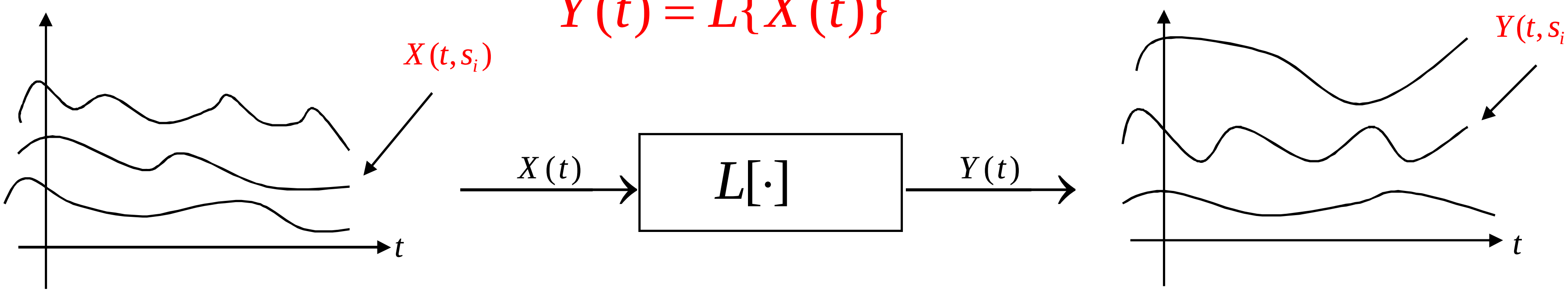
6. Linear Filtering of a Random Signal

Linear System

$$L\{a_1X(t_1)+a_2X(t_2)\}=a_1L\{X(t_1)\}+a_2L\{X(t_2)\}.$$

$$Y(t)=L\{X(t)\}$$



- Our goal is to study the output process statistics in terms of the input process statistics and the system function.

Deterministic System

Deterministic Systems

- Memoryless Systems

  $$Y(t) = g[X(t)]$$

- Systems with Memory
  - Time-varying systems
  - Time-Invariant systems
  - Linear systems

    $$Y(t) = L[X(t)]$$

  - Time-Invariant systems + Linear systems → Linear-Time Invariant (LTI) systems

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t) = \int_{-\infty}^{+\infty} h(t-\tau)X(\tau)d\tau$$

LTI system

$$= \int_{-\infty}^{+\infty} h(\tau)X(t-\tau)d\tau. = X(t) * h(t)$$

Memoryless Systems

The output $Y(t)$ in this case depends only on the present value of the input $X(t)$. i.e., $Y(t) = g\{X(t)\}$.

Strict-sense stationary input → Memoryless system → Strict-sense stationary output.

Wide-sense stationary input → Memoryless system → Need *not* be stationary in any sense.

$X(t)$ stationary Gaussian with $R_{XX}(\tau)$ → Memoryless system → $Y(t)$ stationary, but *not* Gaussian with $R_{XY}(\tau) = \eta R_{XX}(\tau)$.

Linear Time-Invariant Systems

Time-Invariant System

Shift in the input results in the same shift in the output.

$$Y(t) = L\{X(t)\} \Rightarrow L\{X(t-t_0)\} = Y(t-t_0)$$

Linear Time-Invariant System

A linear system with time-invariant property.

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \qquad h(t) = L\{\delta(t)\}$$

