RETARDED POTENTIALS

At point r =(x,y,z), integrate over charges at positions r'

$$\phi(\vec{r},t) = \int \frac{[\rho]}{|\vec{r}-\vec{r}'|} d^3\vec{r}' \qquad (7)$$

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int \frac{[\vec{j}]}{|\vec{r}-\vec{r}'|} d^3\vec{r}' \qquad (8)$$

where [ρ]→ evaluate ρ at retarded time:

$$[\rho] = \rho\left(\vec{r}', t - \frac{1}{c}|\vec{r}-\vec{r}'|\right)$$

Similar for [j]

- So potentials at point $\vec{r}$ and time t are affected by conditions at

  point $\vec{r}\,'$ at a retarded time, $t-\frac{1}{c}\left|\vec{r}-\vec{r}\,'\right|$

- Given a charge and current density,

  find retarded potentials ϕ and $\vec{A}$ by means of (7) and (8)

  Then use (1) and (2) to derive E, B

  Fourier transform → spectrum

Radiation from Moving Charges

<u>The Liénard-Wiechart Potentials</u>

Retarded potentials of *single*, *moving* charges

Charge q

moves along trajectory $$\vec{r} = \vec{r}_0(t)$$

velocity at time t is $$\vec{u}(t) = \dot{\vec{r}}_o(t)$$

charge density $$\rho(\vec{r},t) = q\delta(\vec{r} - \vec{r}_0(t))$$

current density $$\vec{j}(\vec{r},t) = q\vec{u}(t)\delta(\vec{r} - \vec{r}_0(t))$$

delta function

Can integrate over volume d^3r to get total charge and current

$$q = \int \rho(\vec{r},t)d^3\vec{r}$$

$$q\vec{u} = \int \vec{j}(\vec{r},t)d^3\vec{r}$$

What is the scalar potential for a moving charge?

Recall
$$\phi(\vec{r},t) = \int \frac{[\rho]}{|\vec{r}-\vec{r}'|} d^3\vec{r}'$$
where [ ] denotes evaluation at retarded time

$$\phi(\vec{r},t) = \int d^3\vec{r}' \int dt' \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|} \delta\left(t' - t + \frac{|\vec{r}-\vec{r}'|}{c} \right)$$

light travel time between $\vec{r}$ and $\vec{r}'$

Substitute
$$\rho(\vec{r}',t') = q\delta\left(\vec{r}' - \vec{r}_0(t')\right)$$

and integrate
$$\int d^3\vec{r}'$$

$$\phi(\vec{r},t) = \int d^3\vec{r}\,' \int dt' \frac{q\delta(\vec{r}\,' - \vec{r}_0(t'))}{\left|\vec{r} - \vec{r}\,'\right|} \delta\left(t' - t + \frac{\left|\vec{r} - \vec{r}\,'\right|}{c} \right)$$

$$= q \int dt' \frac{\delta(t' - t + \frac{\left|\vec{r} - \vec{r}\,'\right|}{c})}{\left|\vec{r} - \vec{r}_0(t')\right|}$$

Now let $\vec{R}(t') = \vec{r} - \vec{r}_0(t')$ vector

$R(t') = \left|\vec{R}(t')\right|$ scalar

then

$$\phi(\vec{r},t) = q \int R^{-1}(t') \delta(t' - t + \frac{R(t')}{c}) dt'$$

Now change variables again $t'' = t' - t + \frac{R(t')}{c}$

then $$dt'' = dt' + \frac{1}{c}\dot{R}(t')dt'$$

Velocity $\vec{u}(t') = \dot{\vec{r}}_0(t)$

$$\dot{\vec{R}}(t') = \frac{d}{dt'}\left(\vec{r} - \vec{r}_0(t)\right)$$

$$= -\vec{u}(t')$$

dot both sides → $R^2(t') = \vec{R}^2(t')$

$$2R(t')\dot{R}(t') = -2\vec{R}(t')\cdot\vec{u}(t')$$

Also define unit vector $\vec{n} = \frac{\vec{R}}{R}$

$$\Rightarrow \quad dt'' = dt' + \frac{1}{c}\dot{R}(t')dt'$$

$$= \left[1 + \frac{1}{c}\dot{R}(t')\right]dt'$$

$$= \left[1 - \frac{1}{c}\vec{n}(t')\cdot\vec{u}(t')\right]dt'$$

so

$$\phi(\vec{r},t) = q\int \frac{R^{-1}(t')}{1-\frac{1}{c}\vec{n}(t')\cdot\vec{u}(t')}\delta(t'')dt''$$

This means evaluate integral at t''=0, or t'=t(retard)

So...

$$\phi(\vec{r},t) = \frac{q}{\kappa(t_{retard})R(t_{retard})}$$

where

$$\kappa(t_{retard}) = \kappa(t') = 1 - \tfrac{1}{c}\vec{n}(t')\cdot\vec{u}(t')$$ beaming factor

or, in the bracket notation:

$$\phi(\vec{r},t) = \left[\frac{q}{\kappa R}\right]$$ Liénard-Wiechart scalar potential

Similarly, one can show for the vector potential:

$$\vec{A} = \left[\frac{q\vec{u}}{c\kappa R}\right]$$ Liénard-Wiechart vector potential

Given the potentials

$$\phi(\vec{r},t) = \left[\frac{q}{\kappa R} \right] \qquad \vec{A} = \left[\frac{q\vec{u}}{c\kappa R} \right]$$

one can use

$$\vec{B} = \nabla \times \vec{A} \qquad \vec{E} = -\nabla \phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

to derive E and B.

