The Fourier Series

Plan of talk

- Evolution of Fourier Series
- Introduction
- Periodic function
- Fourier series for period T
- Even and Odd functions
- Fourier series for even and odd function
- Conclusion

Evolution of Fourier Series

- When the French mathematician Joseph Fourier (1768–1830) was trying to solve a problem in heat conduction, he needed to express a function as an infinite series of sine and cosine functions:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

$$= \frac{a_0}{2} + a_1 \cos \omega t + a_2 \cos 2\omega t + ...$$

$$+ b_1 \sin \omega t + b_2 \sin 2\omega t + ...$$

The above series is a *trigonometric series, later called as Fourier series.*

- Interest in studying Fourier Series in the field of Science and Engineering is increased because this provides an important tool in solving problems that involve ordinary and partial differential equations.
- The theory of Fourier series is rather complicated, but the application of these series is simple.

Introduction

A Fourier series is an expansion of a ***periodic function*** $f(t)$ in terms of an infinite sum of cosine and sine series

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\omega t+b_n\sin n\omega t)$$

In other words, any periodic function can be resolved as a summation of constant value, cosine and sine functions as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$

$$= \frac{a_0}{2} \quad + (a_1 \cos \omega t + b_1 \sin \omega t)$$

$$+ (a_2 \cos 2\omega t + b_2 \sin 2\omega t)$$

$$+ (a_3 \cos 3\omega t + b_3 \sin 3\omega t) \quad + \ldots$$

Periodic function

- If at equal intervals of abscissa t, the value of each ordinate f(t) repeat itself, then y = f(t) is said to be a periodic function having period T, i.e.,

$$f(t) = f(t+T) \quad \text{for all } t.$$

Examples: Sin t, cos t are periodic functions of period 2π.

Fourier series for period T

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \qquad (1)$$

where $\omega = \frac{2\pi}{T} = \text{Fundamental frequency}$

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t\,dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\,dt$$

*we can also use the integrals limit $\int_{-T/2}^{T/2}$.

Determination of Fourier Coefficients

To find the value of a_0 we have to integrate both sides of Eqn. (1) from 0 to T, then

$$\int_0^T f(t)dt = \frac{a_0}{2}\int_0^T dt + \int_0^T\left(\sum_{n=1}^{\infty} a_n \cos n\omega t\right)dt + \int_0^T\left(\sum_{n=1}^{\infty} b_n \sin n\omega t\right)dt$$

$$= \frac{a_0 T}{2} + 0 + 0 = \frac{a_0 T}{2}$$

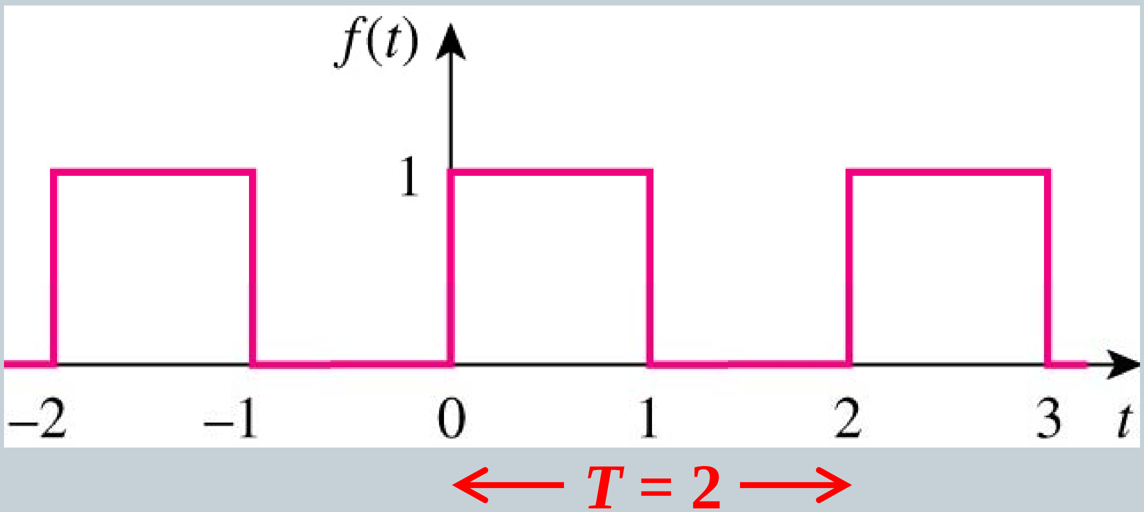
$$\Rightarrow a_0 = \frac{2}{T}\int_0^T f(t)dt \qquad f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$

To find the value of a_n and b_n we have to multipfy both side of Eqn. (1) by cos $n\omega t$ *and sin* $n\omega t$ *respectively and then integrate* from 0 to T

