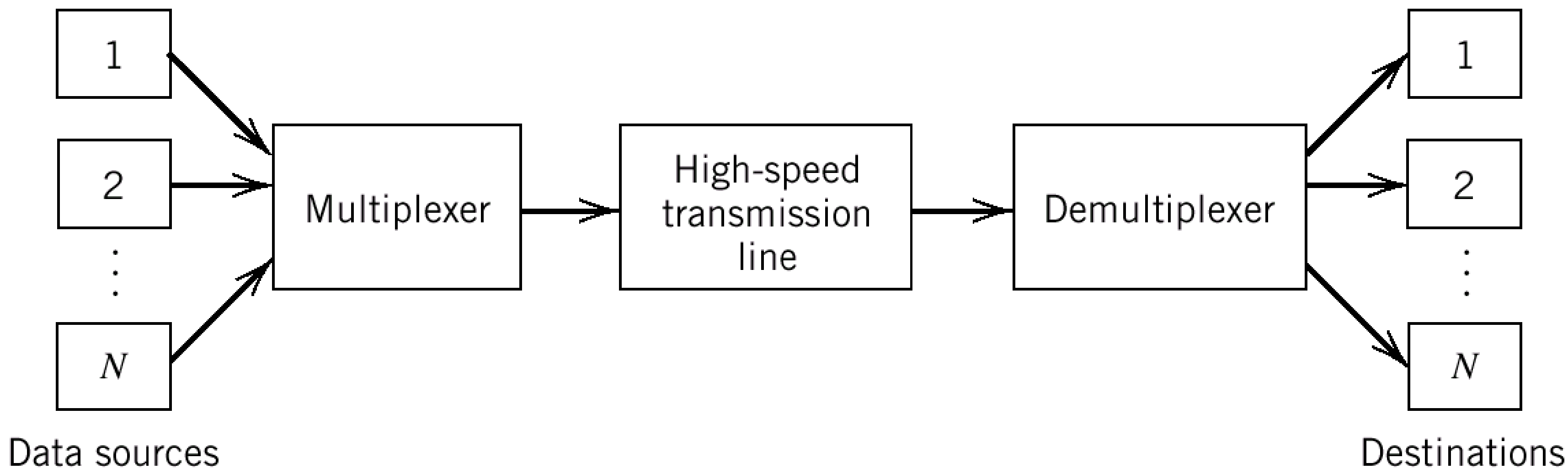


Figure 11.9 Positive portion of the segmented approximation of the μ-law quantizing curve used in the North American DS1 PCM channelizing equipment. (Courtesy ITT Telecommunications, Raleigh, NC.)

3.10 Digital Multiplexers

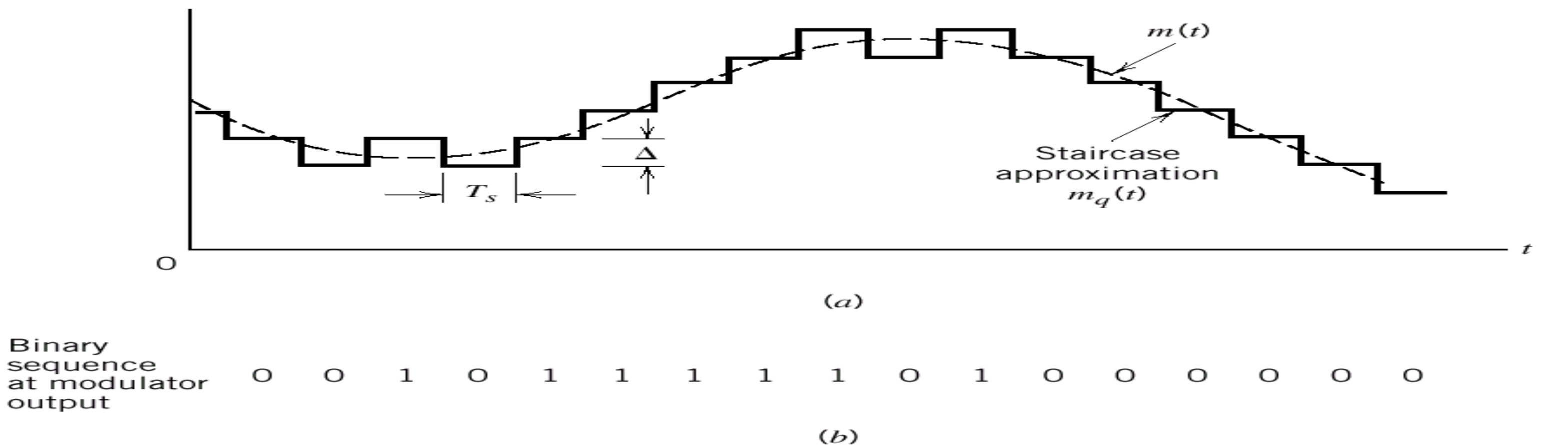


3.11 Virtues, Limitations and Modifications of PCM

Advantages of PCM

1. Robustness to noise and interference
2. Efficient regeneration
3. Efficient SNR and bandwidth trade-off
4. Uniform format
5. Ease add and drop
6. Secure

3.12 Delta Modulation (DM) (Simplicity)



Let $m[n] = m(nT_s)$, $n = 0, \pm 1, \pm 2, \ldots$

where T_s is the sampling period and $m(nT_s)$ is a sample of $m(t)$.

