Differential Equations

Ordinary. An ordinary differential equation is one in which there is only one independent variable (in our case, t). As a result there is no need for *partial* derivatives.

Constant coefficients. The coefficients $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are constant, independent of the variable t.

Linear. A differential equation is linear if it contains only terms of the first degree in $x(t)$ and all its higher derivatives, as given by Eq. 4.3. For example, the equation

$$3x'(t) + 2x(t) = \sin t$$

is a linear differential equation. On the other hand,

$$3[x'(t)]^2 + 2x(t)\,x'(t) + 4x(t) = 5t$$

is nonlinear, because the terms $[x'(t)]^2$ and $x(t)\,x'(t)$ are nonlinear by the definition just given.

Homogeneous linear differential equation

First, let us find the solution to the equation

$$x'(t) - 2x(t) = 0 \tag{4.8}$$

Now, with a little prestidigitation, we *assume* the solution to be of the form

$$x(t) = Ce^{2t} \tag{4.9}$$

where C is any arbitrary constant. Let us check to see whether $x(t) = Ce^{2t}$ is truly a solution of Eq. 4.8. Substituting the assumed solution in Eq. 4.8, we obtain

$$2Ce^{2t} - 2Ce^{2t} = 0 \tag{4.10}$$

It can be shown, in general, that the solutions of homogeneous, linear differential equations consist of exponential terms of the form $C_i e^{p_i t}$. To obtain the solution of any differential equation, we substitute Ce^{pt} for $x(t)$ in the equation and determine those values of p for which the equation is zero. In other words, given the general equation

$$a_n x^{(n)}(t) + \cdots + a_1 x'(t) + a_0 x(t) = 0 \qquad (4.11)$$

we let $x(t) = Ce^{pt}$, so that Eq. 4.11 becomes

$$Ce^{pt}(a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0) = 0 \qquad (4.12)$$

Since e^{pt} cannot be zero except at $p = -\infty$, the only nontrivial solutions for Eq. 4.12 occur when the polynomial

$$H(p) = a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0 = 0 \qquad (4.13)$$

Equation 4.13 is often referred to as the *characteristic equation*, and is denoted symbolically in this discussion as $H(p)$. The characteristic equation is zero only at its roots. Therefore, let us factor $H(p)$ to give

$$H(p) = a_n(p - p_0)(p - p_1)\cdots(p - p_{n-1}) \tag{4.14}$$

From Eq. 4.14, we note that $C_0e^{p_0t}, C_1e^{p_1t}, \ldots, C_{n-1}e^{p_{n-1}t}$ are all solutions of Eq. 4.11. By the *superposition principle*, the total solution is a linear combination of all the individual solutions. Therefore, the total solution of the differential equation is

$$x(t) = C_0e^{p_0t} + C_1e^{p_1t} + \cdots + C_{n-1}e^{p_{n-1}t} \tag{4.15}$$

Example 4.1 Find the solution for

$$x''(t) + 5x'(t) + 4x(t) = 0 \tag{4.17}$$

given the initial conditions

$$x(0+) = 2 \qquad x'(0+) = -1$$

Solution. From the given equation, we first obtain the characteristic equation

$$H(p) = p^2 + 5p + 4 = 0 \tag{4.18}$$

which factors into

$$(p + 4)(p + 1) = 0 \tag{4.19}$$

The roots of the characteristic equation (referred to here as *characteristic values*) are $p = -1$; $p = -4$. Then $x(t)$ takes the form

$$x(t) = C_1e^{-t} + C_2e^{-4t} \tag{4.20}$$

From the initial condition $x(0+) = 2$, we obtain the equation

$$x(0+) = 2 = C_1 + C_2 \tag{4.21}$$

In order to solve for C_1 and C_2 explicitly, we need the additional initial condition $x'(0+) = -1$. Taking the derivative of $x(t)$ in Eq. 4.20, we have