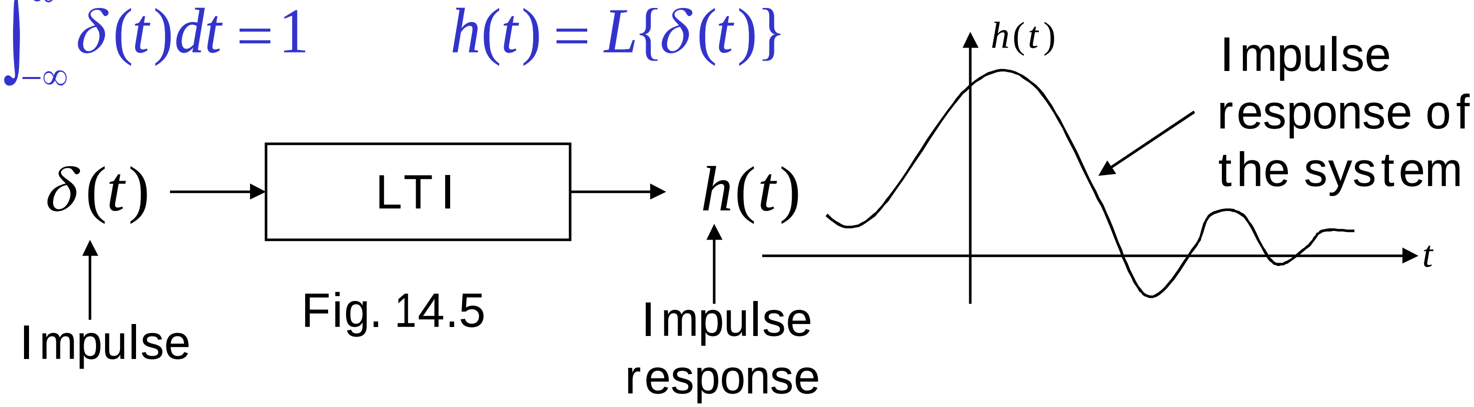
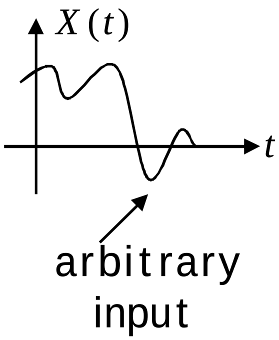
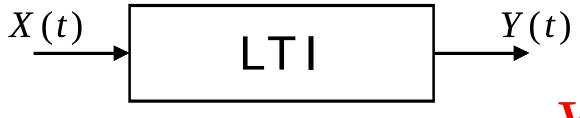
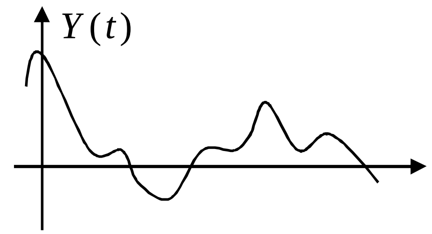


Fig. 14.5

Linear Filtering of a Random Signal







$$Y(t) = \int_{-\infty}^{+\infty} h(t-\tau)X(\tau)d\tau$$
$$= \int_{-\infty}^{+\infty} h(\tau)X(t-\tau)d\tau$$

$$X(t) = \int_{-\infty}^{+\infty} X(\tau)\delta(t-\tau)d\tau$$

$$Y(t) = L\{X(t)\} = L\{\int_{-\infty}^{+\infty} X(\tau)\delta(t-\tau)d\tau\}$$
$$= \int_{-\infty}^{+\infty} L\{X(\tau)\delta(t-\tau)d\tau\}$$
$$= \int_{-\infty}^{+\infty} X(\tau)L\{\delta(t-\tau)\}d\tau$$
$$= \int_{-\infty}^{+\infty} X(\tau)h(t-\tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)X(t-\tau)d\tau.$$

By Linearity

By Time-invariance

Theorem 6.1

$$E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau\right] = \int_{-\infty}^{\infty} h(\tau)E[X(t-\tau)]d\tau$$

$$= E[X(t)] * h(t)$$

Theorem 6.2

If the input to an LTI filter with impulse response $h(t)$ is a wide sense stationary process $X(t)$, the output $Y(t)$ has the following properties:

(a) $Y(t)$ is a WSS process with expected value

$$\mu_Y = E[Y(t)] = \mu_X \int_{-\infty}^{\infty} h(\tau)d\tau$$

autocorrelation function

$$R_Y(\tau) = \int_{-\infty}^{\infty} h(u) \int_{-\infty}^{\infty} h(v) R_X(\tau + u - v)dudv$$

(b) $X(t)$ and $Y(t)$ are jointly WSS and have I/O cross- correlation by

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} h(u)R_X(\tau - u)du = R_X(\tau) * h(\tau)$$

(c) The output autocorrelation is related to the I/O cross-correlation by

$$R_Y(\tau) = \int_{-\infty}^{\infty} h(-w)R_{XY}(\tau - w)dw$$

$$= R_{XY}(\tau) * h(-\tau)$$

Example 6.1

$X(t)$, a WSS stochastic process with expected value $\mu_X = 10$ volts, is the input to an LTI filter with

$$h(t) = \begin{cases} e^{5t} & 0 \le t \le 0.1 \text{ sec}, \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected value of the filter output process $Y(t)$?