The error signal is

$$e[n] = m[n] - m_q[n-1] \qquad (3.52)$$

$$e_q[n] = \Delta \operatorname{sgn}(e[n]) \qquad (3.53)$$

$$m_q[n] = m_q[n-1] + e_q[n] \qquad (3.54)$$

where $m_q[n]$ is the quantizer output , $e_q[n]$ is the quantized version of $e[n]$, and Δ is the step size

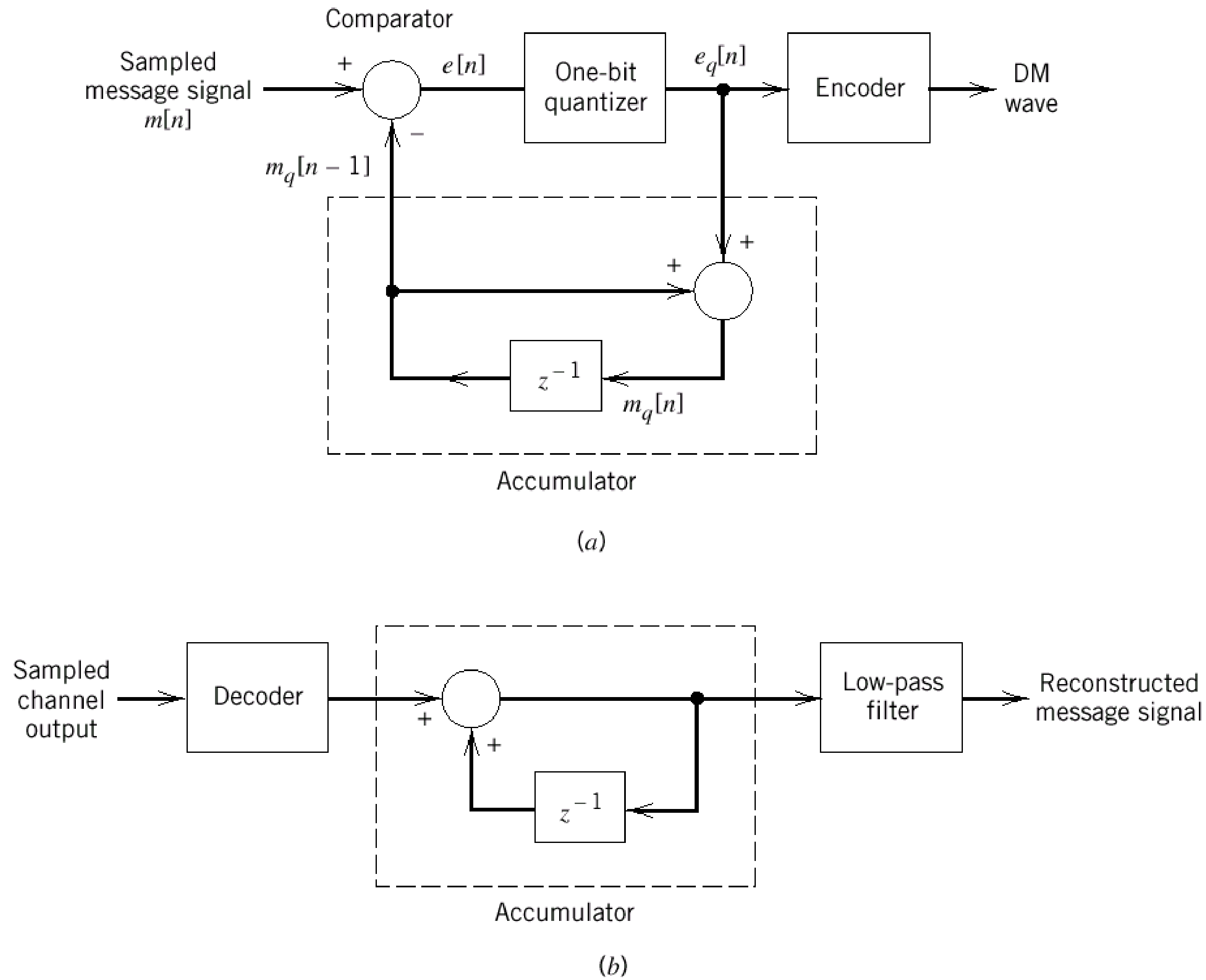


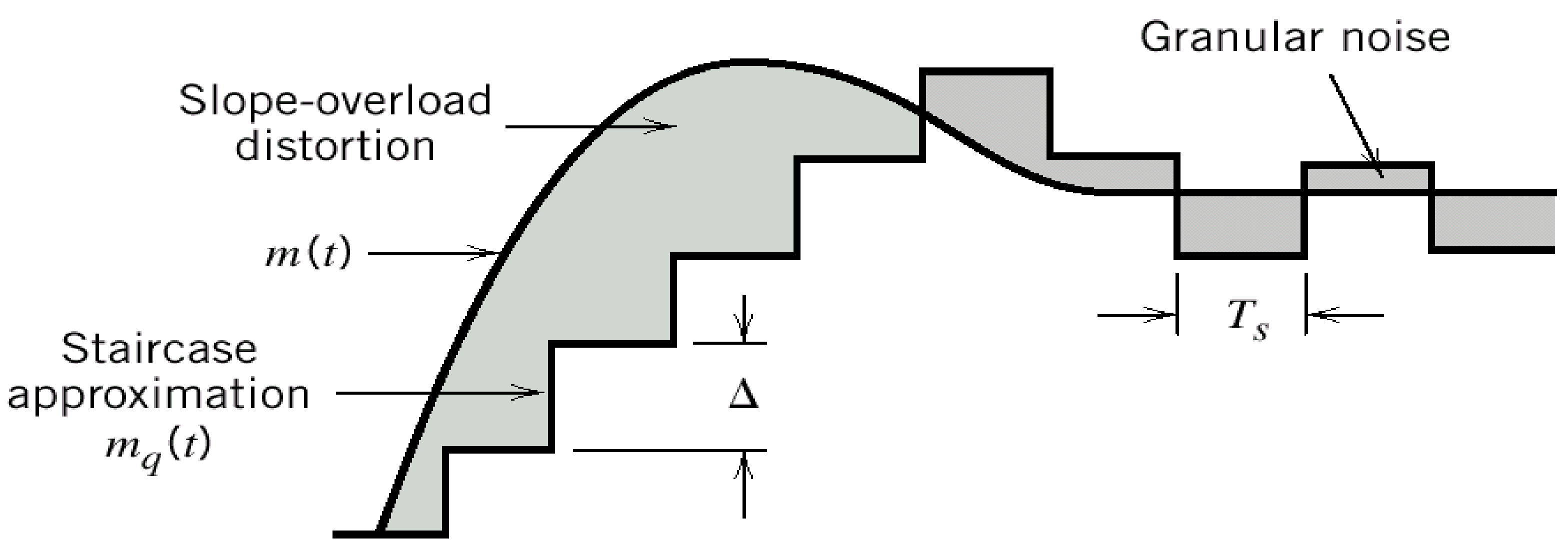
Figure 3.23 DM system. (*a*) Transmitter. (*b*) Receiver.

The modulator consists of a comparator, a quantizer, and an accumulator

The output of the accumulator is

$$m_q[n] = \Delta \sum_{i=1}^{n} \mathrm{sgn}(e[i])$$

$$= \sum_{i=1}^{n} e_q[i] \qquad (3.55)$$



Two types of quantization errors :

Slope overload distortion and **granular noise**

Slope Overload Distortion and Granular Noise

Denote the quantization error by $q[n]$,

$$m_q[n] = m[n] - q[n] \qquad (3.56)$$

Recall (3.52), we have

$$e[n] = m[n] - m[n-1] - q[n-1] \qquad (3.57)$$

Except for $q[n-1]$, the quantizer input is a first backward difference of the input signal **(differentiator)**

To avoid slope-overload distortion, we require

(slope) $$\frac{\Delta}{T_s} \geq \max\left|\frac{dm(t)}{dt}\right| \qquad (3.58)$$

On the other hand, granular noise occurs when step size Δ is too large relative to the local slope of $m(t)$.

Delta-Sigma modulation (sigma-delta modulation)

The $\Delta - \Sigma$ modulation which has an **integrator** can relieve the draw back of delta modulation (**differentiator**)

Beneficial effects of using integrator:

1. Pre-emphasize the low-frequency content
2. Increase correlation between adjacent samples
   (reduce the variance of the error signal at the quantizer input)
3. Simplify receiver design

Because the transmitter has an integrator , the receiver consists simply of a low-pass filter.

(The differentiator in the conventional DM receiver is cancelled by the integrator)

