LAPLACE TRANSFORM

THE LAPLACE TRANSFORM

The Laplace transform of a function of time $f(t)$ is defined as

$$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$$

where s is the complex frequency variable

$$s = \sigma + j\omega$$

The inverse transform $\mathcal{L}^{-1}[F(s)]$ is

$$f(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)e^{st}\,ds$$

PROPERTIES OF LAPLACE TRANSFORM

Linearity

The transform of a finite sum of time functions is the sum of the transforms of the individual functions, that is

$$\mathcal{L}\left[\sum_i f_i(t)\right] = \sum_i \mathcal{L}[f_i(t)]$$

This property follows readily from the definition of the Laplace transform.

Real differentiation

Given that $\mathcal{L}[f(t)] = F(s)$, then

$$\mathcal{L}\left[\frac{df}{dt}\right] = s\,F(s) - f(0-)$$

where $f(0-)$ is the value of $f(t)$ at $t = 0-$.

Proof. By definition,

$$\mathcal{L}[f'(t)] = \int_{0-}^{\infty} e^{-st} f'(t)\,dt$$

Integrating Eq. 6.19 by parts, we have

$$\mathcal{L}[f'(t)] = e^{-st} f(t)\Big|_{0-}^{\infty} + s\int_{0-}^{\infty} f(t)e^{-st}\,dt$$

Since $e^{-st} \to 0$ as $t \to \infty$, and because the integral on the right-hand side is $\mathcal{L}[f(t)] = F(s)$, we have

$$\mathcal{L}[f'(t)] = s\,F(s) - f(0-)$$

Similarly, we can show for the nth derivative

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0-) - s^{n-2} f'(0-) - \cdots - f^{(n-1)}(0-)$$

Real integration

If $\mathcal{L}[f(t)] = F(s)$, then the Laplace transform of the integral of $f(t)$ is $F(s)$ divided by s, that is,

$$\mathcal{L}\left[\int_{0-}^{t} f(\tau)\, d\tau\right] = \frac{F(s)}{s}$$

Proof. By definition,

$$\mathcal{L}\left[\int_{0-}^{t} f(\tau)\, d\tau\right] = \int_{0-}^{\infty} e^{-st}\left[\int_{0-}^{t} f(\tau)\, d\tau\right] dt$$

Integrating by parts, we obtain

$$\mathcal{L}\left[\int_{0-}^{t} f(\tau)\, d\tau\right] = -\frac{e^{-st}}{s}\int_{0-}^{t} f(\tau)\, d\tau\Big|_{0-}^{\infty} + \frac{1}{s}\int_{0-}^{\infty} e^{-st} f(t)\, dt$$

Since $e^{-st} \to 0$ as $t \to \infty$, and since

$$\int_{0-}^{t} f(\tau)\, d\tau\Big|_{t=0-} = 0$$

we then have

$$\mathcal{L}\left[\int_{0-}^{t} f(\tau)\, d\tau\right] = \frac{F(s)}{s}$$

Differentiation by s

Differentiation by s in the complex frequency domain corresponds to multiplication by t in the domain, that is,

$$\mathcal{L}[tf(t)] = -\frac{d\,F(s)}{ds}$$

Proof. From the definition of the Laplace transform, we see that

$$\frac{d\,F(s)}{ds} = \int_{0-}^{\infty} f(t)\frac{d}{ds}e^{-st}\,dt = -\int_{0-}^{\infty} t\,f(t)e^{-st}\,dt = -\mathcal{L}[t\,f(t)]$$

Example Given $f(t) = e^{-\alpha t}$, whose transform is

$$F(s) = \frac{1}{s+\alpha}$$

let us find $\mathcal{L}[te^{-\alpha t}]$. By the preceding theorem, we find that

$$\mathcal{L}[te^{-\alpha t}] = -\frac{d}{ds}\left(\frac{1}{s+\alpha}\right) = \frac{1}{(s+\alpha)^2}$$

Similarly, we can show that

$$\mathcal{L}[t^n e^{-\alpha t}] = \frac{n!}{(s+\alpha)^{n+1}}$$

where n is a positive integer.

Complex translation

By the complex translation property, if $F(s) = \mathcal{L}[f(t)]$, then

$$F(s-a) = \mathcal{L}[e^{at} f(t)]$$

where a is a complex number.

