Random processes - basic concepts

Random processes - basic concepts

- Topics :

- Concepts of deterministic and random processes

  stationarity, ergodicity

- Basic properties of a single random process

  mean, standard deviation, auto-correlation, spectral density

- Joint properties of two or more random processes

  correlation, covariance, cross spectral density, simple input-output relations

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- Deterministic and random processes :

- both continuous functions of time (usually), mathematical concepts

- deterministic processes :

  physical process is represented by explicit mathematical relation

- Example :

  response of a single mass-spring-damper in free vibration in laboratory

- Random processes :

result of a large number of separate causes. Described in probabilistic terms and by properties which are averages

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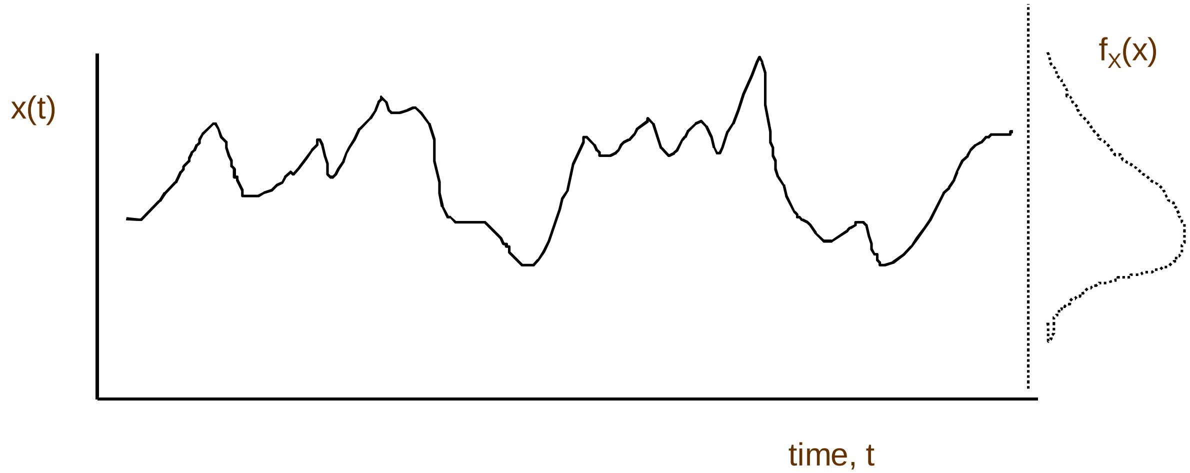
- random processes :



- The probability density function describes the general distribution of the magnitude of the random process, but it gives no information on the time or frequency content of the process

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- # Averaging and stationarity :

- Underlying process
- Sample records which are individual representations of the underlying process
- Ensemble averaging :

  properties of the process are obtained by averaging over a collection or ‘ensemble’ of sample records using values at corresponding times
- Time averaging :

  properties are obtained by averaging over a single record in time

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- Stationary random process :

- Ensemble averages do not vary with time

- Ergodic process :

  stationary process in which averages from a single record are the same as those obtained from averaging over the ensemble

  Most stationary random processes can be treated as ergodic

Wind loading from extra - tropical synoptic gales can be treated as *stationary* random processes

Wind loading from hurricanes - stationary over shorter periods <2 hours
- non stationary over the duration of the storm

Wind loading from thunderstorms, tornadoes - *non stationary*

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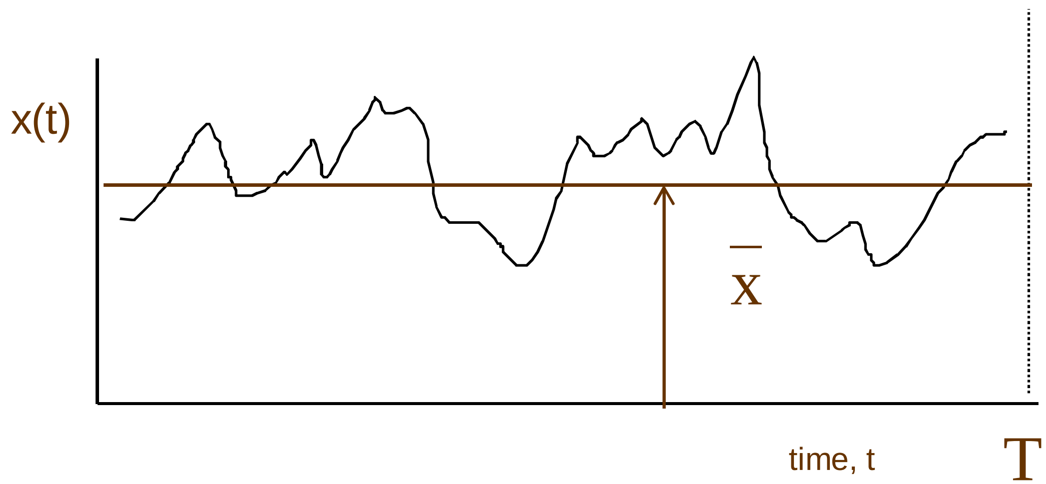
- Mean value :