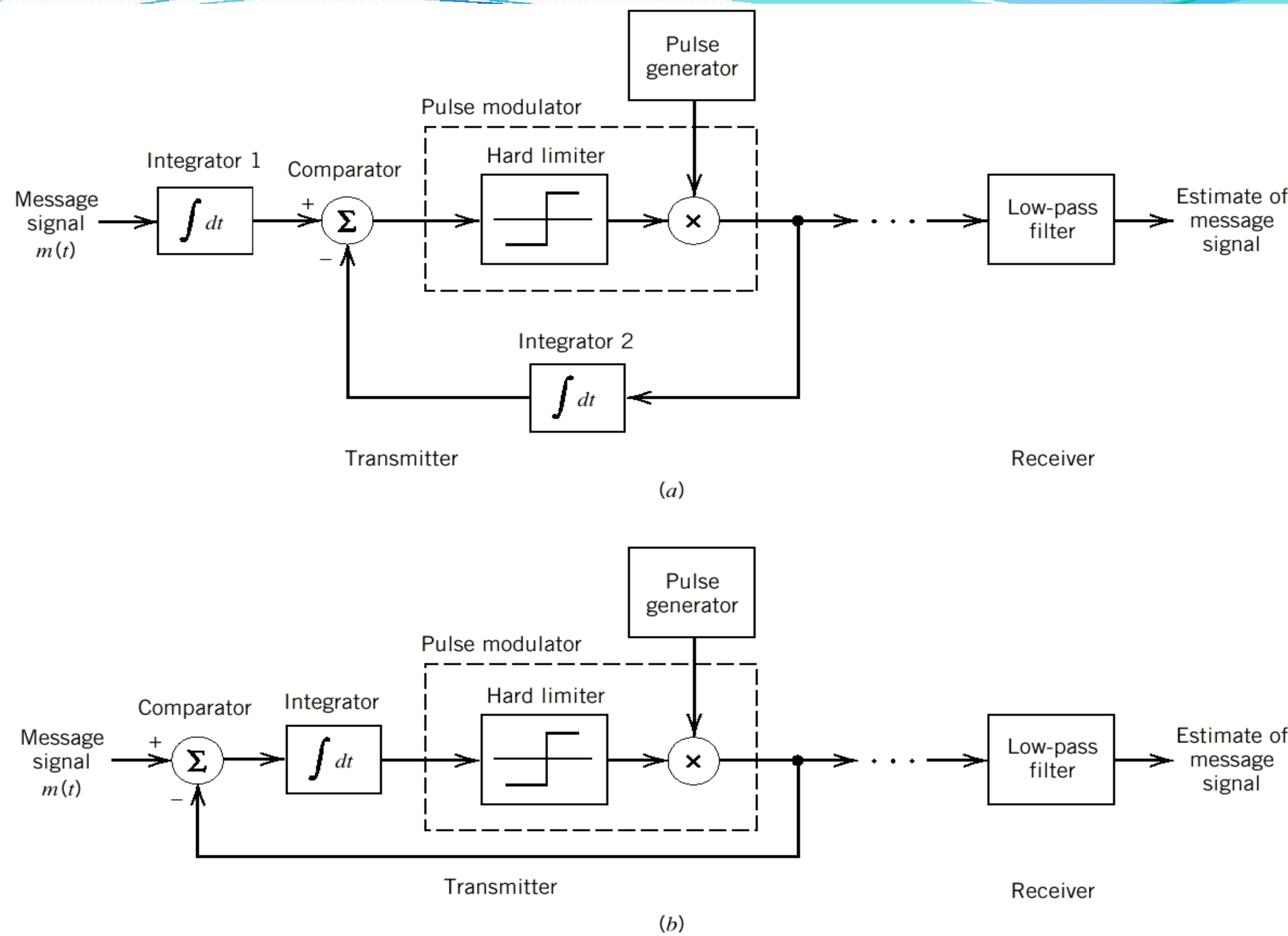


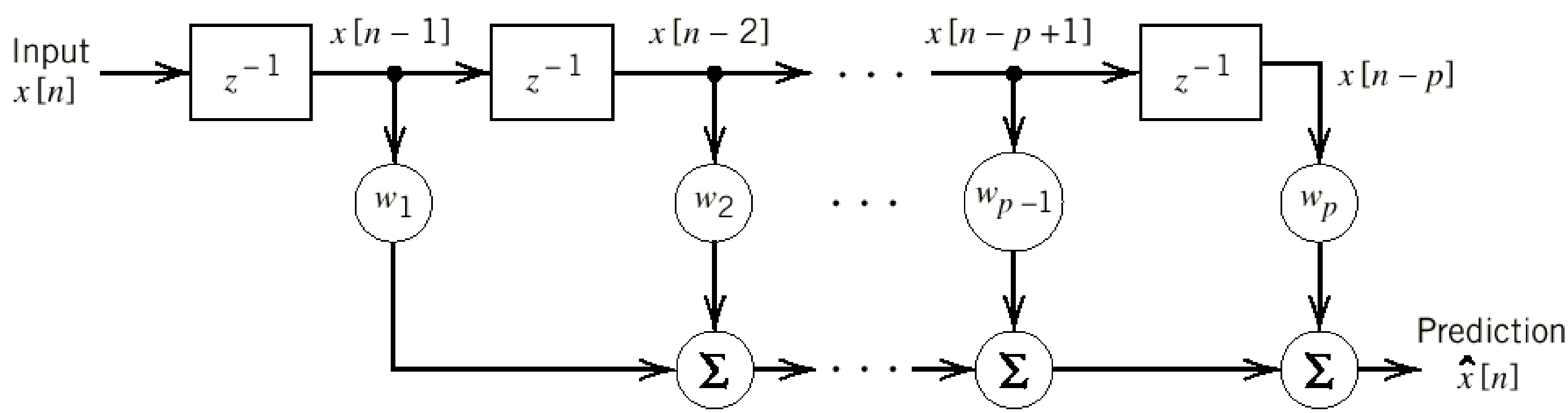
Figure 3.25 Two equivalent versions of delta-sigma modulation system.

