CYCLIC CODES

Cyclic codes are of interest and importance because

- They posses rich algebraic structure that can be utilized in a variety of ways.
- They have extremely concise specifications.
- They can be efficiently implemented using simple *shift registers*.
- Many practically important codes are cyclic.

Convolution codes allow to encode streams od data (bits).

IMPORTANT NOTE

- In order to specify a binary code with 2^k codewords of length *n* one may need
- to write down
- 2^k
- codewords of length n.

- In order to specify a linear binary code with 2^k codewords of length *n* it is sufficient
- to write down
- k
- codewords of length n.

- In order to specify a binary cyclic code with 2^k codewords of length *n* it is sufficient
- to write down
- *1*
- codeword of length *n*.

BASIC DEFINITION AND EXAMPLES

- Definition A code C is cyclic if
- (i) C is a linear code;
- (ii) any cyclic shift of a codeword is also a codeword, i.e. whenever $a_0, \ldots a_{n-1} \in C$, then also $a_{n-1} a_0 \ldots a_{n-2} \in C$.

Example

(i) Code $C = \{000, 101, 011, 110\}$ is cyclic.

(ii) Hamming code *Ham*(3, 2): with the generator matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

is equivalent to a cyclic code.

(iii) The binary linear code {0000, 1001, 0110, 1111} is not a cyclic, but it is equivalent to a cyclic code.

(iv) Is Hamming code *Ham*(2, 3) with the generator matrix

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

(a) cyclic?

(b) equivalent to a cyclic code?

FREQUENCY of CYCLIC CODES

- Comparing with linear codes, the cyclic codes are quite scarce. For, example there are 11 811 linear (7,3) linear binary codes, but only two of them are cyclic.

- Trivial cyclic codes. For any field F and any integer $n >= 3$ there are always the following cyclic codes of length n over F:
  - No-information code - code consisting of just one all-zero codeword.
  - Repetition code - code consisting of codewords $(a, a, \ldots, a)$ for $a \in F$.
  - Single-parity-check code - code consisting of all codewords with parity 0.
  - No-parity code - code consisting of all codewords of length n

- For some cases, for example for $n = 19$ and $F = GF(2)$, the above four trivial cyclic codes are the only cyclic codes.

EXAMPLE of a CYCLIC CODE

- The code with the generator matrix

$$G = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

- has codewords
- $c_1 = 1011100 \qquad c_2 = 0101110 \qquad c_3 = 0010111$
- $c_1 + c_2 = 1110010 \quad c_1 + c_3 = 1001011 \quad c_2 + c_3 = 0111001$
- $c_1 + c_2 + c_3 = 1100101$
- and it is cyclic because the right shifts have the following impacts
- $c_1 \to c_2, \qquad c_2 \to c_3, \qquad c_3 \to c_1 + c_3$
- $c_1 + c_2 \to c_2 + c_3, \quad c_1 + c_3 \to c_1 + c_2 + c_3, \qquad c_2 + c_3 \to c_1$
- $c_1 + c_2 + c_3 \to c_1 + c_2$

POLYNOMIALS over GF(q)

- A codeword of a cyclic code is usually denoted
  - $a_0\, a_1 \ldots a_{n-1}$
- and to each such a codeword the polynomial
  - $a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1}$
- is associated.
- $F_q[x]$ denotes the set of all polynomials over $GF(q)$.
- $deg\,(f(x))$ = the largest m such that x^m has a non-zero coefficient in $f(x)$.

Multiplication of polynomials If $f(x), g(x) \in F_q[x]$, then

$$deg\,(f(x)\, g(x)) = deg\,(f(x)) + deg\,(g(x)).$$

Division of polynomials For every pair of polynomials $a(x), b(x) \neq 0$ in $F_q[x]$ there exists a unique pair of polynomials $q(x), r(x)$ in $F_q[x]$ such that

$$a(x) = q(x)b(x) + r(x),\ deg\,(r(x)) < deg\,(b(x)).$$

Example Divide $x^3 + x + 1$ by $x^2 + x + 1$ in $F_2[x]$.

