Properties of the Fourier Transform

Basic Properties of the Fourier Transform

- **Linearity Property:** Given signals $x_1(t)$ and $x_2(t)$ with the Fourier transforms

$$\mathcal{F}[x_1(t)] = X_1(f)$$

$$\mathcal{F}[x_2(t)] = X_2(f).$$

The Fourier transform of $\alpha x_1(t) + \beta x_2(t)$ is

$$\mathcal{F}[\alpha x_1(t) + \beta x_2(t)] = \alpha X_1(f) + \beta X_2(f).$$

- **Duality Property:**

If $X(f) = \mathcal{F}[x(t)]$, then $x(f) = \mathcal{F}[X(-t)]$ and $x(-f) = \mathcal{F}[X(t)]$.

Proof:

$$\begin{aligned}\mathcal{F}[X(-t)] &= \int_{-\infty}^{\infty} X(-t) e^{-j2\pi ft} dt \\ &= \int_{-\infty}^{\infty} X(t) e^{j2\pi ft} dt \\ &= x(f).\end{aligned}$$

$$\begin{aligned}\mathcal{F}[X(t)] &= \int_{-\infty}^{\infty} X(t) e^{-j2\pi ft} dt \\ &= \int_{-\infty}^{\infty} X(t) e^{j2\pi(-f)t} dt \\ &= x(-f).\end{aligned}$$

- **Time Shift Property:** A shift of t_0 in the time origin causes a phase shift of $-2\pi f t_0$ in the frequency domain.

Proof: $$\mathcal{F}[x(t-t_0)] = e^{-j2\pi f t_0}\mathcal{F}[x(t)].$$

$$\mathcal{F}[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0)e^{-j2\pi ft}dt.$$

Let $t' = t - t_0$

$$\mathcal{F}[x(t-t_0)] = \int_{-\infty}^{\infty} x(t')e^{-j2\pi f(t'+t_0)}dt'$$

$$= e^{-j2\pi f t_0}\int_{-\infty}^{\infty} x(t')e^{-j2\pi ft'}dt'$$

$$= e^{-j2\pi f t_0}\mathcal{F}[x(t)] = e^{-j2\pi f t_0}X(f).$$

- **Scaling Property:** For any real $a \neq 0$, we have

$$\mathcal{F}[x(at)] = \frac{1}{|a|} X\left(\frac{f}{a}\right).$$

- Proof:

Case 1:

Let $a > 0$; we have

$$\mathcal{F}[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j2\pi ft} dt.$$

$t' = at$ $\quad dt = (1/a)dt'$

$$\mathcal{F}[x(at)] = \frac{1}{a}\int_{-\infty}^{\infty} x(t') e^{-j2\pi(f/a)t'} dt' = \frac{1}{a} X\left(\frac{f}{a}\right).$$

Case 2: $a < 0$

$$\mathcal{F}[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j2\pi ft} dt.$$

Let $t' = at$; we have

$$dt = (1/a)dt'$$

$$\mathcal{F}[x(at)] = \frac{1}{a}\int_{\infty}^{-\infty} x(t') e^{-j2\pi(f/a)t'} dt' = -\frac{1}{a} X\left(\frac{f}{a}\right).$$

- **Convolution Property:** If the signal $x(t)$ and $y(t)$ both possess Fourier transforms, then

Proof: $$\mathcal{F}\,[x(t) * y(t)] = \mathcal{F}\,[x(t)]\;\mathcal{F}\,[y(t)] = X(f)\,Y(f).$$

Convolution

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\,y(t-\tau)\,d\tau$$

$$\mathcal{F}\,[x(t) * y(t)] = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau)\,y(t-\tau)\,d\tau\right) e^{-j2\pi ft}\,dt$$

$$= \int_{-\infty}^{\infty} x(\tau)\left(\int_{-\infty}^{\infty} y(t-\tau)\,e^{-j2\pi ft}\,dt\right) d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\left(e^{-j2\pi f\tau}\,Y(f)\right) d\tau$$

$$= Y(f)\int_{-\infty}^{\infty} x(\tau)\,e^{-j2\pi f\tau}\,d\tau$$

$$= X(f)\,Y(f).$$

- **Modulation Property:** The Fourier transform of $x(t)\,e^{j2\pi f_0 t}$ $X(f-f_0)$
  is , and the Fourier transform of

  is $$x(t)\cos(2\pi f_0 t) = x(t)\frac{1}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right)$$

  $$\frac{1}{2}X(f-f_0) + \frac{1}{2}X(f+f_0).$$

Proof:

$$\mathcal{F}\,[x(t)e^{j2\pi f_0 t}] = \int_{-\infty}^{\infty} x(t)e^{j2\pi f_0 t}e^{-j2\pi f t}\,dt$$

$$= \int_{-\infty}^{\infty} x(t)\,e^{-j2\pi(f-f_0)t}\,dt$$

$$= X(f-f_0).$$

- **Parseval's Property:** If the Fourier transforms of $x(t)$ and $y(t)$ are denoted by $X(f)$ and $Y(f)$, respectively, then

$$\int_{-\infty}^{\infty} x(t)\, y^{*}(t)\, dt = \int_{-\infty}^{\infty} X(f)\, Y^{*}(f)\, df.$$