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t\,dt \qquad b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\,dt$$

Solution

First, determine the period & describe the one period of the function:



$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases} \qquad f(t+2) = f(t)$$

Then, obtain the coefficients a_0, a_n and b_n:

$$a_0 = \frac{2}{T}\int_0^T f(t)dt = \frac{2}{2}\int_0^2 f(t)dt = \int_0^1 1dt + \int_1^2 0dt = 1 - 0 = 1$$

Or, since $\int_a^b f(t)dt$ is the total area below graph $y = f(t)$ over the interval $[a,b]$, hence

$$a_0 = \frac{2}{T}\int_0^T f(t)dt = \frac{2}{T}\times\begin{pmatrix}\text{Area below graph}\\ \text{over}\,[0,T]\end{pmatrix} = \frac{2}{2}\times(1\times 1) = 1$$

$$a_n = \frac{2}{T}\int_0^2 f(t)\cos n\omega t dt$$

$$= \int_0^1 1\cos n\pi t dt + \int_1^2 0 dt = \left[\frac{\sin n\pi t}{n\pi}\right]_0^1 = \frac{\sin n\pi}{n\pi}$$

Notice that n is integer which leads $\sin n\pi = 0$, since $\sin\pi = \sin 2\pi = \sin 3\pi = \ldots = 0$

Therefore, $a_n = 0$.

$$b_n = \frac{2}{T}\int_0^2 f(t)\sin n\omega t dt$$

$$= \int_0^1 1\sin n\pi t dt + \int_1^2 0 dt = \left[-\frac{\cos n\pi t}{n\pi}\right]_0^1 = \frac{1-\cos n\pi}{n\pi}$$

Notice that $\cos\pi = \cos 3\pi = \cos 5\pi = \ldots = -1$
$\cos 2\pi = \cos 4\pi = \cos 6\pi = \ldots = 1$

or $\cos n\pi = (-1)^n$

Therefore, $b_n = \frac{1-(-1)^n}{n\pi} = \begin{cases} 2/n\pi & , \quad n \text{ odd} \\ 0 & , \quad n \text{ even} \end{cases}$

Finally,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \left[\frac{1-(-1)^n}{n\pi} \right] \sin n\pi t$$

$$= \frac{1}{2} + \frac{2}{\pi} \sin \pi t + \frac{2}{3\pi} \sin 3\pi t + \frac{2}{5\pi} \sin 5\pi t + \ldots$$

[Supplementary]

- The sum of the Fourier series terms can evolve (progress) into the original waveform

From Example 1, we obtain

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sin \pi t + \frac{2}{3\pi}\sin 3\pi t + \frac{2}{5\pi}\sin 5\pi t + \ldots$$

It can be demonstrated that the sum will lead to the square wave:

Some helpful identities

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x$$

For n integers,

$$\sin n\pi = 0 \qquad \cos n\pi = (-1)^n$$

$$\sin 2n\pi = 0 \qquad \cos 2n\pi = 1$$

$$\sin \frac{n\pi}{2} = 0 \quad (if\ \ n\, is\, even) \qquad \cos \frac{n\pi}{2} = 0 \quad (if\ \ n\, is\, odd)$$

$$= \pm 1 \quad (if\ \ n\, is\, odd) \qquad = \pm 1 \quad (if\ \ n\, is\, even)$$

Even and Odd Functions

- Symmetry functions:

  (i) **even** symmetry

  (ii) **odd** symmetry

Even symmetry

- Any function $f(t)$ is **even** if its plot is symmetrical about the vertical axis, i.e.

$$f(-t) = f(t)$$

Odd symmetry

- Any function $f(t)$ is **odd** if its plot is antisymmetrical about the vertical axis, i.e.

$$f(-t) = -f(t)$$

Even and odd functions

The product properties of even and odd functions are:

- (even) × (even) = (even)
- (odd) × (odd) = (even)
- (even) × (odd) = (odd)
- (odd) × (even) = (odd)

Fourier series for even and odd functions in the interval (-T/2 to T/2)

Case I: when f(t) is an even function

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt = \frac{4}{T}\int_{0}^{T/2} f(t)dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega t\, dt = \frac{4}{T}\int_{0}^{T/2} f(t)\cos n\omega t\, dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega t\, dt = 0$$

The Fourier series for the even function is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t$$

Fourier series for even and odd functions in the interval (-T/2 to T/2)

Case II: when f(t) is an Odd function

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt = 0$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega t\, dt = 0$$

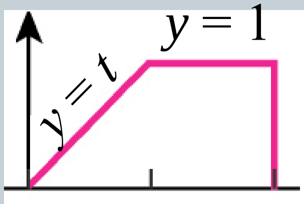
$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega t\, dt = \frac{4}{T}\int_{0}^{T/2} f(t)\sin n\omega t\, dt$$