$$x'(t) = -C_1e^{-t} - 4C_2e^{-4t} \tag{4.22}$$

At $t = 0+$, $x'(t)$ is

$$x'(0+) = -1 = -C_1 - 4C_2 \tag{4.23}$$

Solving Eqs. 4.21 and 4.23 simultaneously, we find that

$$C_1 = \tfrac{7}{3} \qquad C_2 = -\tfrac{1}{3}$$

Thus the final solution is

$$x(t) = \tfrac{7}{3}e^{-t} - \tfrac{1}{3}e^{-4t} \tag{4.24}$$

Roots of $H(p)$	Forms of Solution
1. Single real root, $p = p_0$	$e^{p_0 t}$
2. Root of multiplicity, k, $(p - p_1)^k$	$C_0 e^{p_1 t} + C_1 t e^{p_1 t} + \cdots + C_{k-1} t^{k-1} e^{p_1 t}$
3. Complex roots at $p_{2,3} = \sigma \pm j\omega$	$M_1 e^{\sigma t} \cos \omega t + M_2 e^{\sigma t} \sin \omega t$ or $M e^{\sigma t} \sin (\omega t + \phi)$
4. Complex roots of multiplicity k at $p_{4,5} = \sigma \pm j\omega$	$M_0 e^{\sigma t} \cos \omega t + M_1 t e^{\sigma t} \cos \omega t + \cdots + M_{k-1} t^{k-1} e^{\sigma t} \cos \omega t + N_0 e^{\sigma t} \sin \omega t + N_1 t e^{\sigma t} \sin \omega t + \cdots + N_{k-1} t^{k-1} e^{\sigma t} \sin \omega t$

Non-Homogeneous differential equation

- A non-homogeneous differential equation is one in which the forcing function is not identically zero for all t.
- Obtain the solution x(t) of an equation with constant coefficients

$$a_n x^{(n)}(t) + a_{n-1} x^{(n-1)}(t) + \cdots + a_0 x(t) = f(t) \qquad (4.55)$$

Let $x_p(t)$ be a particular solution for Eq. 4.55, and let $x_c(t)$ be the solution of the homogeneous equation obtained by letting $f(t) = 0$ in Eq. 4.55. It is readily seen that

$$x(t) = x_p(t) + x_c(t) \qquad (4.56)$$

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is also a solution of Eq. 4.55. According to the *uniqueness theorem*, the solution $x(t)$ in Eq. 4.56 is the unique solution for the nonhomogeneous differential equation *if* it satisfies the specified initial conditions at $t = 0+$.[1] In Eq. 4.56, $x_p(t)$ is the *particular integral*; $x_c(t)$ is the *complementary function*; and $x(t)$ is the *total solution*.

Since we already know how to find the complementary function $x_c(t)$, we now have to find the particular integral $x_p(t)$. In solving for $x_p(t)$, a very reliable rule of thumb is that $x_p(t)$ usually takes the *same form* as the forcing function if $f(t)$ can be expressed as a sum of exponential functions. Specifically, $x_p(t)$ assumes the form of $f(t)$ plus all its derivatives. For example, if $f(t) = \alpha \sin \omega t$, then $x_p(t)$ takes the form

$$x_p(t) = A \sin \omega t + B \cos \omega t$$

The only unknowns that must be determined are the coefficients A and B of the terms in $x_p(t)$. The method for obtaining $x_p(t)$ is appropriately called the *method of undetermined coefficients* or unknown coefficients.

In illustrating the method of unknown coefficients, let us take $f(t)$ to be

$$f(t) = \alpha e^{\beta t} \tag{4.57}$$

where α and β are arbitrary constants. We then assume $x_p(t)$ to have a similar form, that is,

$$x_p(t) = Ae^{\beta t} \tag{4.58}$$

and A is the unknown coefficient. To determine A, we simply substitute the assumed solution $x_p(t)$ into the differential equation. Thus,

$$Ae^{\beta t}(a_n\beta^n + a_{n-1}\beta^{n-1} + \cdots + a_1\beta + a_0) = \alpha e^{\beta t} \tag{4.59}$$

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We see that the polynomial within the parentheses is the characteristic equation $H(p)$ with $p = \beta$. Consequently, the unknown coefficient is obtained as

$$A = \frac{\alpha}{H(\beta)} \tag{4.60}$$

provided that $H(\beta) \neq 0$.