Definition Let $f(x)$ be a fixed polynomial in $F_q[x]$. Two polynomials $g(x), h(x)$ are said to be congruent modulo $f(x)$, notation

$$g(x) \equiv h(x)\ (\mathrm{mod}\ f(x)),$$

if $g(x) - h(x)$ is divisible by $f(x)$.

RING of POLYNOMIALS

- The set of polynomials in $F_q[x]$ of degree less than *deg* (f(x)), with addition and multiplication modulo f(x) forms a **ring denoted $F_q[x]/f(x)$**.
- Example Calculate $(x + 1)^2$ in $F_2[x] \ / \ (x^2 + x + 1)$. It holds
  - $(x + 1)^2 = x^2 + 2x + 1 \equiv x^2 + 1 \equiv x \pmod{x^2 + x + 1}$.
- How many elements has $F_q[x] \ / \ f(x)$?
- Result $| \ F_q[x] \ / \ f(x) \ | = q^{\deg (f(x))}$.
- Example Addition and multiplication in $F_2[x] \ / \ (x^2 + x + 1)$

+	0	1	x	1 + x
0	0	1	x	1 + x
1	1	0	1 + x	x
x	x	1 + x	0	1
1 + x	1 + x	x	1	0

•	0	1	x	1 + x
0	0	0	0	0
1	0	1	X	1 + x
x	0	x	1 + x	1
1 + x	0	1 + x	1	x

Definition A polynomial f(x) in $F_q[x]$ is said to be reducible if f(x) = a(x)b(x), where a(x), b(x) $\in F_q[x]$ and

$$deg\ (a(x)) < deg\ (f(x)), \qquad deg\ (b(x)) < deg\ (f(x)).$$

If f(x) is not reducible, it is irreducible in $F_q[x]$.

Theorem The ring $F_q[x] \ / \ f(x)$ is a field if f(x) is irreducible in $F_q[x]$.

FIELD R_n, $R_n = F_q[x] / (x^n - 1)$

- Computation modulo $x^n - 1$

- Since $x^n \equiv 1 \pmod{x^n - 1}$ we can compute f(x) mod x^n -1 as follow:
- In f(x) replace x^n by 1, x^{n+1} by x, x^{n+2} by x^2, x^{n+3} by x^3, …

- Identification of words with polynomials

  - $a_0 a_1 \ldots a_{n-1} \leftrightarrow a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1}$

- Multiplication by x in R_n corresponds to a single cyclic shift
  - $x (a_0 + a_1 x + \ldots a_{n-1} x^{n-1}) = a_{n-1} + a_0 x + a_1 x^2 + \ldots + a_{n-2} x^{n-1}$

Algebraic characterization of cyclic codes

- Theorem A code C is cyclic if C satisfies two conditions
- (i) $a(x), b(x) \in C \Rightarrow a(x) + b(x) \in C$
- (ii) $a(x) \in C$, $r(x) \in R_n \Rightarrow r(x)a(x) \in C$

- Proof
- (1) Let C be a cyclic code. C is linear $\Rightarrow$ (i) holds.
- (ii) Let $a(x) \in C$, $r(x) = r_0 + r_1 x + \ldots + r_{n-1}x^{n-1}$
  - $r(x)a(x) = r_0 a(x) + r_1 x a(x) + \ldots + r_{n-1}x^{n-1}a(x)$
- is in C by (i) because summands are cyclic shifts of $a(x)$.
- (2) Let (i) and (ii) hold
- • Taking $r(x)$ to be a scalar the conditions imply linearity of C.
- • Taking $r(x) = x$ the conditions imply cyclicity of C.

CONSTRUCTION of CYCLIC CODES

- Notation If $f(x) \in R_n$, then
  - $\langle f(x) \rangle = \{r(x)f(x) \mid r(x) \in R_n\}$
- (multiplication is modulo $x^n - 1$).
- Theorem For any $f(x) \in R_n$, the set $\langle f(x) \rangle$ is a cyclic code (generated by f).
- Proof We check conditions (i) and (ii) of the previous theorem.
- (i) If $a(x)f(x) \in \langle f(x) \rangle$ and $b(x)f(x) \in \langle f(x) \rangle$, then
  - $a(x)f(x) + b(x)f(x) = (a(x) + b(x))\, f(x) \in \langle f(x) \rangle$
- (ii) If $a(x)f(x) \in \langle f(x) \rangle$, $r(x) \in R_n$, then
  - $r(x)\,(a(x)f(x)) = (r(x)a(x))\, f(x) \in \langle f(x) \rangle$.