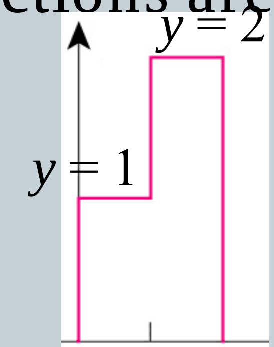
The Fourier series for the even function is

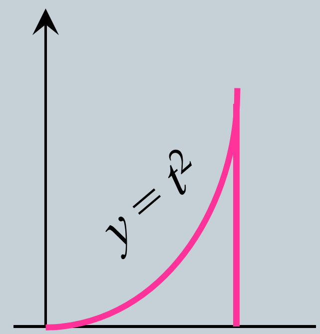
$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t$$

Conclusion

- Fourier series only support periodic functions
- In real application, many functions are non-periodic
- The non-periodic functions are often can be defined over finite intervals, e.g.







Therefore, any non-periodic function must be **extended to a periodic function** first, before computing its Fourier series representation

Normally, we prefer symmetry (even or odd) periodic extension instead of normal periodic extension, since symmetry function will provide zero coefficient of either a_n or b_n.

Assignment-1

Q. 1 If $f(x)=\left(\frac{\pi-x}{2}\right)^2$ in the interval $0 < x < 2\pi$, show that

$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ hence obtain the following relations:

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \ldots\ldots\ldots\ldots = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \ldots\ldots\ldots\ldots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \ldots\ldots\ldots\ldots = \frac{\pi^2}{8}$

Q.2 :Expand $f(x) = |\cos x|$ as a fourier series $-\pi < x < \pi$.

Q.3: Expand $f(x) = |\sin x|$ as a Fourier series in the interval, $-\pi < x < \pi$

Q. 4 Obtain a fourier series to represent e^{-ax} from $x = -\pi$ to $x = \pi$.hence derive series for $\frac{\pi}{\sinh \pi}$

Q.5 : Prove that $\cos wx = \frac{2w \sin w\pi}{\pi}\left(\frac{1}{2w^2} + \frac{\cos x}{1^2 - w^2} - \frac{\cos 2x}{2^2 - w^2} + \ldots\ldots\right)$ in the interval$(-\pi, \pi)$

Where w is a non- integral ,hence deduce that

$\pi \operatorname{cosec}(wx)=\sum_{n=0}^{\infty}(-1)^n \left[\frac{1}{n+w}+\frac{1}{n+1-w}\right]$

Assignment-2

Q. 1:Find the fourier series of the function

$$F(x) = \begin{cases} x^2, & 0 \le x \le \pi \\ -x^2, & -\pi \le x \le 0 \end{cases}$$

Q.2:Find fourier series for $f(x)=\begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 < x < \pi \end{cases}$

And hence deduce that

(i) $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots\ldots = \frac{1}{2}$

(ii) $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\ldots\ldots = \frac{\pi - 2}{4}$

Q.3: Obtain the Fourier series for the function

$$F(x)=\begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2 - x), & 1 \le x \le 2 \end{cases}$$

Q.4: Find the Fourier of the function $f(x) = \begin{cases} x, & -1 \le x \le 0 \\ x + 2, & 0 \le x \le 1 \end{cases}$

where $f(x + 2) = f(x)$

Q.5: Find the half-range series for the function $f(x)=(x-1)^2$

In the interval $0 < x < 1$ and show that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \ldots\ldots\ldots\ldots = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \ldots\ldots\ldots\ldots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \ldots\ldots\ldots\ldots = \frac{\pi^2}{8}$

Assignment-3

Q.1 Obtain the Fourier Series of a Function

$f(x) = x\sin(x)$ in the inteval $-\pi < x < \pi$

Q.2: Find the half range sine and cosine series of the function

$$f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < \pi \end{cases}$$

Q.3: Find a series of cosines of multiples of x which will represent

$x\sin x$ in interval $(-\pi, \pi)$ and show that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots\dots\dots = \frac{\pi - 2}{4}$

Q.4: Find fourier series for function $f(x) = \sqrt{1 - \cos x}$ in $(-\pi, \pi)$

Q.5:Draw the graph of wave forms & find fourier series for all wave forms

Q.6:Find the Fourier expansion for $f(x)=\pi x$ from $x=-c$ to $x=c$

Q.7.Find the Fourier series to represent $f(x)=x^2-2$,where $-2 \le x \le 2$

Q.8.Obtain a half-range cosine series for

$$f(x) \begin{cases} kx & for\ 0 < x < \frac{1}{2} \\ k(1-x) & for\ 0 \le x \le 1 \end{cases}$$

Thank you