$$\bar{x} = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)dt$$

- The mean value, $\bar{x}$, is the height of the rectangular area having the same area as that under the function x(t)

- Can also be defined as the first moment of the p.d.f. (ref. Lecture 3)

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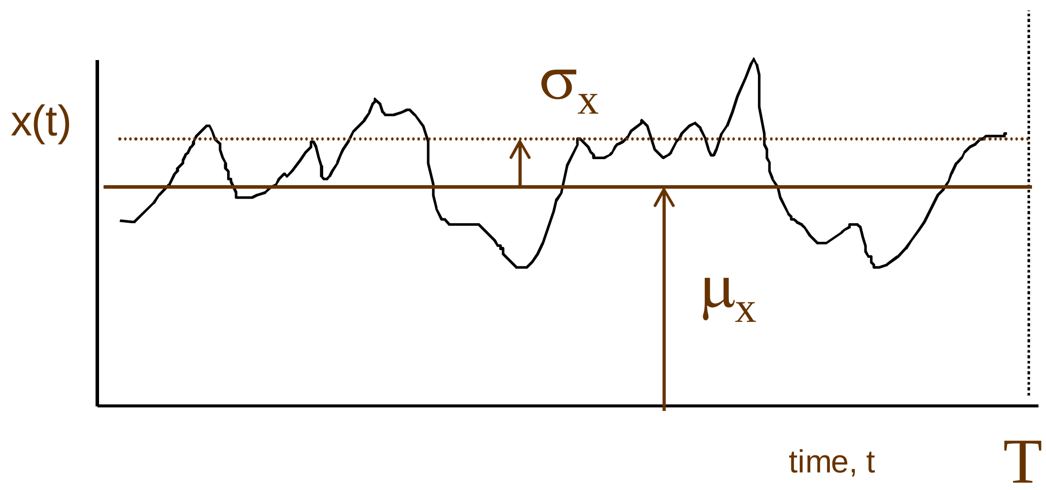
- Mean square value, variance, standard deviation :



mean square value, $\overline{x^2} = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)dt$

variance, $\sigma_x^2 = \overline{\left(x(t) - \overline{x}\right)^2} = \lim_{T\to\infty} \frac{1}{T}\int_0^T [x(t) - \overline{x}]^2 dt$

(average of the square of the deviation of x(t) from the mean value, $\overline{x}$)

standard deviation, σ_x, is the square root of the variance

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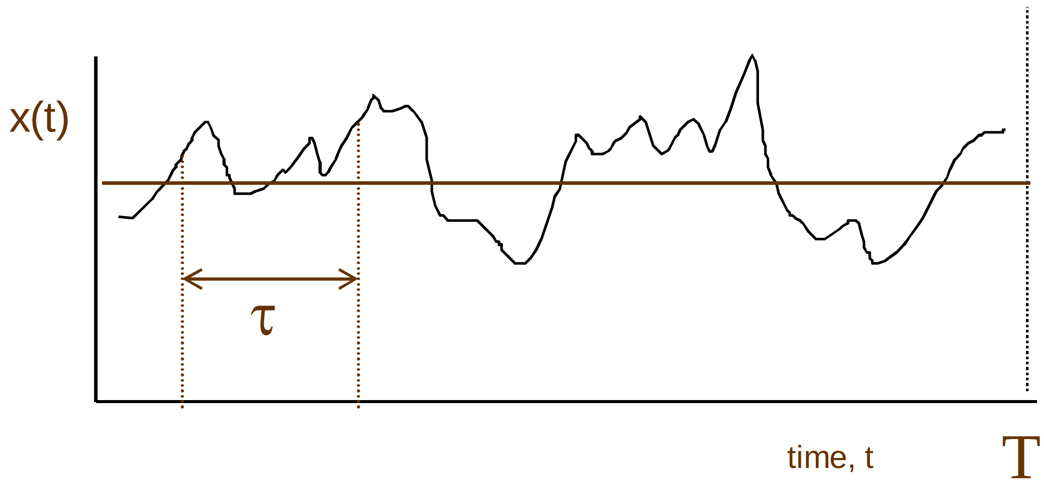
- Autocorrelation :