3.13 Linear Prediction (to reduce the sampling rate)

Consider a finite-duration impulse response (FIR) discrete-time filter which consists of three blocks :

1. Set of p ($\boldsymbol{p}$**: prediction order**) unit-delay elements (z^{-1})

2. Set of multipliers with coefficients $w_1, w_2, \ldots w_p$



The filter output (The linear predition of the input) is

$$\hat{x}[n] = \sum_{k=1}^{p} w_k \, x(n-k) \qquad (3.59)$$

The prediction error is

$$e[n] = x[n] - \hat{x}[n] \qquad (3.60)$$

Let the index of performance be

$$J = E\left[e^2[n]\right] \text{ (mean square error)} \qquad (3.61)$$

Find $w_1, w_2, \ldots, w_p$ to minimize J

From (3.59) (3.60) and (3.61) we have

$$J = E\left[x^2[n]\right] - 2\sum_{k=1}^{p} w_k E\left[x[n]x[n-k]\right]$$

$$+ \sum_{j=1}^{p}\sum_{k=1}^{p} w_j w_k E\left[x[n-j]x[n-k]\right] \qquad (3.62)$$

Assume $X(t)$ is stationary process with zero mean $(E[x[n]]=0)$

$$\sigma_X^2 = E\left[x^2[n]\right] - \left(E\left[x[n]\right]\right)^2$$
$$= E\left[x^2[n]\right]$$

The autocorrelation

$$R_X(\tau = kT_s) = R_X[k] = E\left[x[n]x[n-k]\right]$$

We may simplify J as

$$J = \sigma_X^2 - 2\sum_{k=1}^{p} w_k R_X[k] + \sum_{j=1}^{p}\sum_{k=1}^{p} w_j w_k R_X[k-j] \qquad (3.63)$$

$$\frac{\partial J}{\partial w_k} = -2R_X[k] + 2\sum_{j=1}^{p} w_j R_X[k-j] = 0$$

$$\sum_{j=1}^{p} w_j R_X[k-j] = R_X[k] = R_X[-k], \; k = 1, 2, \ldots, p \; (3.64)$$

(3.64) are called Wiener - Hopf equations

For convenience, we may rewrite the Wiener-Hopf equations as , if $\mathbf{R}_X^{-1}$ exists $\quad \mathbf{w}_0 = \mathbf{R}_X^{-1}\mathbf{r}_X \qquad (3.66)$

where $\quad \mathbf{w}_0 = \left[w_1, w_2, \ldots, w_p\right]^T$

$$\mathbf{r}_X = [R_X[1], R_X[2], \ldots, R_X[p]]^T$$

$$\mathbf{R}_X = \begin{bmatrix} R_X[0] & R_X[1] & \cdots & R_X[p-1] \\ R_X[1] & R_X[0] & \cdots & R_X[p-2] \\ \vdots & \vdots & & \vdots \\ R_X[p-1] & R_X[p-2] & \cdots & R_X[0] \end{bmatrix}$$

The filter coefficients are uniquely determined by

$$R_X[0], R_X[1], \cdots, R_X[p]$$

Substituting (3.64) into (3.63) yields

$$J_{\min} = \sigma_X^2 - 2\sum_{k=1}^{p} w_k R_X[k] + \sum_{k=1}^{p} w_k R_X[k]$$

$$= \sigma_X^2 - \sum_{k=1}^{p} w_k R_X[k]$$

$$= \sigma_X^2 - \mathbf{r}_X^T \mathbf{w}_0 = \sigma_X^2 - \mathbf{r}_X^T \mathbf{R}_X^{-1} \mathbf{r}_X \qquad (3.67)$$

$\because \mathbf{r}_X^T \mathbf{R}_X^{-1} \mathbf{r}_X \ge 0, \therefore J_{\min}$ is always less than σ_X^2

Linear adaptive prediction (If $R_x[k]$ for varying k is not available)

The predictor is adaptive in the follow sense

1. Compute $w_k, k = 1,2,\ldots, p,$ starting any initial values
2. Do iteration using the method of steepest descent

Define the gradient vector

$$g_k = \frac{\partial J}{\partial w_k}\ ,\ k = 1,2,\ldots,p \qquad (3.68)$$

$w_k[n]$ denotes the value at iteration n. Then update $w_k[n+1]$

$$w_k[n+1] = w_k[n] - \frac{1}{2}\mu g_k\ , k = 1,2,\ldots,p \qquad (3.69)$$

where μ is a step-size parameter and $\frac{1}{2}$ is for convenience of presentation.

