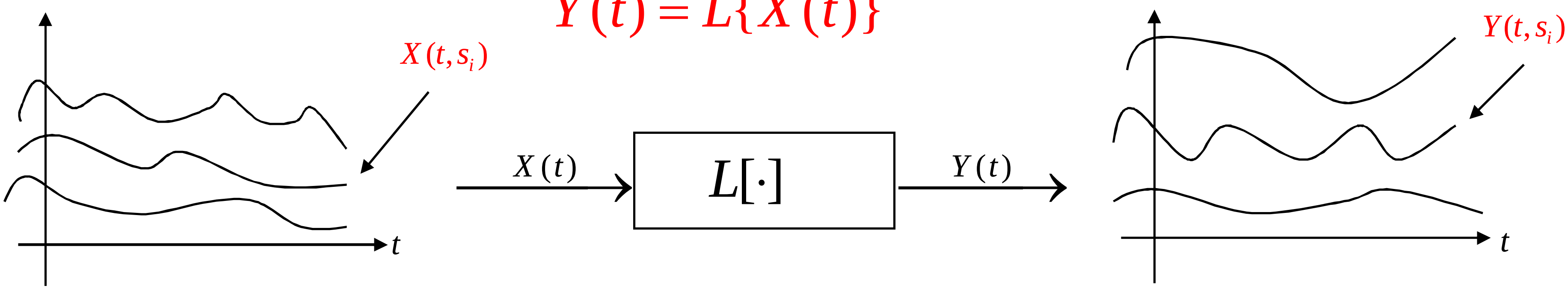
Linear Filtering of a Random Signal

Linear System

$$L\{a_1 X(t_1) + a_2 X(t_2)\} = a_1 L\{X(t_1)\} + a_2 L\{X(t_2)\}.$$

$$Y(t) = L\{X(t)\}$$



- Our goal is to study the output process statistics in terms of the input process statistics and the system function.