- The autocorrelation, or autocovariance, describes the general dependency of x(t) with its value at a short time later, $x(t+\tau)$

$$\rho_x(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left[x(t) - \overline{x}\right]\left[x(t+\tau) - \overline{x}\right] dt$$

The value of $\rho_x(\tau)$ at τ equal to 0 is the variance, σ_x^2

Normalized auto-correlation : $R(\tau) = \rho_x(\tau)/\sigma_x^2$ $\qquad$ $R(0) = 1$

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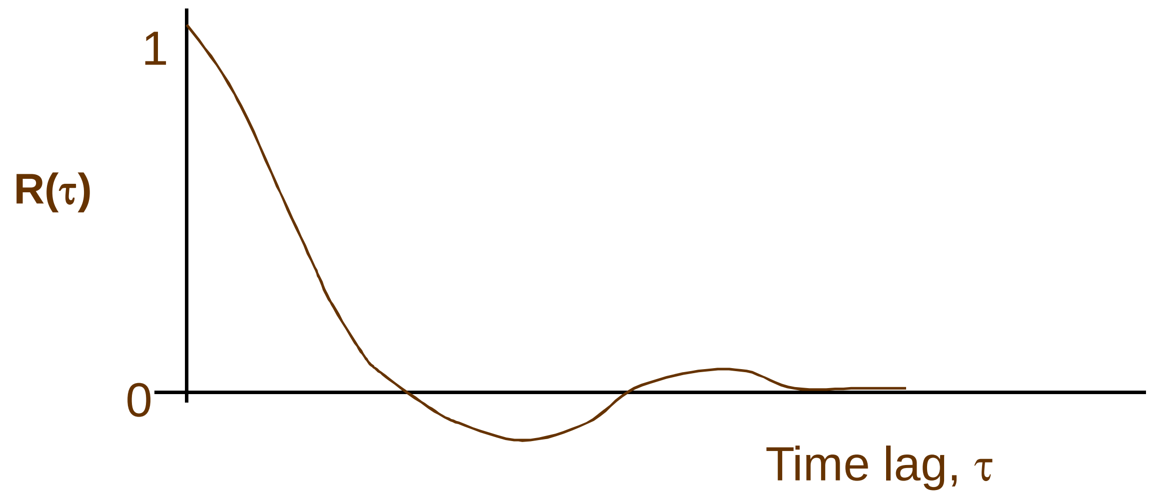
- Autocorrelation :



- The autocorrelation for a random process eventually decays to zero at large τ

- The autocorrelation for a sinusoidal process (deterministic) is a cosine function which does not decay to zero

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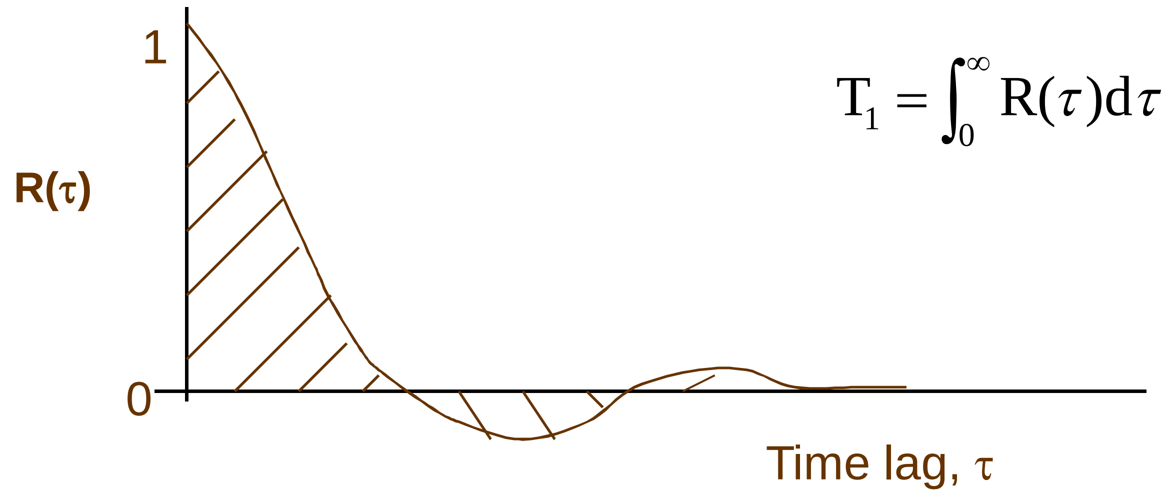
- Autocorrelation :



