RLC Circuit

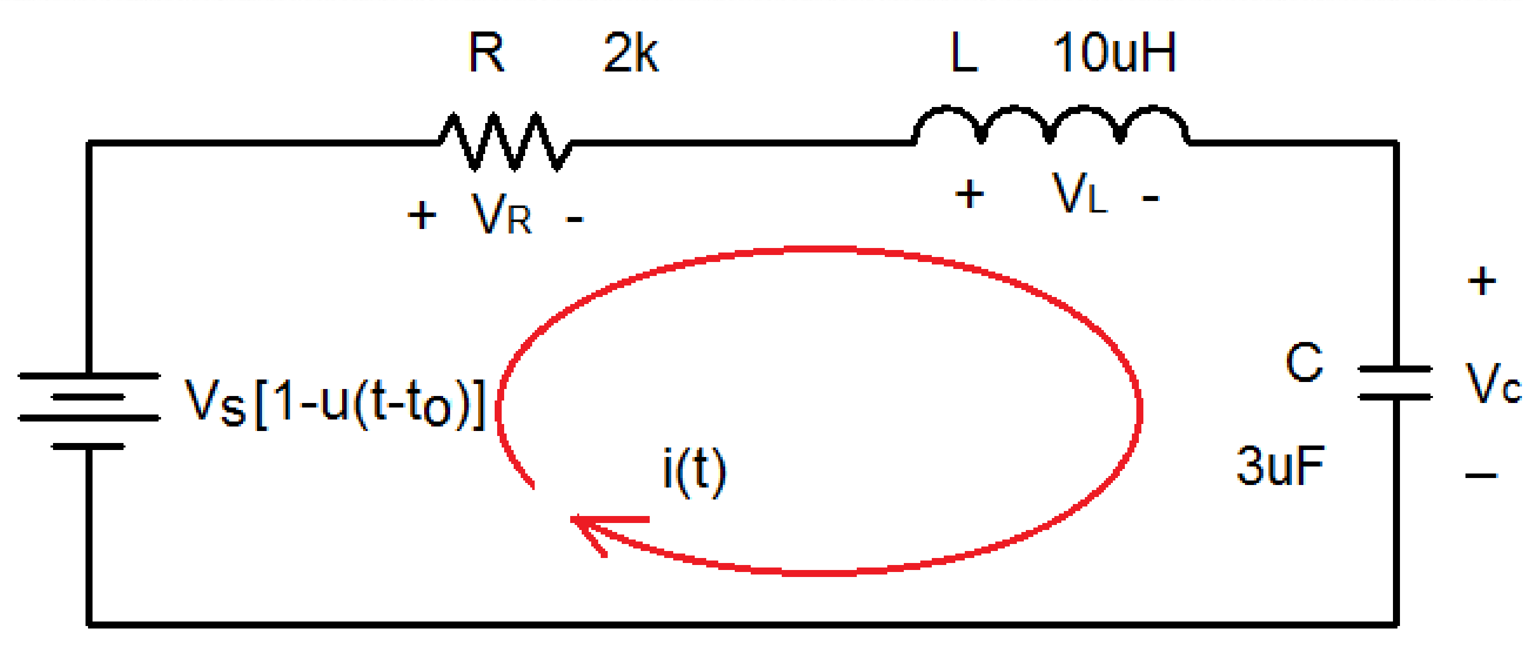
Series RLC Network

Objective of Lecture

- Derive the equations that relate the voltages across a resistor, an inductor, and a capacitor in series as:
  - the unit step function associated with voltage or current source changes from 1 to 0 or
  - a switch disconnects a voltage or current source into the circuit.
- Describe the solution to the 2nd order equations when the condition is:
  - Overdamped
  - Critically Damped
  - Underdamped

Series RLC Network

- With a step function voltage source.