Example $C = \langle 1 + x^2 \rangle$, $n = 3$, $q = 2$.

We have to compute $r(x)(1 + x^2)$ for all $r(x) \in R_3$.

$$R_3 = \{0, 1, x, 1 + x, x^2, 1 + x^2, x + x^2, 1 + x + x^2\}.$$

Result

$$C = \{0, 1 + x, 1 + x^2, x + x^2\}$$

$$C = \{000, 011, 101, 110\}$$

Characterization theorem for cyclic codes

- We show that all cyclic codes C have the form $C = \langle f(x) \rangle$ for some $f(x) \in R_n$.
- Theorem Let C be a non-zero cyclic code in R_n. Then
  - there exists unique monic polynomial $g(x)$ of the smallest degree such that
  - $C = \langle g(x) \rangle$
  - $g(x)$ is a factor of $x^n - 1$.

Proof

(i) Suppose $g(x)$ and $h(x)$ are two monic polynomials in C of the smallest degree. Then the polynomial $g(x) - h(x) \in C$ and it has a smaller degree and a multiplication by a scalar makes out of it a monic polynomial. If $g(x) \neq h(x)$ we get a contradiction.

(ii) Suppose $a(x) \in C$.

Then

$$a(x) = q(x)g(x) + r(x) \qquad (\deg r(x) < \deg g(x))$$

and

$$r(x) = a(x) - q(x)g(x) \in C.$$

By minimality

$$r(x) = 0$$

and therefore $a(x) \in \langle g(x) \rangle$.

Characterization theorem for cyclic codes

- (iii) Clearly,
  - $x^n - 1 = q(x)g(x) + r(x)$ with $\deg r(x) < \deg g(x)$
- and therefore $r(x) \equiv -q(x)g(x) \pmod{x^n - 1}$ and
  - $r(x) \in C \Rightarrow r(x) = 0 \Rightarrow g(x)$ is a factor of $x^n - 1$.

GENERATOR POLYNOMIALS

Definition If for a cyclic code C it holds

$$C = \langle g(x) \rangle,$$

then g is called the **generator polynomial** for the code C.

HOW TO DESIGN CYCLIC CODES?

- The last claim of the previous theorem gives a recipe to get all cyclic codes of given length n.
- Indeed, all we need to do is to find all factors of

$$x^n - 1.$$

- Problem: Find all binary cyclic codes of length 3.
- Solution: Since

$$x^3 - 1 = \underbrace{(x + 1)(x^2 + x + 1)}_{\text{both factors are irreducible in } GF(2)}$$

- we have the following generator polynomials and codes.

Generator polynomials	Code in R_3	Code in $V(3,2)$
1	R_3	$V(3,2)$
$x + 1$	$\{0, 1 + x, x + x^2, 1 + x^2\}$	{000, 110, 011, 101}
$x^2 + x + 1$	$\{0, 1 + x + x^2\}$	{000, 111}
$x^3 - 1$ (= 0)	$\{0\}$	{000}

Design of generator matrices for cyclic codes

- **Theorem** Suppose C is a cyclic code of codewords of length n with the generator polynomial
  - $g(x) = g_0 + g_1x + \ldots + g_rx^r$.
- Then $dim\ (C) = n - r$ and a generator matrix G_1 for C is

$$G_1 = \begin{pmatrix} g_0 & g_1 & g_2 & \ldots & g_r & 0 & 0 & 0 & \ldots & 0 \\ 0 & g_0 & g_1 & g_2 & \ldots & g_r & 0 & 0 & \ldots & 0 \\ 0 & 0 & g_0 & g_1 & g_2 & \ldots & g_r & 0 & \ldots & 0 \\ .. & .. & & & & & & & & .. \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & g_0 & \ldots & g_r \end{pmatrix}$$

Proof

(i) All rows of G_1 are linearly independent.