Sol : *Ans:* $2(e^{0.5}-1)$ V

Example 6.2

A white Gaussian noise process $X(t)$ with autocorrelation function $R_W(\tau) = \eta_0 \delta(\tau)$ is passed through the moving-average filter

$$h(t) = \begin{cases} 1/T & 0 \le t \le T, \\ 0 & \text{otherwise.} \end{cases}$$

For the output $Y(t)$, find the expected value $E[Y(t)]$, the I/O cross-correlation $R_{WY}(\tau)$ and the autocorrelation $R_Y(\tau)$.

Sol :

$$Ans: R_{WY}(\tau) = \begin{cases} \eta_0 / T & 0 \le \tau \le T, \\ 0 & \text{otherwise.} \end{cases} \quad R_Y(\tau) = \begin{cases} \eta_0 (T - |\tau|) / T^2 & |\tau| \le T, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 6.3

If a stationary Gaussian process $X(t)$ is the input to an LTI Filter $h(t)$, the output $Y(t)$ is a stationary Gaussian process with expected value and autocorrelation given by Theorem 6.2.

Example 6.3

For the white noise moving-average process $Y(t)$ in Example 6.2, let $\eta_0 = 10^{-15}$ W/Hz and $T = 10^{-3}$ s. For an arbitrary time t_0, find $P[Y(t_0) > 4\times10^{-6}]$.

Sol :

Ans: $Q(4) = 3.17\times10^{-5}$

Theorem 6.4

The random sequence X_n is obtained by sampling the continuous-time process $X(t)$ at a rate of $1/T_s$ samples per second. If $X(t)$ is a WSS process with expected value $E[X(t)] = \mu_X$ and autocorrelation $R_X(\tau)$, then X_n is a WSS random sequence with expected value $E[X_n] = \mu_X$ and autocorrelation function $R_X[k] = R_X(kT_s)$.

Example 6.4

Continuing Example 6.3, the random sequence Y_n is obtained by sampling the white noise moving-average process $Y(t)$ at a rate of $f_s = 10^4$ samples per second. Derive the autocorrelation function $R_Y[n]$ of Y_n.

Sol :

$$Ans: \quad R_Y[n] = \begin{cases} 10^{-6}(1-0.1|n|) & |n| \le 10, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 6.5

If the input to a discrete-time LTI filter with impulse response h_n is a WSS random sequence, X_n, the output Y_n has the following properties.

(a) Y_n is a WSS random sequence with expected value

and autocorrelation function

$$\mu_Y = E[Y_n] = \mu_X \sum_{n=-\infty}^{\infty} h_n.$$

(b) Y_n and X_n are jointly WSS with I/O cross-correlation

$$R_Y[n] = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h_i h_j R_X[n+i-j].$$

(c) The output autocorrelation is related to the I/O cross-correlation by

$$R_{XY}[n] = \sum_{i=-\infty}^{\infty} h_i R_X[n-i].$$

$$R_Y[n] = \sum_{i=-\infty}^{\infty} h_{-i} R_{XY}[n-i].$$

Example 6.5

A WSS random sequence, X_n, with $\mu_X = 1$ and auto-correlation function $R_X[n]$ is the input to the order $M-1$ discrete-time moving-average filter h_n where

$$h_n = \begin{cases} 1/M & n = 0,1,\cdots,M-1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad R_X[n] = \begin{cases} 4 & n = 0, \\ 2 & n = \pm 1, \\ 0 & |n| \geq 2. \end{cases}$$

For the case $M = 2$, find the following properties of the output random sequence Y_n : the expected value μ_Y, the autocorrelation $R_Y[n]$, and the variance $Var[Y_n]$.

Sol :

$$Ans: R_Y[n] = \begin{cases} 3 & n = 0, \\ 2 & |n| = 1, \\ 1/2 & |n| = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Example 6.6

A WSS random sequence, X_n, with $\mu_X = 0$ and auto-correlation function $R_X[n] = \sigma^2\delta_n$ is passed through the order $M-1$ discrete-time moving-average filter h_n where

$$h_n = \begin{cases} 1/M & n = 0,1,\cdots,M-1, \\ 0 & \text{otherwise.} \end{cases}$$

Find the output autocorrelation $R_Y[n]$.