Boundary Conditions

- You must determine the initial condition of the inductor and capacitor at $t < t_o$ and then find the final conditions at $t = \infty s$.
  - Since the voltage source has a magnitude of 0V at $t < t_o$
    - $i(t_o^-) = i_L(t_o^-) = 0A$ and $v_C(t_o^-) = Vs$
    - **$v_L(t_o^-) = 0V$ and $i_C(t_o^-) = 0A$**
  - Once the steady state is reached after the voltage source has a magnitude of Vs at $t > t_o$, replace the capacitor with an open circuit and the inductor with a short circuit.
    - $i(\infty s) = i_L(\infty s) = 0A$ and $v_C(\infty s) = 0V$
    - **$v_L(\infty s) = 0V$ and $i_C(\infty s) = 0A$**

Selection of Parameter

- Initial Conditions
  - $i(t_o^-) = i_L(t_o^-) = 0A$ and $v_C(t_o^-)$ **= Vs**
  - $v_L(t_o^-) = 0V$ and $i_C(t_o^-) = 0A$
- Final Conditions
  - $i(\infty s) = i_L(\infty s) = 0A$ and **$v_C(\infty s) = 0V$**
  - $v_L(\infty s) = 0V$ and $i_C(\infty s) = 0A$

- Since the voltage across the capacitor is the only parameter that has a non-zero boundary condition, the first set of solutions will be for $v_C(t)$.

Kirchhoff's Voltage Law

$$\sum v(t) = 0$$

$$v_C(t) + L\frac{di_L(t)}{dt} + Ri_L = 0$$

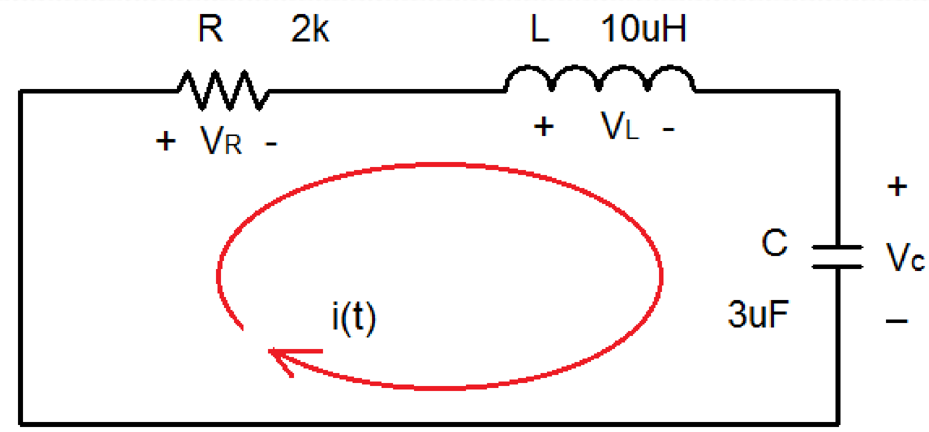
$$i_C(t) = C\frac{dv_C(t)}{dt}$$

$$i_L(t) = i_C(t)$$

$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = 0$$

$$\frac{d^2v_C(t)}{dt^2} + \frac{R}{L}\frac{dv_C(t)}{dt} + \frac{1}{LC}v_C(t) = 0$$

$$v_C(t-t_o) = v_t(t-t_o) \text{ when } t > t_o$$



General Solution

Let $v_C(t) = Ae^{s\Delta t}$

$$As^2e^{s\Delta t} + \frac{AR}{L}se^{s\Delta t} + \frac{A}{LC}e^{s\Delta t} = 0$$

$$Ae^{s\Delta t}(s^2 + \frac{R}{L}s + \frac{1}{LC}) = 0$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

General Solution (con't)

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

General Solution (con't)