(ii) The $n - r$ rows of G represent codewords

$$g(x), xg(x), x^2g(x), \ldots, x^{n-r-1}g(x) \qquad (*)$$

(iii) It remains to show that every codeword in C can be expressed as a linear combination of vectors from (*).

Inded, if $a(x) \in C$, then

$$a(x) = q(x)g(x).$$

Since $deg\ a(x) < n$ we have $deg\ q(x) < n - r$.

Hence

$$q(x)g(x) = (q_0 + q_1x + \ldots + q_{n-r-1}x^{n-r-1})g(x)$$
$$= q_0g(x) + q_1xg(x) + \ldots + q_{n-r-1}x^{n-r-1}g(x).$$

EXAMPLE

- The task is to determine all ternary codes of length 4 and generators for them.
- Factorization of $x^4 - 1$ over $GF(3)$ has the form
  - $x^4 - 1 = (x - 1)(x^3 + x^2 + x + 1) = (x - 1)(x + 1)(x^2 + 1)$
- Therefore there are $2^3 = 8$ divisors of $x^4 - 1$ and each generates a cyclic code.

Generator polynomial	Generator matrix
1	I_4
x	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$
$x + 1$	$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$
$x^2 + 1$	$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$
$(x - 1)(x + 1) = x^2 - 1$	$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$
$(x - 1)(x^2 + 1) = x^3 - x^2 + x - 1$	[-1 1 -1 1]
$(x + 1)(x^2 + 1)$	[1 1 1 1]
$x^4 - 1 = 0$	[0 0 0 0]

Check polynomials and parity check matrices for cyclic codes

- Let C be a cyclic $[n,k]$-code with the generator polynomial $g(x)$ (of degree $n - k$). By the last theorem $g(x)$ is a factor of $x^n - 1$. Hence

$$x^n - 1 = g(x)h(x)$$

- for some $h(x)$ of degree k (where $h(x)$ is called the check polynomial of C).
- **Theorem** Let C be a cyclic code in R_n with a generator polynomial $g(x)$ and a check polynomial $h(x)$. Then an $c(x) \in R_n$ is a codeword of C if $c(x)h(x) \equiv 0$ - this and next congruences are modulo $x^n - 1$.

Proof Note, that $g(x)h(x) = x^n - 1 \equiv 0$

(i) $c(x) \in C \Rightarrow c(x) = a(x)g(x)$ for some $a(x) \in R_n$

$$\Rightarrow c(x)h(x) = a(x)\underbrace{g(x)h(x)}_{\equiv 0} \equiv 0.$$

(ii) $c(x)h(x) \equiv 0$

$$c(x) = q(x)g(x) + r(x),\ deg\ r(x) < n - k = deg\ g(x)$$

$$c(x)h(x) \equiv 0 \Rightarrow r(x)h(x) \equiv 0 \pmod{x^n - 1}$$

Since $deg\ (r(x)h(x)) < n - k + k = n$, we have $r(x)h(x) = 0$ in $F[x]$ and therefore

$$r(x) = 0 \Rightarrow c(x) = q(x)g(x) \in C.$$

POLYNOMIAL REPRESENTATION of DUAL CODES

- Since *dim* (⟨h(x)⟩) = *n* - *k* = *dim* ($C^\perp$) we might easily be fooled to think that the check polynomial h(x) of the code C generates the dual code $C^\perp$.
- Reality is "slightly different":
- Theorem Suppose C is a cyclic [n,k]-code with the check polynomial
  - $h(x) = h_0 + h_1x + \ldots + h_kx^k$,
- then
- (i) a parity-check matrix for C is

$$H = \begin{pmatrix} h_k & h_{k-1} & \ldots & h_0 & 0 & \ldots & 0 \\ 0 & h_k & \ldots & h_1 & h_0 & \ldots & 0 \\ .. & .. & & & & & \\ 0 & 0 & \ldots & 0 & h_k & \ldots & h_0 \end{pmatrix}$$

- (ii) $C^\perp$ is the cyclic code generated by the polynomial

$$\bar{h}(x) = h_k + h_{k-1}x + \ldots + h_0x^k$$

- i.e. the reciprocal polynomial of h(x).

