Convolution Integral

Given two functions $f_1(t)$ and $f_2(t)$, which are zero for $t < 0$, the convolution theorem states that if the transform of $f_1(t)$ is $F_1(s)$, and if the transform of $f_2(t)$ is $F_2(s)$, the transform of the *convolution* of $f_1(t)$ and $f_2(t)$ is the product of the individual transforms, $F_1(s)\,F_2(s)$, that is,

$$\mathcal{L}\left[\int_0^t f_1(t-\tau) f_2(\tau)\, d\tau\right] = F_1(s)\, F_2(s) \tag{7.97}$$

where the integral

$$\int_0^t f_1(t-\tau) f_2(\tau)\, d\tau$$

is the *convolution integral* or *folding integral*, and is denoted operationally as

$$\int_0^t f_1(t-\tau) f_2(\tau)\, d\tau = f_1(t)^* f_2(t) \tag{7.98}$$

Convolution integral(cont...)

Proof. Let us prove that $\mathfrak{L}[f_1 * f_2] = F_1 F_2$. We begin by writing

$$\mathfrak{L}[f_1(t) * f_2(t)] = \int_0^\infty e^{-st}\left[\int_0^t f_1(t-\tau) f_2(\tau)\, d\tau\right] dt \tag{7.99}$$

From the definition of the shifted step function

$$\begin{aligned} u(t-\tau) &= 1 \qquad \tau \le t \\ &= 0 \qquad \tau > t \end{aligned} \tag{7.100}$$

Cont.....

we have the identity

$$\int_0^t f_1(t-\tau) f_2(\tau)\, d\tau = \int_0^\infty f_1(t-\tau)\, u(t-\tau) f_2(\tau)\, d\tau \qquad (7.101)$$

Then Eq. 7.99 can be written as

$$\mathcal{L}[f_1(t) * f_2(t)] = \int_0^\infty e^{-st} \int_0^\infty f_1(t-\tau)\, u(t-\tau) f_2(\tau)\, d\tau\, dt \qquad (7.102)$$

If we let $x = t - \tau$ so that

$$e^{-st} = e^{-s(x+\tau)} \qquad (7.103)$$

Cont.....

then Eq. 7.102 becomes

$$\mathfrak{L}[f_1(t)^* f_2(t)] = \int_0^\infty \int_0^\infty f_1(x)\, u(x) f_2(\tau) e^{-s\tau} e^{-sx}\, d\tau\, dx$$

$$= \int_0^\infty f_1(x)\, u(x) e^{-sx}\, dx \int_0^\infty f_2(\tau) e^{-s\tau}\, d\tau$$

$$= F_1(s)\, F_2(s) \tag{7.104}$$

The separation of the double integral in Eq. 7.104 into a product of two integrals is based upon a property of integrals known as the *separability property.*[2]

THE DUHAMEL SUPERPOSITION INTEGRAL

this section we will study the *Duhamel superposition integral*, which also describes an input-output relationship for a system. The superposition integral requires the step response $\alpha(t)$ to characterize the system behavior.

We plan to derive the superposition integral in two different ways.

The simplest is examined first. We begin with the excitation-response relationship

$$R(s) = E(s)\,H(s) \tag{7.128}$$

Multiplying and dividing by s gives

$$R(s) = \frac{H(s)}{s} \cdot s\,E(s) \tag{7.129}$$

Taking inverse transforms of both sides gives

$$\begin{aligned} \mathcal{L}^{-1}[R(s)] &= \mathcal{L}^{-1}\left[\frac{H(s)}{s} \cdot s\,E(s)\right] \\ &= \mathcal{L}^{-1}\left[\frac{H(s)}{s}\right] * \mathcal{L}^{-1}[s\,E(s)] \end{aligned} \tag{7.130}$$

which then yields

$$\begin{aligned} r(t) &= \alpha(t) * [e'(t) + e(0-)\,\delta(t)] \\ &= e(0-)\,\alpha(t) + \int_{0-}^{t} e'(\tau)\,\alpha(t-\tau)\,d\tau \end{aligned} \tag{7.131}$$

where $e'(t)$ is the derivative of $e(t)$, $e(0-)$ is the value of $e(t)$ at $t = 0-$, and $\alpha(t)$ is the step response of the system. Equation 7.131 is usually referred to as the *Duhamel superposition integral.*