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2}$$

$$\alpha = \frac{R}{2L}$$

$$\omega_o = \frac{1}{\sqrt{LC}}$$

$$s^2 + 2\alpha s + \omega_o^2 = 0$$

General Solution (con't)

$$v_{C1}(t) = A_1 e^{s_1 \Delta t}$$

$$v_{C2}(t) = A_2 e^{s_2 \Delta t}$$

$$v_C(t) = v_{C1}(t) + v_{C2}(t) = A_1 e^{s_1 \Delta t} + A_2 e^{s_2 \Delta t}$$

Solve for Coefficients A_1 and A_2

- Use the boundary conditions at t_o^- and t = ∞s to solve for A1 and A2.

$$v_C(t_o^-) = V_S$$

  - Since the voltage across a capacitor must be a continuous function of time.

$$v_C(t_o^-) = v_C(t_o^+) = v_{C1}(t_o^+) + v_{C2}(t_o^+) = V_S$$

$$A_1 e^{s_1(0s)} + A_2 e^{s_2(0s)} = A_1 + A_2 = V_S$$

  - Also know that

$$i_C(t_o) = C\frac{dv_C(t_o)}{dt} = \frac{d}{dt}\left[v_{C1}(t_o) + v_{C2}(t_o)\right] = 0$$

$$s_1 A_1 e^{s_1(0s)} + s_2 A_2 e^{s_2(0s)} = s_1 A_1 + s_2 A_2 = 0$$

Overdamped Case

- $\alpha > \omega_o$
  - implies that $C > 4L/R^2$

    s_1 and s_2 are negative and real numbers

$$v_C(t) = A_1 e^{s_1 \Delta t} + A_2 e^{s_2 \Delta t}$$

Critically Damped Case

- $\alpha = \omega_o$
  - implies that $C = 4L/R^2$

    $s_1 = s_2 = -\alpha = -R/2L$

$$v_C(t) = A_1 e^{-\alpha\Delta t} + A_2 \Delta t e^{-\alpha\Delta t}$$

Underdamped Case

- $\alpha < \omega_o$
  - implies that $C < 4L/R^2$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} = -\alpha + j\omega_d$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2} = -\alpha - j\omega_d$$

$$\omega_d = \sqrt{\omega_o^2 - \alpha^2}$$

  - $j = \sqrt{-1}$, i is used by the mathematicians for imaginary numbers

$$v_C(t) = e^{-\alpha\Delta t}(A_1 e^{j\omega_d \Delta t} + A_2 e^{-j\omega_d \Delta t})$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$v_C(t) = e^{-\alpha\Delta t}[A_1(\cos\omega_d \Delta t + j\sin\omega_d \Delta t) + A_2(\cos\omega_d \Delta t - j\sin\omega_d \Delta t)]$$