$$T_1 = \int_0^\infty R(\tau)\,d\tau$$

- The area under the normalized autocorrelation function for the fluctuating wind velocity measured at a point is a measure of the average time scale of the eddies being carried passed the measurement point, say T_1
- If we assume that the eddies are being swept passed at the mean velocity, $\bar{U}.T_1$ is a measure of the average length scale of the eddies
- This is known as the 'integral length scale', denoted by ℓ_u

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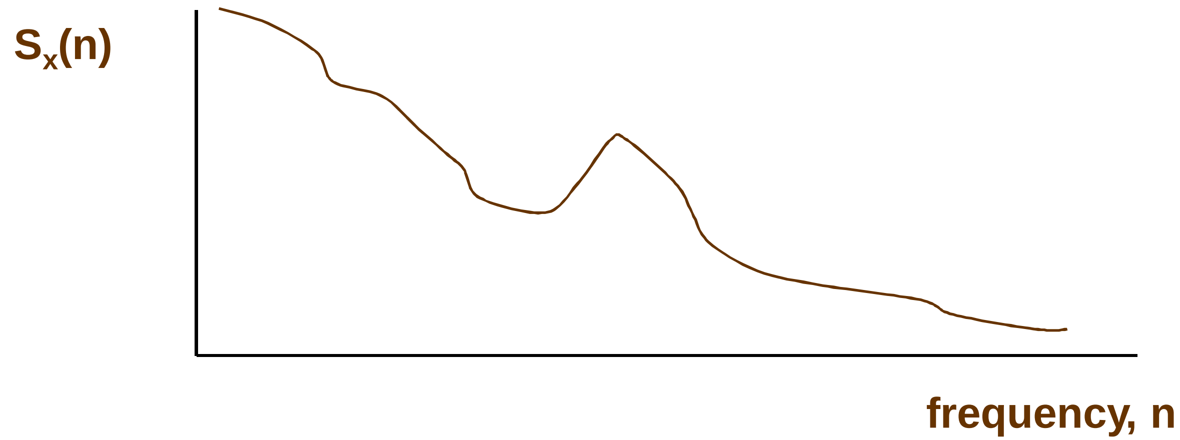
- Spectral density :



- The spectral density, (auto-spectral density, power spectral density, spectrum) describes the average frequency content of a random process, x(t)

Basic relationship (1) : $$\sigma_x^2 = \int_0^\infty S_x(n)\,dn$$

The quantity $S_x(n).\delta n$ represents the contribution to σ_x^2 from the frequency increment δn

Units of $S_x(n)$: [units of x]^2 . sec

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- Spectral density :

Basic relationship (2) :

$$S_x(n) = \lim_{T\to\infty}\left[\frac{2}{T}\left|X_T(n)\right|^2\right]$$

Where $X_T(n)$ is the Fourier Transform of the process x(t) taken over the time interval $-T/2<t<+T/2$

The above relationship is the basis for the usual method of obtaining the spectral density of experimental data

Usually a Fast Fourier Transform (FFT) algorithm is used

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- Spectral density :

Basic relationship (3) : $$S_x(n) = 2\int_{-\infty}^{\infty} \rho_x(\tau) e^{-i2\pi n\tau} d\tau$$

The spectral density is twice the Fourier Transform of the autocorrelation function

Inverse relationship :

$$\rho_x(\tau) = \mathrm{Real}\left\{\int_0^{\infty} S_x(n) e^{i2\pi n\tau} dn\right\} = \int_0^{\infty} S_x(n)\cos(2\pi n\tau) dn$$

Thus the spectral density and auto-correlation are closely linked -

they basically provide the same information about the process x(t)