Differentiating (3.63), we have

$$g_k = \frac{\partial J}{\partial w_k} = -2R_X[k] + 2\sum_{j=1}^{P} w_j R_X[k-j]$$

$$= -2E[x[n]x[n-k]] + 2\sum_{j=1}^{p} w_j E[x[n-j]x[n-k]], k = 1,2,\ldots, p \qquad (3.70)$$

To simplify the computing we use $x[n]x[n-k]$ for E[x[n] x[n - k]] (ignore the expectation)

$$\hat{g}_k[n] = -2x[n]x[n-k] + 2\sum_{j=1}^{p} w_j[n]x[n-j]x[n-k], k = 1,2,\ldots, p \qquad (3.71)$$

Substituting (3.71) into (3.69)

$$\hat{w}_k[n+1] = \hat{w}_k[n] + \mu x[n-k]\left(x[n] - \sum_{j=1}^{p} \hat{w}_j[n]x[n-j]\right)$$

$$= \hat{w}_k[n] + \mu x[n-k]e[n] \quad , \; k = 1,2,\ldots, p \qquad (3.72)$$

where $e[n] = x[n] - \sum_{j=1}^{p} \hat{w}_j[n]x[n-j]$ by (3.59) + (3.60) (3.73)

The above equations are called lease - mean - square algorithm

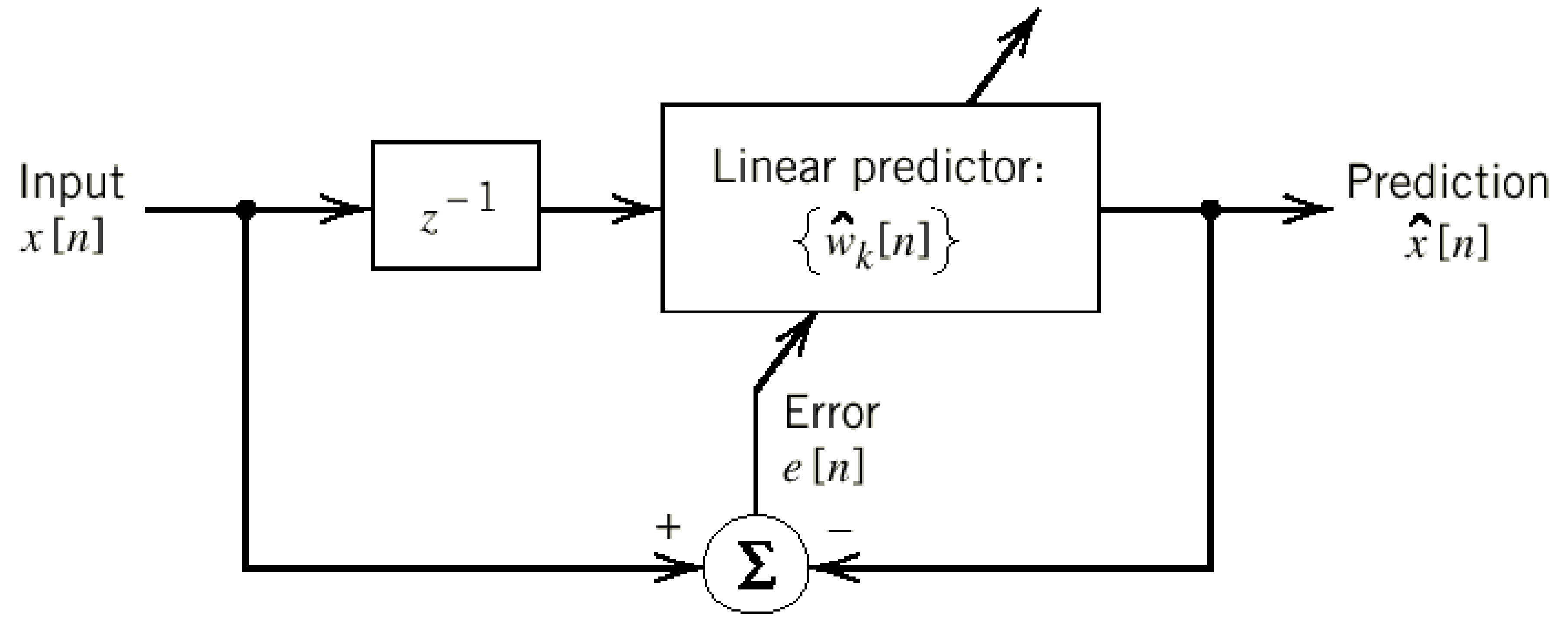


Figure 3.27
Block diagram illustrating the linear adaptive prediction process.

3.14 Differential Pulse-Code Modulation (DPCM)

Usually **PCM** has the sampling rate higher than the **Nyquist rate** .The encode signal contains redundant information. **DPCM** can efficiently remove this redundancy.