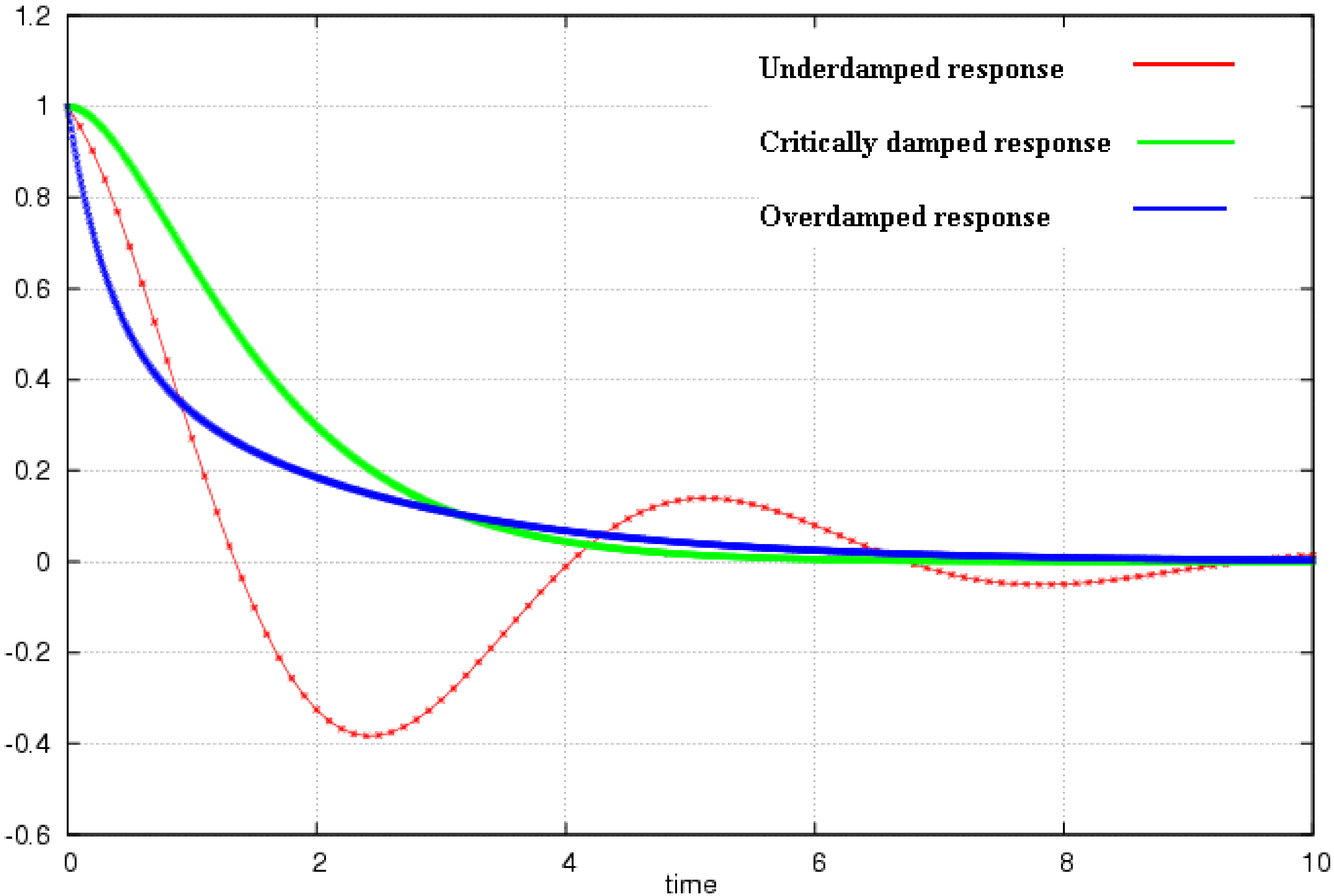
$$v_C(t) = e^{-\alpha\Delta t}[(A_1 + A_2)\cos\omega_d \Delta t + j(A_1 - A_2)\sin\omega_d \Delta t]$$

$$v_C(t) = e^{-\alpha\Delta t}[B_1 \cos\omega_d \Delta t + jB_2 \sin\omega_d \Delta t]$$

$$B_1 = A_1 + A_2 \qquad B_2 = A_1 - A_2$$

Angular Frequencies

- ω_o is called the undamped natural frequency
  - The frequency at which the energy stored in the capacitor flows to the inductor and then flows back to the capacitor. If $R = 0\Omega$, this will occur forever.
- ω_d is called the damped natural frequency
  - Since the resistance of R is not usually equal to zero, some energy will be dissipated through the resistor as energy is transferred between the inductor and capacitor.
    - α determined the rate of the damping response.



Underdamped response
Critically damped response
Overdamped response
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0
2
4
6
8
10
time

Properties of RLC network

- Behavior of RLC network is described as damping, which is a gradual loss of the initial stored energy
  - The resistor R causes the loss
  - α determined the rate of the damping response
    - If R = 0, the circuit is loss-less and energy is shifted back and forth between the inductor and capacitor forever at the natural frequency.
  - Oscillatory response of a lossy RLC network is possible because the energy in the inductor and capacitor can be transferred from one component to the other.
    - Underdamped response is a damped oscillation, which is called ringing.