Sol :

$$Ans: \quad R_Y[n] = \begin{cases} \sigma^2(M-|n|)/M^2 & |n| \le (M-1), \\ 0 & \text{otherwise.} \end{cases}$$

Example 6.7

A first-order discrete-time integrator with WSS input sequence X_n has output $Y_n = X_n + 0.8Y_{n-1}$. What is the impulse response h_n ?

Sol :

$$Ans: \quad R_Y[n] = \begin{cases} 0.8^n & n = 0,1,2,\cdots \\ 0 & \text{otherwise.} \end{cases}$$

Example 6.8

Continuing Example 6.7, suppose the WSS input X_n with expected value $\mu_X = 0$ and autocorrelation function

$$R_X[n] = \begin{cases} 1 & n = 0, \\ 0.5 & |n| = 1, \\ 0 & |n| \geq 2. \end{cases}$$

is the input to the first-order integrator h_n . Find the second moment, $E[Y_n^2]$, of the output.

Theorem 6.6

If X_n is a WSS process with expected value μ and auto-correlation function $R_X[k]$, then the vector $\vec{X}_n$ has correlation matrix $\vec{R}_{\vec{X}_n}$ and expected value $E[\vec{X}_n]$ given by

$$\vec{R}_{\vec{X}_n} = \begin{bmatrix} R_X[0] & R_X[1] & \cdots & R_X[M-1] \\ R_X[1] & R_X[0] & \cdots & \vdots \\ \vdots & \vdots & \ddots & R_X[1] \\ R_X[M-1] & \cdots & R_X[1] & R_X[0] \end{bmatrix}, \quad E[\vec{X}_n] = \mu \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

where $\vec{X}_n = [X_{n-M+1} \;\; X_{n-M+2} \;\; \cdots \;\; X_n]^T$ is the M-dimensional vector.

$$\vec{R}_{\vec{X}_n} = E\left[\vec{X}_n \vec{X}_n^T\right]$$

Example 6.9

The WSS sequence X_n has autocorrelation function $R_X[n]$ as given in Example 6.5. Find the correlation matrix of

Sol :

$$\vec{X}_{33} = \begin{bmatrix} X_{30} & X_{31} & X_{32} & X_{33} \end{bmatrix}.$$

$$R_X[n] = \begin{cases} 4 & n = 0, \\ 2 & n = \pm 1, \\ 0 & |n| \geq 2. \end{cases}$$

Example 6.10

The order $M-1$ averaging filter h_n given in Example 6.6 can be represented by the M element vector

$$\vec{h} = \frac{1}{M}\begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T, \quad h_n = \begin{cases} 1/M & n = 0,1,\cdots,M-1, \\ 0 & \text{otherwise.} \end{cases}$$

The input is $\vec{X} = \begin{bmatrix}X_0 & X_1 & \cdots & X_{L-1}\end{bmatrix}^T$, then

The output vector

$$\vec{Y} = \begin{bmatrix}Y_0 & Y_1 & \cdots & Y_{L+M-2}\end{bmatrix}^T \qquad \vec{Y} = \vec{H}\cdot\vec{X}$$

$$\underbrace{\begin{bmatrix} Y_0 \\ \vdots \\ Y_{M-1} \\ \vdots \\ \vdots \\ Y_{L-1} \\ \vdots \\ Y_{L+M-2} \end{bmatrix}}_{\vec{Y}} = \underbrace{\begin{bmatrix} h_0 & & & & \\ \vdots & \ddots & & & \\ h_{M-1} & \cdots & h_0 & & \\ & \ddots & & \ddots & \\ & & \ddots & & \ddots \\ & & h_{M-1} & \cdots & h_0 \\ & & & \ddots & \vdots \\ & & & & h_{M-1} \end{bmatrix}}_{\vec{H}} \bullet \underbrace{\begin{bmatrix} X_0 \\ \vdots \\ X_{M-1} \\ \vdots \\ \vdots \\ X_{L-M} \\ \vdots \\ X_{L-1} \end{bmatrix}}_{\vec{X}}$$