- proof:

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} X(u)e^{j2\pi ut}du\right)\left(\int_{-\infty}^{\infty} Y(v)e^{j2\pi vt}dv\right)^* dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(u)\,Y^*(v)\,e^{j2\pi ut}\ e^{-j2\pi vt}dt\,dv\,du$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(u)\,Y^*(v)\left(\int_{-\infty}^{\infty} e^{j2\pi(u-v)t}dt\right)dv\,du$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(u)\,Y^*(v)\,\delta(u-v)\,dv\,du$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(u)\,Y^*(u)\,\delta(u-v)\,dv\,du$$

$$= \int_{-\infty}^{\infty} X(u)Y^*(u)du$$

$$= \int_{-\infty}^{\infty} X(f)Y^*(f)df.$$

- **Rayleigh's Property:** If *X(f)* is the Fourier transform of *x(t)*, then

Proof: $$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df .$$

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt = \int_{-\infty}^{\infty} X(f) X^*(f)\, df = \int_{-\infty}^{\infty} |X(f)|^2 \, df .$$

Parseval's Property

- **Autocorrelation Property:** The (time) autocorrelation function of the aperiodic signal *x(t)* is denoted by $R_x(\tau)$ and is defined by

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt.$$

The autocorrelation property states that

$$\mathcal{F}\,[R_x(\tau)] = \left|X(f)\right|^2.$$

- **Differentiation Property:** The Fourier transform of the derivative of a signal can be obtained from the relation

$$\mathcal{F}\left[\frac{d}{dt}x(t)\right] = j2\pi f\; X(f).$$

- **Integration Property:** The Fourier transform of the integral of a signal can be determined from the relation

$$\mathcal{F}\left[\int_{-\infty}^{t} x(\tau)d\tau\right] = \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f).$$

- **Moments Property:** If $\mathcal{F}[x(t)] = X(f).$, then $\int_{-\infty}^{\infty} t^n x(t)dt$, the *n*th moment of *x(t),* can be obtained from the relation

$$\int_{-\infty}^{\infty} t^n x(t)dt = \left(\frac{j}{2\pi}\right)^n \frac{d^n}{df^n} X(f)\bigg|_{f=0}.$$

TABLE 2.1 TABLE OF FOURIER TRANSFORMS

Time Domain ($x(t)$)	Frequency Domain ($X(f)$)
$\delta(t)$	1
1	$\delta(f)$
$\delta(t-t_0)$	$e^{-j2\pi f t_0}$
$e^{j2\pi f_0 t}$	$\delta(f-f_0)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}\delta(f-f_0)+\frac{1}{2}\delta(f+f_0)$
$\sin(2\pi f_0 t)$	$-\frac{1}{2j}\delta(f+f_0)+\frac{1}{2j}\delta(f-f_0)$
$\Pi(t)=\begin{cases}1, & \lvert t\rvert<\frac{1}{2}\\ \frac{1}{2}, & t=\pm\frac{1}{2}\\ 0, & \text{otherwise}\end{cases}$	$\text{sinc}(f)$
$\text{sinc}(t)$	$\Pi(f)$
$\Lambda(t)=\begin{cases}t+1, & -1\le t<0\\ -t+1, & 0\le t<1\\ 0, & \text{otherwise}\end{cases}$	$\text{sinc}^2(f)$
$\text{sinc}^2(t)$	$\Lambda(f)$
$e^{-\alpha t}u_{-1}(t), \alpha>0$	$\frac{1}{\alpha+j2\pi f}$
$te^{-\alpha t}u_{-1}(t), \alpha>0$	$\frac{1}{(\alpha+j2\pi f)^2}$
$e^{-\alpha\lvert t\rvert}$	$\frac{2\alpha}{\alpha^2+(2\pi f)^2}$
$e^{-\pi t^2}$	$e^{-\pi f^2}$
$\text{sgn}(t)=\begin{cases}1, & t>0\\ -1, & t<0\\ 0, & t=0\end{cases}$	$1/(j\pi f)$
$u_{-1}(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$\delta'(t)$	$j2\pi f$
$\delta^{(n)}(t)$	$(j2\pi f)^n$
$\frac{1}{t}$	$-j\pi\,\text{sgn}(f)$
$\sum_{n=-\infty}^{n=+\infty}\delta(t-nT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{n=+\infty}\delta\left(f-\frac{n}{T_0}\right)$