Properties of RLC network

- Critically damped circuits reach the final steady state in the shortest amount of time as compared to overdamped and underdamped circuits.
  - However, the initial change of an overdamped or underdamped circuit may be greater than that obtained using a critically damped circuit.

Set of Solutions when t > t_o

- There are three different solutions which depend on the magnitudes of the coefficients of the $\frac{dv_C(t)}{dt}$ and the $v_C(t)$ terms.
  - To determine which one to use, you need to calculate the natural angular frequency of the series RLC network and the term α.

$$\omega_o = \frac{1}{\sqrt{LC}}$$

$$\alpha = \frac{R}{2L}$$

Transient Solutions when t > t_o

- Overdamped response ($\alpha > \omega_o$)

$$v_C(t) = A_1 e^{s_1 \Delta t} + A_2 e^{s_2 \Delta t}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

where $\Delta t = t - t_o$

- Critically damped response ($\alpha = \omega_o$)

$$v_C(t) = (A_1 + A_2 \Delta t) e^{-\alpha \Delta t}$$

- Underdamped response ($\alpha < \omega_o$)

$$v_C(t) = [A_1 \cos(\omega_d \Delta t) + A_2 \sin(\omega_d \Delta t)] e^{-\alpha \Delta t}$$

$$\omega_d = \sqrt{\omega_o^2 - \alpha^2}$$

Find Coefficients

- After you have selected the form for the solution based upon the values of ω_o and α
  - Solve for the coefficients in the equation by evaluating the equation at $t = t_o^-$ and $t = \infty s$ using the initial and final boundary conditions for the voltage across the capacitor.
    - $v_C(t_o^-)$ **= Vs**
    - $\mathbf{v_C(\infty s) = 0V}$

Other Voltages and Currents

- Once the voltage across the capacitor is known, the following equations for the case where $t > t_o$ can be used to find:

$$i_C(t) = C\frac{dv_C(t)}{dt}$$

$$i(t) = i_C(t) = i_L(t) = i_R(t)$$

$$v_L(t) = L\frac{di_L(t)}{dt}$$

$$v_R(t) = Ri_R(t)$$

Solutions when $t < t_o$

- The initial conditions of all of the components are the solutions for all times $-\infty s < t < t_o$.
  - $v_C(t) = Vs$
  - $i_C(t) = 0A$

  - $v_L(t) = 0V$
  - $i_L(t) = 0A$

  - $v_R(t) = 0V$
  - $i_R(t) = 0A$

Summary

- The set of solutions when $t > t_o$ for the voltage across the capacitor in a RLC network in series was obtained.
  - Selection of equations is determine by comparing the natural frequency ω_o to α.
  - Coefficients are found by evaluating the equation and its first derivation at $t = t_o^-$ and $t = \infty s$.
  - The voltage across the capacitor is equal to the initial condition when $t < t_o$
- Using the relationships between current and voltage, the current through the capacitor and the voltages and currents for the inductor and resistor can be calculated.