POLYNOMIAL REPRESENTATION of DUAL CODES

- Proof A polynomial $c(x) = c_0 + c_1x + \dots + c_{n-1}x^{n-1}$ represents a code from C if $c(x)h(x) = 0$. For $c(x)h(x)$ to be 0 the coefficients at $x^k, \dots, x^{n-1}$ must be zero, i.e.

$$\begin{aligned} c_0h_k + c_1h_{k-1} + \dots + c_kh_0 &= 0 \\ c_1h_k + c_2h_{k-1} + \dots + c_{k+1}h_0 &= 0 \\ \dots & \quad \dots \\ c_{n-k-1}h_k + c_{n-k}h_{k-1} + \dots + c_{n-1}h_0 &= 0 \end{aligned}$$

- Therefore, any codeword $c_0 c_1 \dots c_{n-1} \in C$ is orthogonal to the word $h_k h_{k-1} \dots h_0 00 \dots 0$ and to its cyclic shifts.
- Rows of the matrix H are therefore in $C^\wedge$. Moreover, since $h_k = 1$, these row-vectors are linearly independent. Their number is $n - k = dim (C^\wedge)$. Hence H is a generator matrix for $C^\wedge$, i.e. a parity-check matrix for C.
- In order to show that $C^\wedge$ is a cyclic code generated by the polynomial

$$\bar{h}(x) = h_k + h_{k-1}x + \dots + h_0x^k$$

- it is sufficient to show that $\bar{h}(x)$ is a factor of $x^n - 1$.
- Observe that $\bar{h}(x) = x^k h(x^{-1})$ and since $h(x^{-1})g(x^{-1}) = (x^{-1})^n - 1$
- we have that $x^k h(x^{-1}) x^{n-k} g(x^{-1}) = x^n(x^{-n} - 1) = 1 - x^n$
- and therefore $\bar{h}(x)$ is indeed a factor of $x^n - 1$.

ENCODING with CYCLIC CODES I

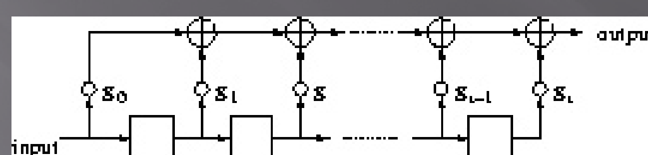
- Encoding using a cyclic code can be done by a multiplication of two polynomials - a message polynomial and the generating polynomial for the cyclic code.
- Let C be an (n,k)-code over an field F with the generator polynomial
- $g(x) = g_0 + \quad g_1 x + \ldots + g_{r-1} x^{r-1}$ of degree $r = n - k$.
- If a message vector m is represented by a polynomial $m(x)$ of degree k and m is encoded by

$$m \Rightarrow c = mG_1,$$

- then the following relation between $m(x)$ and $c(x)$ holds

$$c(x) = m(x)g(x).$$

- Such an encoding can be realized by the shift register shown in Figure below, where input is the k-bit message to be encoded followed by $n - k$ 0' and the output will be the encoded message.



- Shift-register encodings of cyclic codes. Small circles represent multiplication by the corresponding constant, $\oplus$ nodes represent modular addition, squares are delay elements

ENCODING of CYCLIC CODES II

- Another method for encoding of cyclic codes is based on the following (so called systematic) representation of the generator and parity-check matrices for cyclic codes.
- Theorem Let C be an (n,k)-code with generator polynomial $g(x)$ and $r = n - k$. For $i = 0,1,\ldots,k - 1$, let $G_{2,i}$ be the length n vector whose polynomial is $G_{2,i}(x) = x^{r+I} - x^{r+I} \bmod g(x)$. Then the $k * n$ matrix G_2 with row vectors $G_{2,I}$ is a generator matrix for C.
- Moreover, if $H_{2,J}$ is the length n vector corresponding to polynomial $H_{2,J}(x) = x^j \bmod g(x)$, then the $r * n$ matrix H_2 with row vectors $H_{2,J}$ is a parity check matrix for C. If the message vector m is encoded by
  - $m \Rightarrow c = mG_2$,
- then the relation between corresponding polynomials is
  - $c(x) = x^r m(x) - [x^r m(x)] \bmod g(x)$.
- On this basis one can construct the following shift-register encoder for the case of a systematic representation of the generator for a cyclic code:

- Shift-register encoder for systematic representation of cyclic codes. Switch A is closed for first k ticks and closed for last r ticks; switch B is down for first k ticks and up for last r ticks.