Deterministic System

Deterministic Systems

Memoryless Systems

$$Y(t) = g[X(t)]$$

Systems with Memory

- Time-varying systems
- Time-Invariant systems
- Linear systems

$$Y(t) = L[X(t)]$$

Time-Invariant systems + Linear systems → Linear-Time Invariant (LTI) systems

$X(t)$ → $h(t)$ →

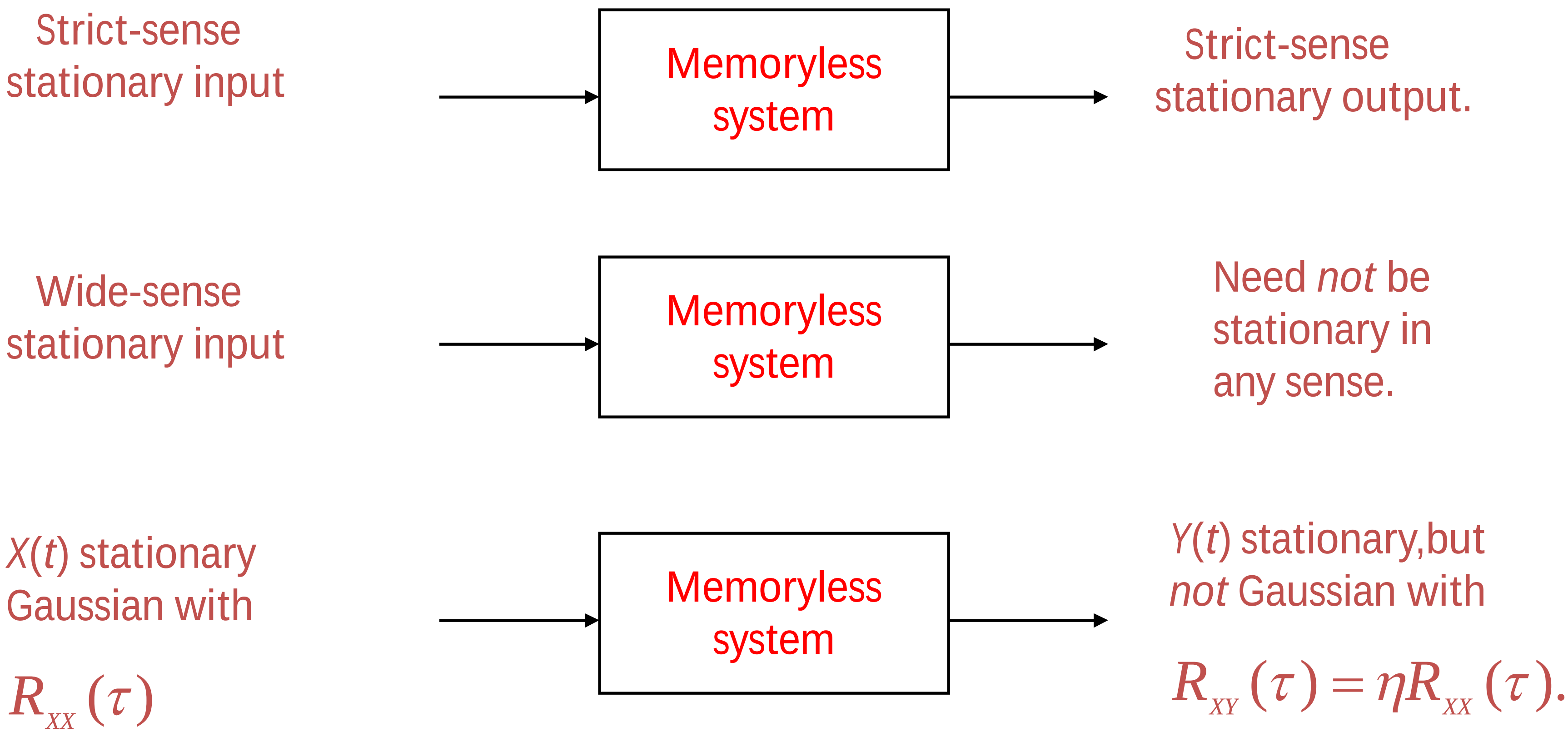
LTI system

$$Y(t) = \int_{-\infty}^{+\infty} h(t-\tau)X(\tau)d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau)X(t-\tau)d\tau. = X(t) * h(t)$$

Memoryless Systems

The output $Y(t)$ in this case depends only on the present value of the input $X(t)$. i.e., $Y(t) = g\{X(t)\}$.



Linear Time-Invariant Systems

Time-Invariant System

Shift in the input results in the same shift in the output.

$$Y(t) = L\{X(t)\} \Rightarrow L\{X(t-t_0)\} = Y(t-t_0)$$

Linear Time-Invariant System

A linear system with time-invariant property.

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \qquad h(t) = L\{\delta(t)\}$$