Source-Free RLC Circuit

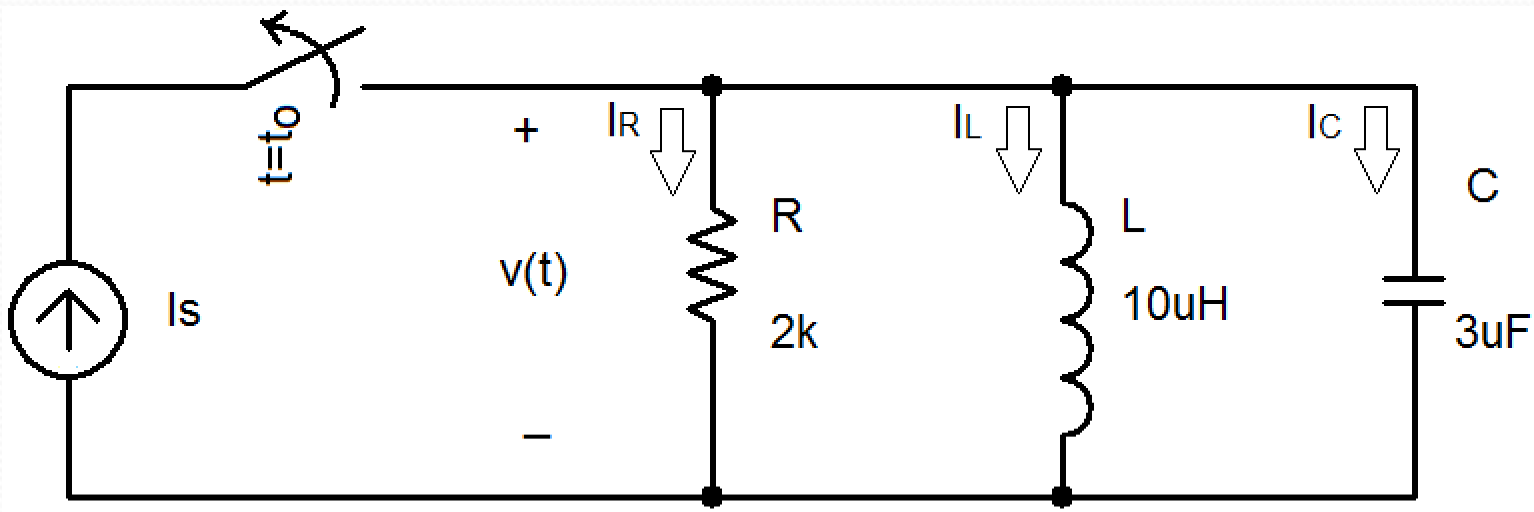
Parallel RLC Network

Objective of Lecture

- Derive the equations that relate the voltages across a resistor, an inductor, and a capacitor in parallel as:
  - the unit step function associated with voltage or current source changes from 1 to 0 or
  - a switch disconnects a voltage or current source into the circuit.
- Describe the solution to the 2nd order equations when the condition is:
  - Overdamped
  - Critically Damped
  - Underdamped

RLC Network

- A parallel RLC network where the current source is switched out of the circuit at $t = t_o$.



Boundary Conditions

- You must determine the initial condition of the inductor and capacitor at $t < t_o$ and then find the final conditions at $t = \infty s$.
  - Since the voltage source has a magnitude of 0V at $t < t_o$
    - $i_L(t_o^-) = Is$ and $v(t_o^-) = v_C(t_o^-) = 0V$
    - **$v_L(t_o^-) = 0V$ and $i_C(t_o^-) = 0A$**
  - Once the steady state is reached after the voltage source has a magnitude of Vs at $t > t_o$, replace the capacitor with an open circuit and the inductor with a short circuit.
    - $i_L(\infty s) = 0A$ and $v(\infty s) = v_C(\infty s) = 0V$
    - **$v_L(\infty s) = 0V$ and $i_C(\infty s) = 0A$**

Selection of Parameter

- Initial Conditions
  - **$i_L(t_o^-) = Is$** and $v(t_o^-) = v_C(t_o^-) = 0V$
  - $v_L(t_o^-) = 0V$ and $i_C(t_o^-) = 0A$
- Final Conditions
  - **$i_L(\infty s) = 0A$** and $v(\infty s) = v_C(\infty s) = 0V$
  - $v_L(\infty s) = 0V$ and $i_C(\infty s) = 0A$

- Since the current through the inductor is the only parameter that has a non-zero boundary condition, the first set of solutions will be for $i_L(t)$.