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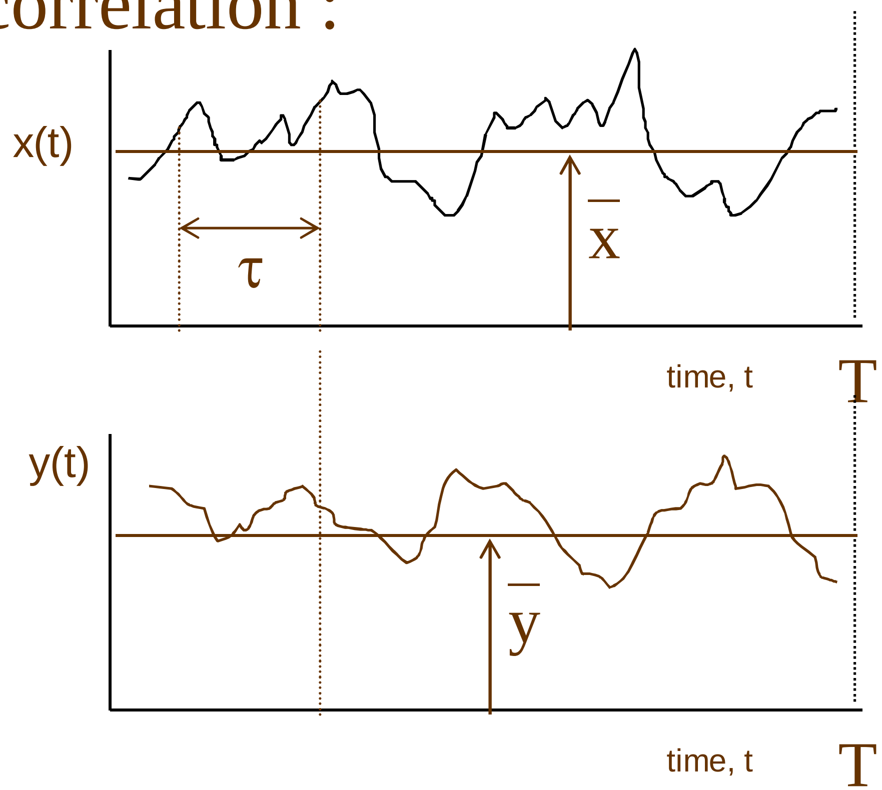
- Cross-correlation :



- The cross-correlation function describes the general dependency of x(t) with another random process y(t+τ), delayed by a time delay, τ

$$c_{xy}(\tau) = \underset{T\to\infty}{\mathrm{Lim}} \frac{1}{T}\int_0^T \left[x(t) - \bar{x}\right]\left[y(t+\tau) - \bar{y}\right]dt$$

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- Covariance :

  - The covariance is the cross correlation function with the time delay, τ, set to zero

$$c_{xy}(0) = \overline{x'(t).y'(t)} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \left[x(t) - \overline{x}\right].\left[y(t) - \overline{y}\right]dt$$

Note that here x'(t) and y'(t) are used to denote the fluctuating parts of x(t) and y(t) (mean parts subtracted)

(Section 3.3.5 in “Wind loading of structures”)

Random processes - basic concepts

- Correlation coefficient :

  - The correlation coefficient, ρ, is the covariance normalized by the standard deviations of x and y

$$\rho = \frac{\overline{x'(t).y'(t)}}{\sigma_x.\sigma_y}$$

When x and y are identical to each other, the value of ρ is +1 (full correlation)

When $y(t)=-x(t)$, the value of ρ is – 1

In general, $-1 < \rho < +1$

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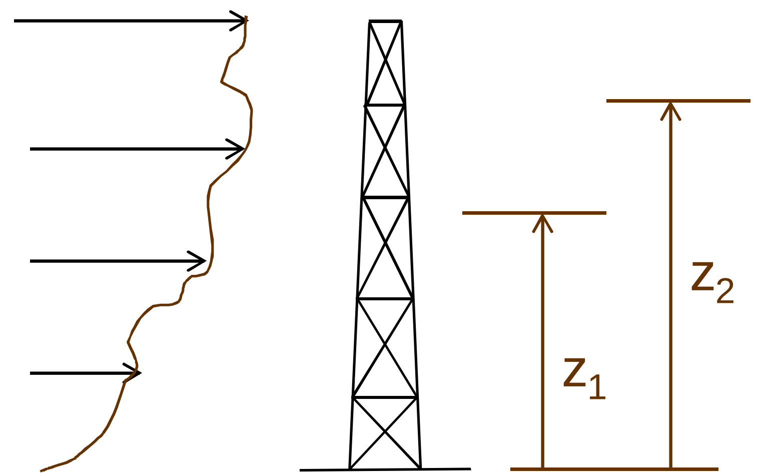
- # Correlation - application :