We'll skip the math and just talk about the result. (see Jackson §14.1)

The Result: The E, B field at point r and time t depends on the retarded position r(ret) and retarded time t(ret) of the charge.

Let

$$\vec{u} = \dot{\vec{r}}_0(t_{ret}) \qquad \text{velocity of charged particle}$$

$$\dot{\vec{u}} = \ddot{\vec{r}}_0(t_{ret}) \qquad \text{acceleration}$$

$$\vec{\beta} \equiv \frac{\vec{u}}{c} \qquad \kappa \equiv 1 - \vec{n}\cdot\vec{\beta}$$

$$\vec{B}(\vec{r},t) = \left[\vec{n} \times \vec{E}(\vec{r},t)\right]$$

$$\vec{E}(\vec{r},t) = q\underbrace{\left[\frac{(\vec{n}-\vec{\beta})(1-\beta^2)}{\kappa^3 R^2}\right]}_{\text{"VELOCITY FIELD"} \;\propto \frac{1}{R^2}\ \text{Coulomb Law}} + \frac{q}{c}\underbrace{\left[\frac{\vec{n}}{\kappa^3 R} \times \left\{(\vec{n}-\vec{\beta}) \times \dot{\vec{\beta}}\right\}\right]}_{\text{"RADIATION FIELD"} \;\propto \frac{1}{R}}$$

Field of particle w/ constant velocity

Transverse field due to acceleration

Qualitative Picture:

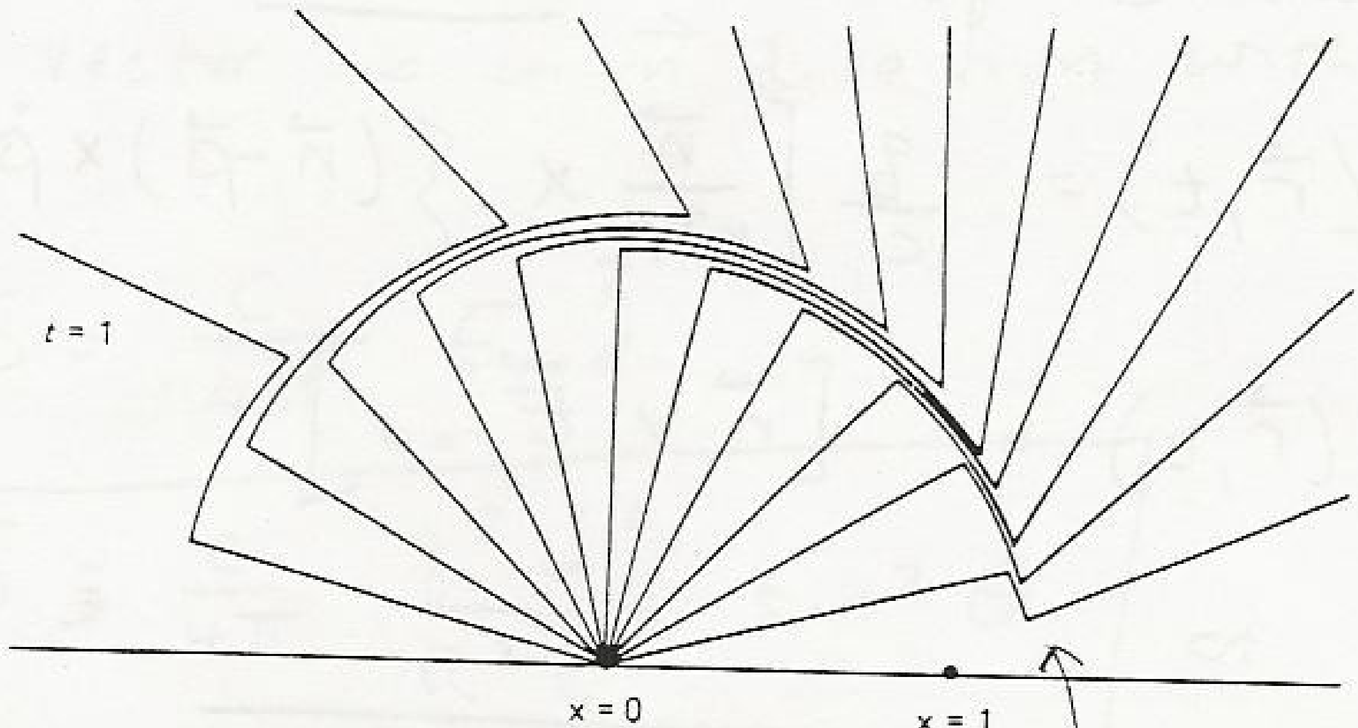

Figure 3.2 Graphical demonstration of the $1/R$ acceleration field. Charged particle moving at uniform velocity in positive x direction is stopped at $x = 0$ and $t = 0$.

transverse “radiation”
field propagates at velocity c