Kirchoff's Current Law

$$i_R(t)+i_L(t)+i_C(t)=0$$

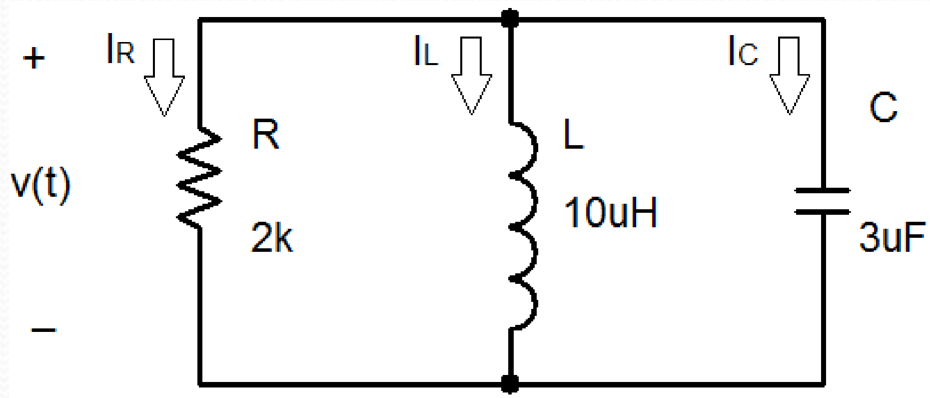
$$v(t)=v_R(t)=v_L(t)=v_C(t)$$

$$\frac{v_R(t)}{R}+i_L(t)+C\frac{dv_C(t)}{dt}=0$$

$$v_L(t)=v(t)=L\frac{di_L(t)}{dt}$$

$$LC\frac{d^2i_L(t)}{dt^2}+\frac{L}{R}\frac{di_L(t)}{dt}+i_L(t)=0$$

$$\frac{d^2i_L(t)}{dt^2}+\frac{1}{RC}\frac{di_L(t)}{dt}+\frac{i_L(t)}{LC}=0$$



General Solution

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2}$$

$$\alpha = \frac{1}{2RC}$$

$$\omega_o = \frac{1}{\sqrt{LC}}$$

$$s^2 + 2\alpha s + \omega_o^2 = 0$$

Note that the equation for the natural frequency of the RLC circuit is the same whether the components are in series or in parallel.

Overdamped Case

- $\alpha > \omega_o$
  - implies that $L > 4R^2C$

  s_1 and s_2 are negative and real numbers

$$i_{L1}(t) = A_1 e^{s_1 \Delta t}$$

$$i_{L2}(t) = A_2 e^{s_2 \Delta t}$$

$$\Delta t = t - t_o$$

$$i_L(t) = i_{L1}(t) + i_{L2}(t) = A_1 e^{s_1 \Delta t} + A_2 e^{s_2 \Delta t}$$

Critically Damped Case

- $\alpha = \omega_o$
  - implies that $L = 4R^2C$

    $s_1 = s_2 = -\alpha = -1/2RC$

$$i_L(t) = A_1 e^{-\alpha\Delta t} + A_2 \Delta t e^{-\alpha\Delta t}$$

Underdamped Case

- $\alpha < \omega_o$
  - implies that $L < 4R^2C$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} = -\alpha + j\omega_d$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2} = -\alpha - j\omega_d$$

$$\omega_d = \sqrt{\omega_o^2 - \alpha^2}$$

$$i_L(t) = e^{-\alpha\Delta t}[A_1 \cos\omega_d \Delta t + A_2 \sin\omega_d \Delta t]$$

Other Voltages and Currents

- Once current through the inductor is known:

$$v_L(t) = L\frac{di_L(t)}{dt}$$

$$v_L(t) = v_C(t) = v_R(t)$$

$$i_C(t) = C\frac{dv_C(t)}{dt}$$

$$i_R(t) = v_R(t)/R$$

Summary

- The set of solutions when $t > t_o$ for the current through the inductor in a RLC network in parallel was obtained.
  - Selection of equations is determine by comparing the natural frequency ω_o to α.
  - Coefficients are found by evaluating the equation and its first derivation at $t = t_o^-$ and $t = \infty$s.
  - The current through the inductor is equal to the initial condition when $t < t_o$
- Using the relationships between current and voltage, the voltage across the inductor and the voltages and currents for the capacitor and resistor can be calculated.