Hamming codes as cyclic codes

- Definition (Again!) Let r be a positive integer and let H be an $r * (2^r - 1)$ matrix whose columns are distinct non-zero vectors of $V(r,2)$. Then the code having H as its parity-check matrix is called binary **Hamming code** denoted by $Ham\ (r,2)$.

- It can be shown that binary Hamming codes are equivalent to cyclic codes.

Theorem The binary Hamming code $Ham\ (r,2)$ is equivalent to a cyclic code.

Definition If p(x) is an irreducible polynomial of degree r such that x is a primitive element of the field $F[x]$ / p(x), then p(x) is called a primitive polynomial.

Theorem If p(x) is a primitive polynomial over $GF(2)$ of degree r, then the cyclic code $\langle p(x) \rangle$ is the code $Ham\ (r,2)$.

Hamming codes as cyclic codes

- **Example** Polynomial $x^3 + x + 1$ is irreducible over $GF(2)$ and x is primitive element of the field $F_2[x] \,/\, (x^3 + x + 1)$.

  - $F_2[x] \,/\, (x^3 + x + 1) =$
  - $\{0, x, x^2, x^3 = x + 1, x^4 = x^2 + x, x^5 = x^2 + x + 1, x^6 = x^2 + 1\}$

- The parity-check matrix for a cyclic version of *Ham* (*3,2*)

$$H = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

PROOF of THEOREM

- The binary Hamming code *Ham* (r,2) is equivalent to a cyclic code.
- It is known from algebra that if p(x) is an irreducible polynomial of degree r, then the ring $F_2[x]$ / p(x) is a field of order 2^r.
- In addition, every finite field has a primitive element. Therefore, there exists an element a of $F_2[x]$ / p(x) such that
  - $F_2[x]$ / p(x) = $\{0, 1, a, a^2, \ldots, a^{2r-2}\}$.
- Let us identify an element $a_0 + a_1 + \ldots a_{r-1}x^{r-1}$ of $F_2[x]$ / p(x) with the column vector
  - $(a_0, a_1, \ldots, a_{r-1})^T$
- and consider the binary r * (2^r -1) matrix
  - $H = [\,1 \;\; a \;\; a^2 \;\ldots\; a^{2^r-2}\,]$.
- Let now C be the binary linear code having H as a parity check matrix.
- Since the columns of H are all distinct non-zero vectors of $V(r,2)$, C = *Ham* (r,2).
- Putting $n = 2^r - 1$ we get
- $C = \{f_0 f_1 \ldots f_{n-1} \in V(n, 2) \mid f_0 + f_1 a + \ldots + f_{n-1} a^{n-1} = 0$ (2)
- $= \{f(x) \in R_n \mid f(a) = 0 \text{ in } F_2[x] / p(x)\}$ (3)
- If $f(x) \in C$ and $r(x) \in R_n$, then $r(x)f(x) \in C$ because
  - $r(a)f(a) = r(a) \bullet 0 = 0$
- and therefore, by one of the previous theorems, this version of *Ham* (r,2) is cyclic.

BCH codes and Reed-Solomon codes

- To the most important cyclic codes for applications belong BCH codes and Reed-Solomon codes.
- Definition A polynomial p is said to be *minimal* for a complex number x in Z_q if $p(x) = 0$ and p is irreducible over Z_q.

Definition A cyclic code of codewords of length n over Z_q, $q = p^r$, p is a prime, is called BCH code[1] of distance d if its generator $g(x)$ is the least common multiple of the minimal polynomials for

$$w^{l}, w^{l+1}, \ldots, w^{l+d-2}$$

for some l, where

ω is the primitive n-th root of unity.

If $n = q^m - 1$ for some m, then the BCH code is called primitive.

Definition A Reed-Solomon code is a primitive BCH code with $n = q - 1$.

Properties:

- Reed-Solomon codes are self-dual.

[1]BHC stands for Bose and Ray-Chaudhuri and Hocquenghem who discovered these codes.