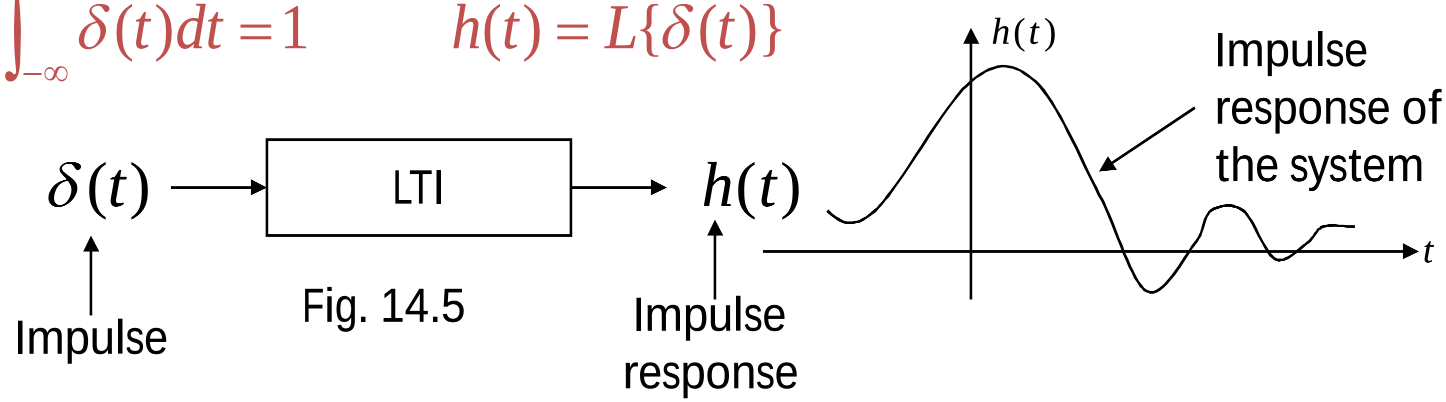
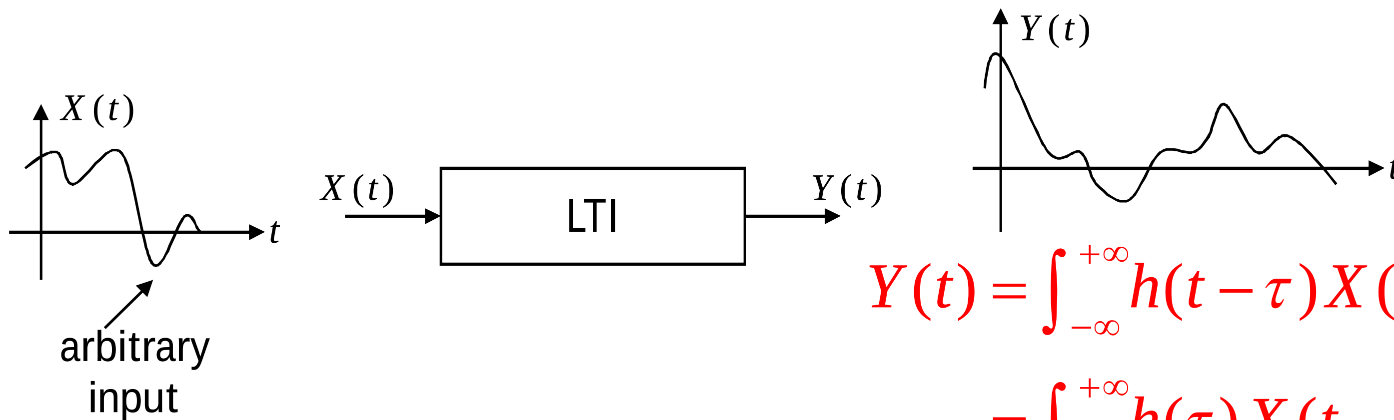


Fig. 14.5

Linear Filtering of a Random Signal



$$Y(t) = \int_{-\infty}^{+\infty} h(t-\tau) X(\tau) d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau) X(t-\tau) d\tau$$

$$X(t) = \int_{-\infty}^{+\infty} X(\tau) \delta(t-\tau) d\tau$$

$$Y(t) = L\{X(t)\} = L\{\int_{-\infty}^{+\infty} X(\tau) \delta(t-\tau) d\tau\}$$

$$= \int_{-\infty}^{+\infty} L\{X(\tau) \delta(t-\tau) d\tau\}$$

By Linearity

$$= \int_{-\infty}^{+\infty} X(\tau) L\{\delta(t-\tau)\} d\tau$$

By Time-invariance

$$= \int_{-\infty}^{+\infty} X(\tau) h(t-\tau) d\tau = \int_{-\infty}^{+\infty} h(\tau) X(t-\tau) d\tau.$$

Theorem 6.1

$$E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau\right] = \int_{-\infty}^{\infty} h(\tau)E[X(t-\tau)]d\tau$$

$$= E[X(t)] * h(t)$$

Theorem 6.2

If the input to an LTI filter with impulse response $h(t)$ is a wide sense stationary process $X(t)$, the output $Y(t)$ has the following properties:

(a) $Y(t)$ is a WSS process with expected value

$$\mu_Y = E[Y(t)] = \mu_X \int_{-\infty}^{\infty} h(\tau)d\tau$$

autocorrelation function

$$R_Y(\tau) = \int_{-\infty}^{\infty} h(u) \int_{-\infty}^{\infty} h(v) R_X(\tau + u - v)du dv$$

(b) $X(t)$ and $Y(t)$ are jointly WSS and have I/O cross- correlation by

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} h(u) R_X(\tau - u)du = R_X(\tau) * h(\tau)$$

(c) The output autocorrelation is related to the I/O cross-correlation by

$$R_Y(\tau) = \int_{-\infty}^{\infty} h(-w) R_{XY}(\tau - w)dw$$

$$= R_{XY}(\tau) * h(-\tau)$$