Proof. By definition,

$$\mathcal{L}[e^{at} f(t)] = \int_{0-}^{\infty} e^{at} f(t) e^{-st}\, dt = \int_{0-}^{\infty} e^{-(s-a)t} f(t)\, dt = F(s-a)$$

Real translation (shifting theorem)

Here we consider the very important concept of the transform of a *shifted* or *delayed* function of time. If $\mathcal{L}[f(t)] = F(s)$, then the transform of the function delayed by time a is

Periodic waveforms

- The laplace transform of a periodic waveforms can be obtained in two ways: (1) Through Summation of an infinite series and (2) Through formula derived below

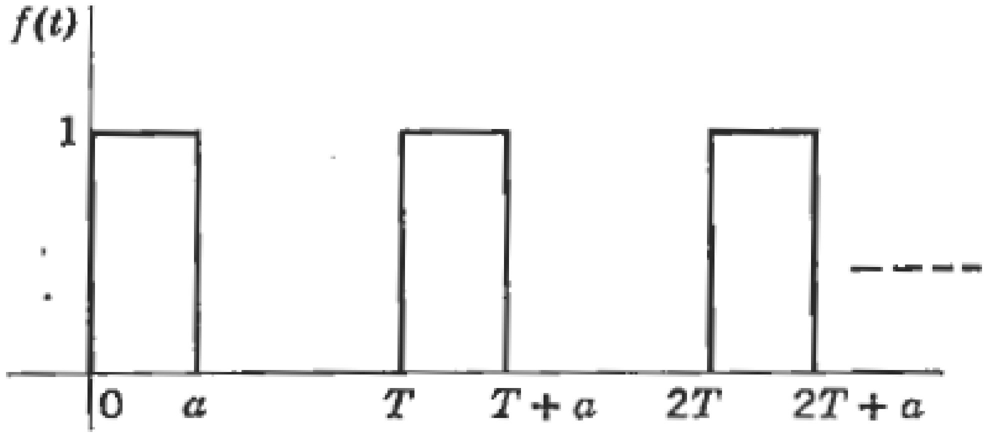
$$\mathfrak{L}[f(t)] = \int_{0-}^{T} f(t)e^{-st}\,dt + \int_{T}^{2T} f(t)e^{-st}\,dt + \cdots + \int_{nT}^{(n+1)T} f(t)e^{-st}\,dt + \cdots$$

$$\mathfrak{L}[f(t)] = \int_{0-}^{T} f(t)e^{-st}\,dt + e^{-sT}\int_{0-}^{T} f(t)e^{-st}\,dt + \cdots + e^{-snT}\int_{0-}^{T} f(t)e^{-st}\,dt + \cdots$$

$$= (1 + e^{-sT} + e^{-s2T} + \cdots)\int_{0-}^{T} f(t)e^{-st}\,dt$$

$$= \frac{1}{1 - e^{-sT}}\int_{0-}^{T} f(t)e^{-st}\,dt$$

- Given the periodic pulse train in fig. Determine its Laplace transform.



Periodic pulse Train

Solution:

$$F(s) = \frac{1}{1 - e^{-sT}} \int_{0-}^{a} e^{-st}\, dt$$

$$= \frac{-1}{s(1 - e^{-sT})} e^{-st} \Big|_{0-}^{a}$$

$$= \frac{1}{s} \frac{1 - e^{-as}}{1 - e^{-sT}}$$

Poles and Zeros

- The poles and zeros of given rational function F(s) can be defined as
- Poles of F(s) is the roots of Denominator of F(s).
- The Zeros of F(s) are defined as the roots of the denominator.
- In the complex s plane, a pole is denoted by a small cross, and a eros by a Small circle
- The Function F(s) is given as $$F(s) = \frac{s(s-1+j1)(s-1-j1)}{(s+1)^2(s+j2)(s-j2)}$$

the poles are at $s = -1$ (double)

