Cross Spectral Density

Definition :

For jointly WSS random processes $X(t)$ and $Y(t)$, the Fourier transform of the cross-correlation yields the *cross spectral density*

$$S_{XY}(f) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j2\pi f\tau} d\tau.$$

For jointly WSS random sequences X_n and Y_n, the Fourier transform of the cross-correlation yields the *cross spectral density*

$$S_{XY}(\phi) = \sum_{k=-\infty}^{\infty} R_{XY}[k] e^{-j2\pi\phi k}.$$

Example 7.7

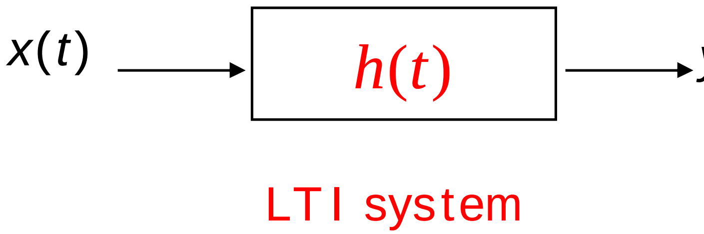
Let $Y(t) = X(t) + N(t)$ where $N(t)$ is a WSS noise process with $\mu_N = 0$.
In this case, when $X(t)$ and $N(t)$ are jointly WSS, we found that

$$R_Y(\tau) = R_X(\tau) + R_{XN}(\tau) + R_{NX}(\tau) + R_N(\tau).$$

Suppose that $X(t)$ and $N(t)$ are independent, find the autocorrelation and power spectral density of the observation $Y(t)$.

Sol : $Ans: S_Y(f) = S_X(f) + S_N(f)$

Frequency Domain Filter Relationships



$$y(t) = \int_{-\infty}^{+\infty} h(t-\tau)\,x(\tau)\,d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau)\,x(t-\tau)\,d\tau. = x(t) * h(t)$$

Time Domain : $Y(t) = X(t) * h(t)$

Frequency Domain : $W(f) = V(f)H(f)$

where $V(f) = F\{X(t)\}$,

$W(f) = F\{Y(t)\}$, and

$H(f) = F\{h(t)\}$.

Theorem 7.5

When a WSS stochastic process $X(t)$ is the input to a LTI filter with transfer function $H(f)$, the power spectral density of the output $Y(t)$ is $S_Y(f) = |H(f)|^2 S_X(f).$

When a WSS random sequence X_n is the input to a LTI filter with transfer function $H(\phi)$, the power spectral density of the output Y_n is $S_Y(\phi) = |H(\phi)|^2 S_X(\phi).$

Example 7.8

A WSS stochastic process $X(t)$ with autocorrelation function

$R_X(\tau) = e^{-b|\tau|}$ is the input to an RC filter with impulse response

$$h(t) = \begin{cases} (1/RC)e^{-t/RC} & t \geq 0, \\ 0 & \text{otherwise.} \end{cases},$$

Assume $b > 0$ and $b \neq 1/RC$, find $S_Y(f)$ and $R_Y(\tau)$, the power spectral density and autocorrelation of the filter output $Y(t)$.

What is the average power of the output stochastic process?

Sol :

$$Ans: S_Y(f) = \frac{2b(1/RC)^2}{[(2\pi f)^2 + (1/RC)^2][(2\pi f)^2 + b^2]}, \quad E[Y^2(t)] = \frac{1/RC}{b + 1/RC}$$

Example 7.9

The random sequence X_n has power sepctral density $S_X(\phi) = 2 + 2\cos(2\pi\phi)$.

This sequence is the input to a filter with impulse response

$$h_n = \begin{cases} 1 & n = 0, \\ -1 & n = -1,\ 1, \\ 0 & \text{otherwise.} \end{cases}$$

Derive $S_Y(\phi)$, the power spectral density function of the output sequence Y_n. What is $E[Y_n^2]$?

Sol :

$Ans: S_Y(\phi) = 2 + 2\cos(6\pi\phi),\ E[Y_n^2] = 2$

Theorem 7.6

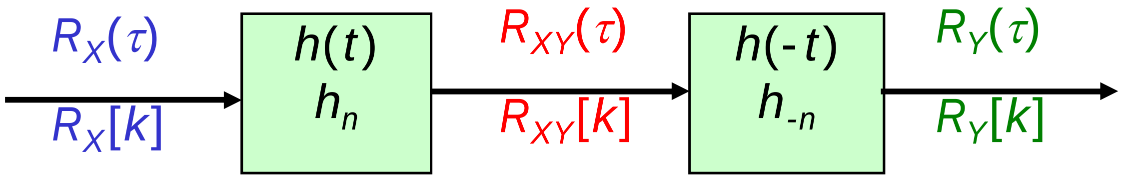
If the WSS stochastic process $X(t)$ is the input to an LTI filter with transfer function $H(f)$, and $Y(t)$ is the filter output, the input-output cross power spectral density function and the output power spectral density function are

$$S_{XY}(f) = H(f)S_X(f), \qquad S_Y(f) = H^*(f)S_{XY}(f).$$

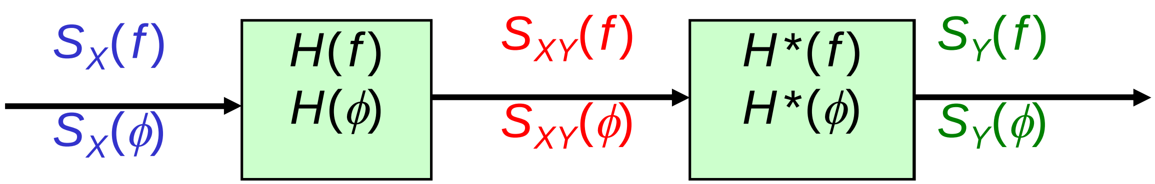
If the WSS stochastic process X_n is the input to a LTI filter with transfer function $H(\phi)$, and Y_n is the filter output, the input-output cross power spectral density function and the output power spectral density function are

$$S_{XY}(\phi) = H(\phi)S_X(\phi), \qquad S_Y(\phi) = H^*(\phi)S_{XY}(\phi).$$

I/O Correlation and Spectral Density Functions



Time Domain



Frequency Domain