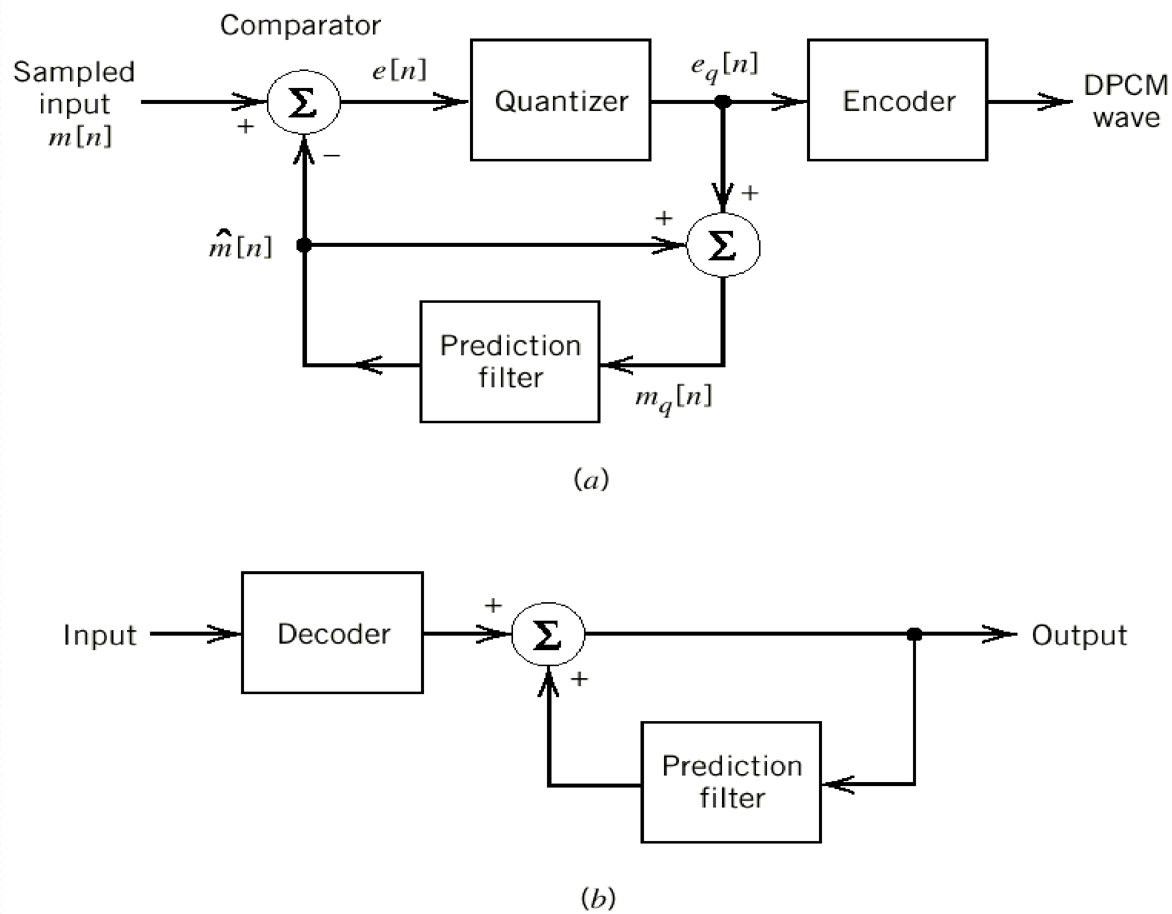


Figure 3.28 DPCM system. (*a*) Transmitter. (*b*) Receiver.

Input signal to the quantizer is defined by:

$$e[n] = m[n] - \hat{m}[n] \qquad (3.74)$$

$\hat{m}[n]$ is a prediction value.

The quantizer output is

$$e_q[n] = e[n] + q[n] \qquad (3.75)$$

where $q[n]$ is quantization error.

The prediction filter input is

$$m_q[n] = \underbrace{\hat{m}[n] + e[n]}_{m[n]} + q[n] \qquad (3.77)$$

From (3.74)

$$\Rightarrow m_q[n] = m[n] + q[n] \qquad (3.78)$$

Processing Gain

The $(\mathrm{SNR})_o$ of the DPCM system is

$$(\mathrm{SNR})_o = \frac{\sigma_M^2}{\sigma_Q^2} \qquad (3.79)$$

where σ_M^2 and σ_Q^2 are variances of $m[n]\,(E[m[n]] = 0)$ and $q[n]$

$$(\mathrm{SNR})_o = \left(\frac{\sigma_M^2}{\sigma_E^2}\right)\left(\frac{\sigma_E^2}{\sigma_Q^2}\right)$$

$$= G_p (\mathrm{SNR})_Q \qquad (3.80)$$

where σ_E^2 is the variance of the predictions error

and the signal-to-quantization noise ratio is

$$(\mathrm{SNR})_Q = \frac{\sigma_E^2}{\sigma_Q^2} \qquad (3.81)$$

Processing Gain, $\mathrm{G}_p = \dfrac{\sigma_M^2}{\sigma_E^2}$ (3.82)

Design a prediction filter to maximize G_p (minimize σ_E^2)

3.15 Adaptive Differential Pulse-Code Modulation (ADPCM)

Need for coding speech at low bit rates , we have two aims in mind:

1. Remove redundancies from the speech signal as far as possible.
2. Assign the available bits in a perceptually efficient manner.