Table 4: **Basic Continuous-Time Fourier Transform Pairs**

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

Table 3: **Properties of the Continuous-Time Fourier Transform**

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Property	**Aperiodic Signal**	**Fourier transform**
	$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time-shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency-shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time-Reversal	$x(-t)$	$X(-j\omega)$
Time- and Frequency-Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}X(j\omega) * Y(j\omega)$
Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \Re e\{X(j\omega)\} = \Re e\{X(-j\omega)\} \\ \Im m\{X(j\omega)\} = -\Im m\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\Re e\{X(j\omega)\}$ $j\Im m\{X(j\omega)\}$

Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Fourier Transform for Periodic Signals

- Let *x(t)* be a periodic signal with period T_0. Let $\{x_n\}$ denote the Fourier series coefficients corresponding to this signal. Then

$$x(t) = \sum_{n=-\infty}^{\infty} x_n \, e^{j2\pi \frac{n}{T_0} t}.$$

- Since

$$\mathcal{F}\left[e^{j2\pi \frac{n}{T_0} t} \right] = \delta\left(f - \frac{n}{T_0} \right),$$

we obtain

$$X(f) = \sum_{n=-\infty}^{\infty} x_n \, \delta\left(f - \frac{n}{T_0} \right).$$

- If we define the truncated signal $x_{T_0}(t)$ as

$$x_{T_0}(t) = \begin{cases} x(t), & -\frac{T_0}{2} < t \le \frac{T_0}{2} \\ 0, & \text{otherwise.} \end{cases}$$

we may have

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} x_{T_0}(t - nT_0) \\ &= x_{T_0}(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_0). \end{aligned}$$

- By using the convolution theorem, we obtain

$$X(f) = X_{T_0}(f)\left[\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T_0}\right)\right]$$

$$= \frac{1}{T_0}\sum_{n=-\infty}^{\infty}X_{T_0}(f)\delta\left(f-\frac{n}{T_0}\right)$$

$$= \frac{1}{T_0}\sum_{n=-\infty}^{\infty}X_{T_0}\left(\frac{n}{T_0}\right)\delta\left(f-\frac{n}{T_0}\right).$$

Comparing this result with $X(f) = \sum_{n=-\infty}^{\infty} x_n\delta\left(f-\frac{n}{T_0}\right),$

we conclude

$$x_n = \frac{1}{T_0}X_{T_0}\left(\frac{n}{T_0}\right).$$

- Alternative way to find the Fourier series coefficients. Given the periodic signal *x(t)*, we carry out the following steps to find :

1. Find the truncated x_n signal .

2. Determine the Fourier transform $x_{T_0}(t)$ of the truncated signal.

3. Evaluate the Fourier transform of the truncated signal $X_{T_0}(f)$ at $f = \frac{n}{T_0}$, to obtain the *n*th harmonic and multiply by $1/T_0$.

- Example 2.2.3: Determine the Fourier series coefficients of the signal x(t) shown in Figure 2.2.

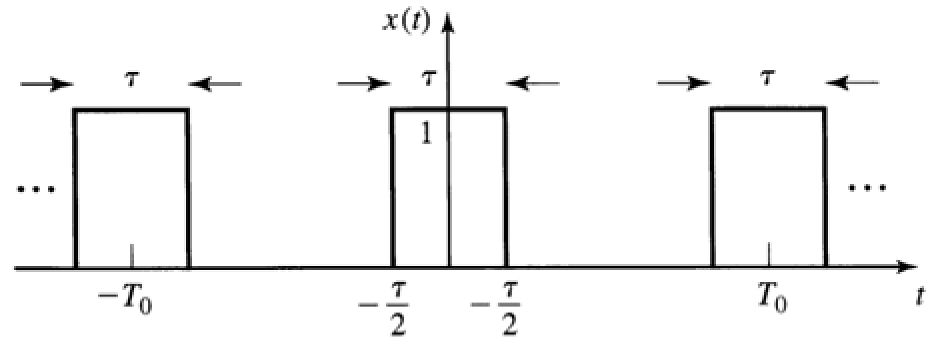

Figure 2.2 Periodic signal $x(t)$.

Solution: The truncated signal is

$$x_{T_0}(t) = \Pi\left(\frac{t}{\tau}\right)$$

and its Fourier transform is

$$X_{T_0}(f) = \tau \ \mathrm{sinc}(\tau f).$$

Therefore,

$$x_n = \frac{\tau}{T_0}\mathrm{sinc}\left(\frac{n\tau}{T_0}\right).$$