- The fluctuating wind loading of a tower depends on the correlation coefficient between wind velocities and hence wind loads, at various heights



For heights, z_1, and z_2 :

$$\rho(z_1, z_2) = \frac{\overline{u'(z_1).u'(z_2)}}{\sigma_u(z_1).\sigma_u(z_2)}$$

Random processes - basic concepts

- Cross spectral density :

By analogy with the spectral density : $$\mathrm{S}_{\mathrm{xy}}(\mathrm{n}) = 2\int_{-\infty}^{\infty} c_{xy}(\tau)\mathrm{e}^{-i2\pi \mathrm{n}\tau}\mathrm{d}\tau$$

The cross spectral density is twice the Fourier Transform of the cross-correlation function for the processes x(t) and y(t)

The cross-spectral density (cross-spectrum) is a complex number :

$$\mathrm{S}_{\mathrm{xy}}(n) = C_{xy}(n) + iQ_{xy}(n)$$

$C_{xy}(n)$ is the co(-incident) spectral density - (in phase)

$Q_{xy}(n)$ is the quad (-rature) spectral density - (out of phase)

- Normalized co- spectral density :

$$\rho_{xy}(\mathrm{n}) = \frac{C_{xy}(n)}{\sqrt{S_x(n).S_y(n)}}$$

It is effectively a correlation coefficient for fluctuations at frequency, n

Application : Excitation of resonant vibration of structures by fluctuating wind forces

If x(t) and y(t) are local fluctuating forces acting at different parts of the structure, $\rho_{xy}(n_1)$ describes how well the forces are correlated ('synchronized') at the structural natural frequency, n_1

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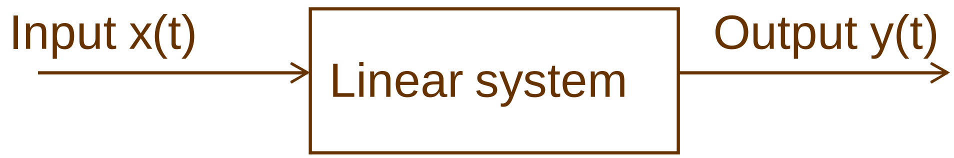
- # Input - output relationships :



There are many cases in which it is of interest to know how an input random process x(t) is modified by a system to give a random output process y(t)

Application : The input is wind force - the output is structural response (e.g. displacement acceleration, stress). The 'system' is the dynamic characteristics of the structure.

Linear system : 1) output resulting from a sum of inputs, is equal to the sum of outputs produced by each input individually (additive property)

Linear system : 2) output produced by a constant times the input, is equal to the constant times the output produced by the input alone (homogeneous property)

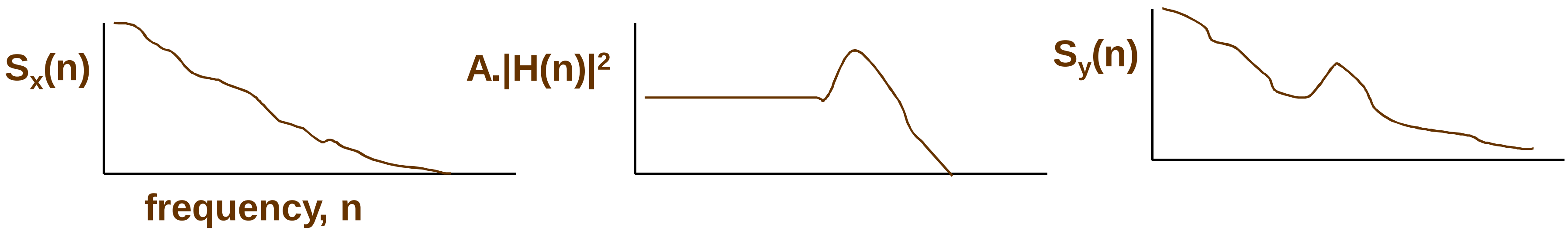
Random processes - basic concepts

- Input - output relationships :

Relation between spectral density of output and spectral density of input :

$$S_y(n) = A . |H(n)|^2 . S_x(n)$$

$|H(n)|^2$ is a transfer function, frequency response function, or 'admittance'



Proof : Bendat & Piersol, Newland