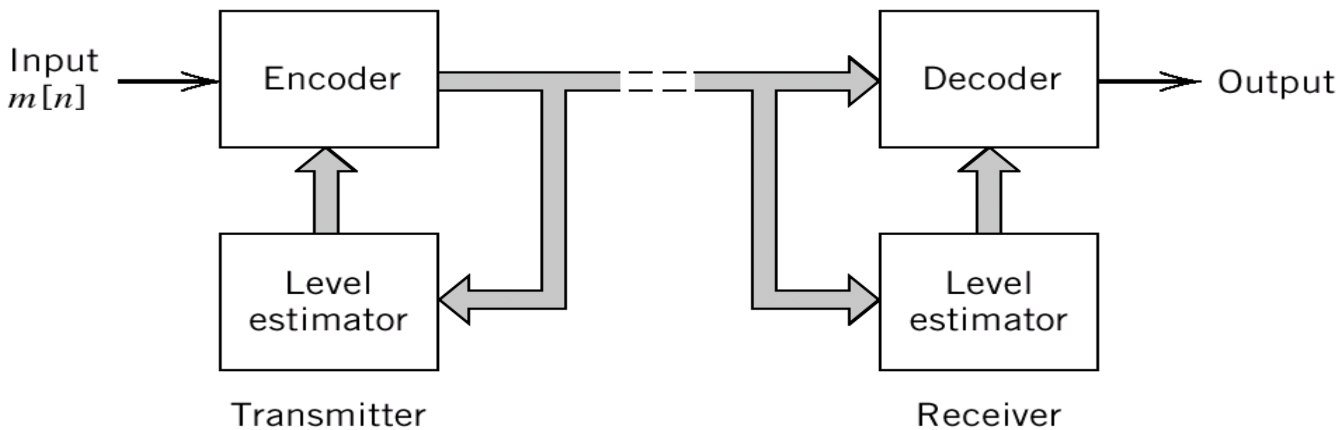


Figure 3.29 Adaptive quantization with backward estimation (AQB).

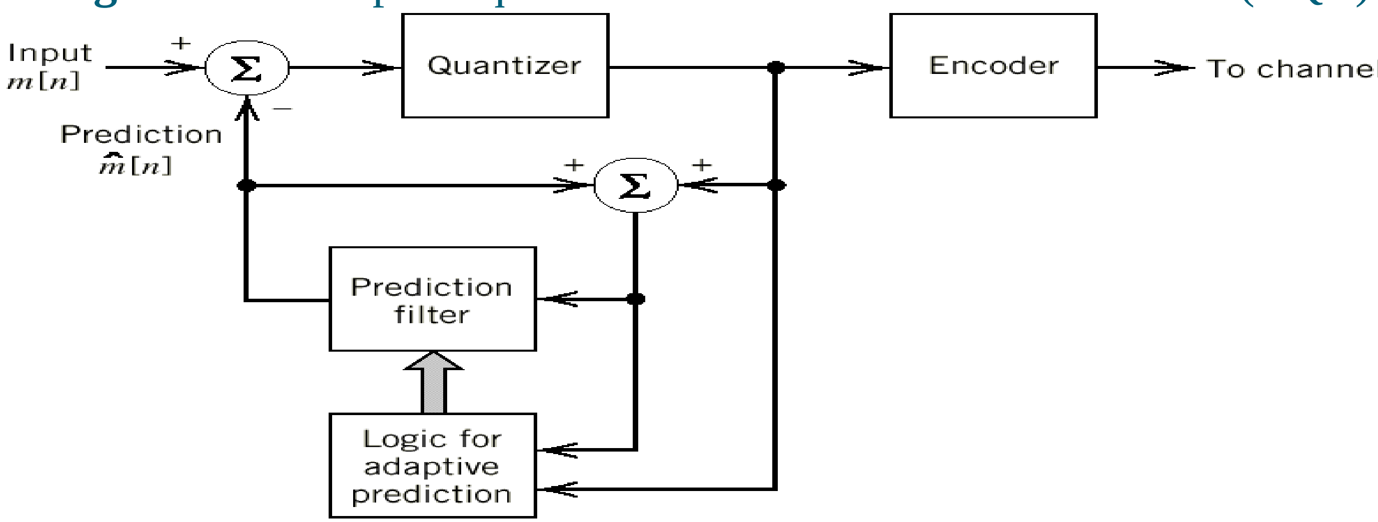


Figure 3.30 Adaptive prediction with backward estimation (APB).