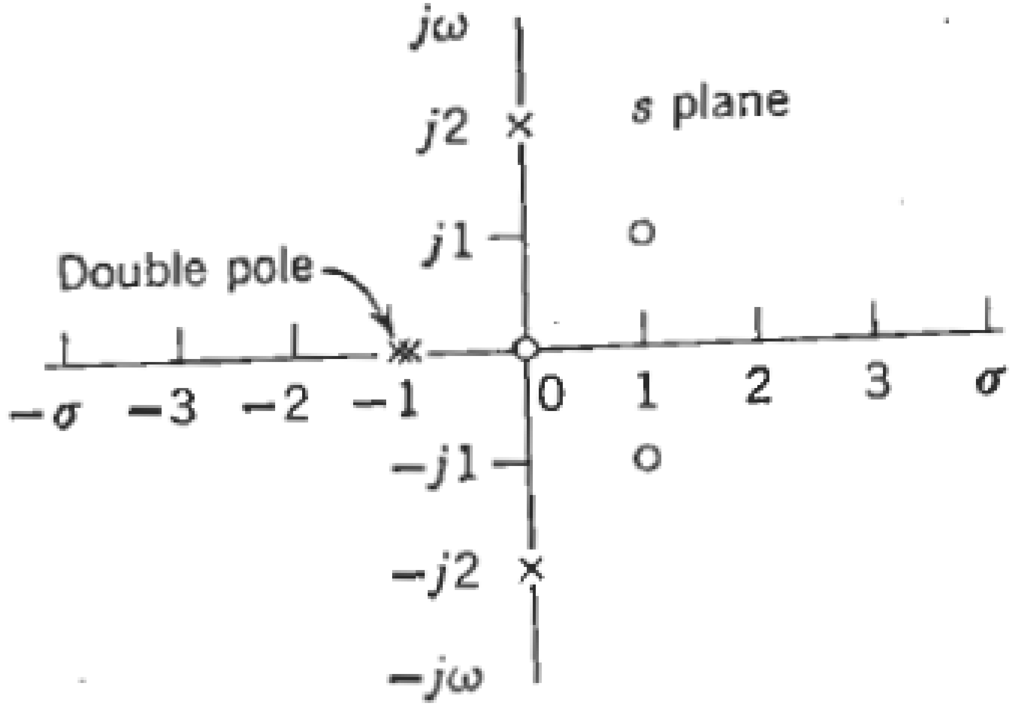
$s = -j2$

$s = +j2$

Continue.....

and the zeros are at

$$s = 0$$
$$s = 1 + j1$$
$$s = 1 - j1$$
$$s = \infty$$



Poles and Zeros plot

Laplace Transform Table

Laplace Transforms

$f(t)$	$F(s)$
1. $f(t)$	$F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$
2. $a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
3. $\frac{d}{dt} f(t)$	$s\,F(s) - f(0-)$
4. $\frac{d^n}{dt^n} f(t)$	$s^n F(s) - \sum_{j=1}^{n} s^{n-j} f^{j-1}(0-)$
5. $\int_{0-}^{t} f(\tau)\,d\tau$	$\frac{1}{s} F(s)$
6. $\int_{0-}^{t}\int_{0-}^{t} f(\tau)\,d\tau\,d\sigma$	$\frac{1}{s^2} F(s)$
7. $(-t)^n f(t)$	$\frac{d^n}{ds^n} F(s)$
8. $f(t-a)\,u(t-a)$	$e^{-as} F(s)$

9.	$e^{at} f(t)$	$F(s - a)$
10.	$\delta(t)$	1
11.	$\frac{d^n}{dt^n} \delta(t)$	s^n
12.	$u(t)$	$\frac{1}{s}$
13.	t	$\frac{1}{s^2}$
14.	$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
15.	$e^{-\alpha t}$	$\frac{1}{s + \alpha}$
16.	$\frac{1}{\beta - \alpha}(e^{-\alpha t} - e^{-\beta t})$	$\frac{1}{(s + \alpha)(s + \beta)}$
17.	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$

18.	$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
19.	$\sinh\ at$	$\dfrac{a}{s^2-a^2}$
20.	$\cosh at$	$\dfrac{s}{s^2-a^2}$
21.	$e^{-\alpha t}\sin \omega t$	$\dfrac{\omega}{(s+\alpha)^2+\omega^2}$
22.	$e^{-\alpha t}\cos \omega t$	$\dfrac{(s+\alpha)}{(s+\alpha)^2+\omega^2}$
23.	$\dfrac{e^{-\alpha t}t^n}{n!}$	$\dfrac{1}{(s+\alpha)^{n+1}}$
24.	$\dfrac{t}{2\omega}\sin \omega t$	$\dfrac{s}{(s^2+\omega^2)^2}$