Discrete Probability Distributions

Random Variables

- Random Variable (RV): A numeric outcome that results from an experiment
- For each element of an experiment's sample space, the random variable can take on exactly one value
- Discrete Random Variable: An RV that can take on only a finite or countably infinite set of outcomes
- Continuous Random Variable: An RV that can take on any value along a continuum (but may be reported "discretely"
- Random Variables are denoted by upper case letters (Y)
- Individual outcomes for RV are denoted by lower case letters (y)

Probability Distributions

- Probability Distribution: Table, Graph, or Formula that describes values a random variable can take on, and its corresponding probability (discrete RV) or density (continuous RV)
- Discrete Probability Distribution: Assigns probabilities (masses) to the individual outcomes
- Continuous Probability Distribution: Assigns density at individual points, probability of ranges can be obtained by integrating density function
- Discrete Probabilities denoted by: $p(y) = P(Y=y)$
- Continuous Densities denoted by: $f(y)$
- Cumulative Distribution Function: $F(y) = P(Y \leq y)$

Discrete Probability Distributions

Probabilit y (Mass) Function :

$$p(y) = P(Y = y)$$

$$p(y) \geq 0 \quad \forall y$$

$$\sum_{\text{all } y} p(y) = 1$$

Cumulative Distributi on Function (CDF) :

$$F(y) = P(Y \leq y)$$

$$F(b) = P(Y \leq b) = \sum_{y=-\infty}^{b} p(y)$$

$$F(-\infty) = 0 \quad F(\infty) = 1$$

$F(y)$ is monotonica lly increasing in y

Example – Rolling 2 Dice (Red/Green)

Y = Sum of the up faces of the two die. Table gives value of *y* for all elements in *S*

Red\Green	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

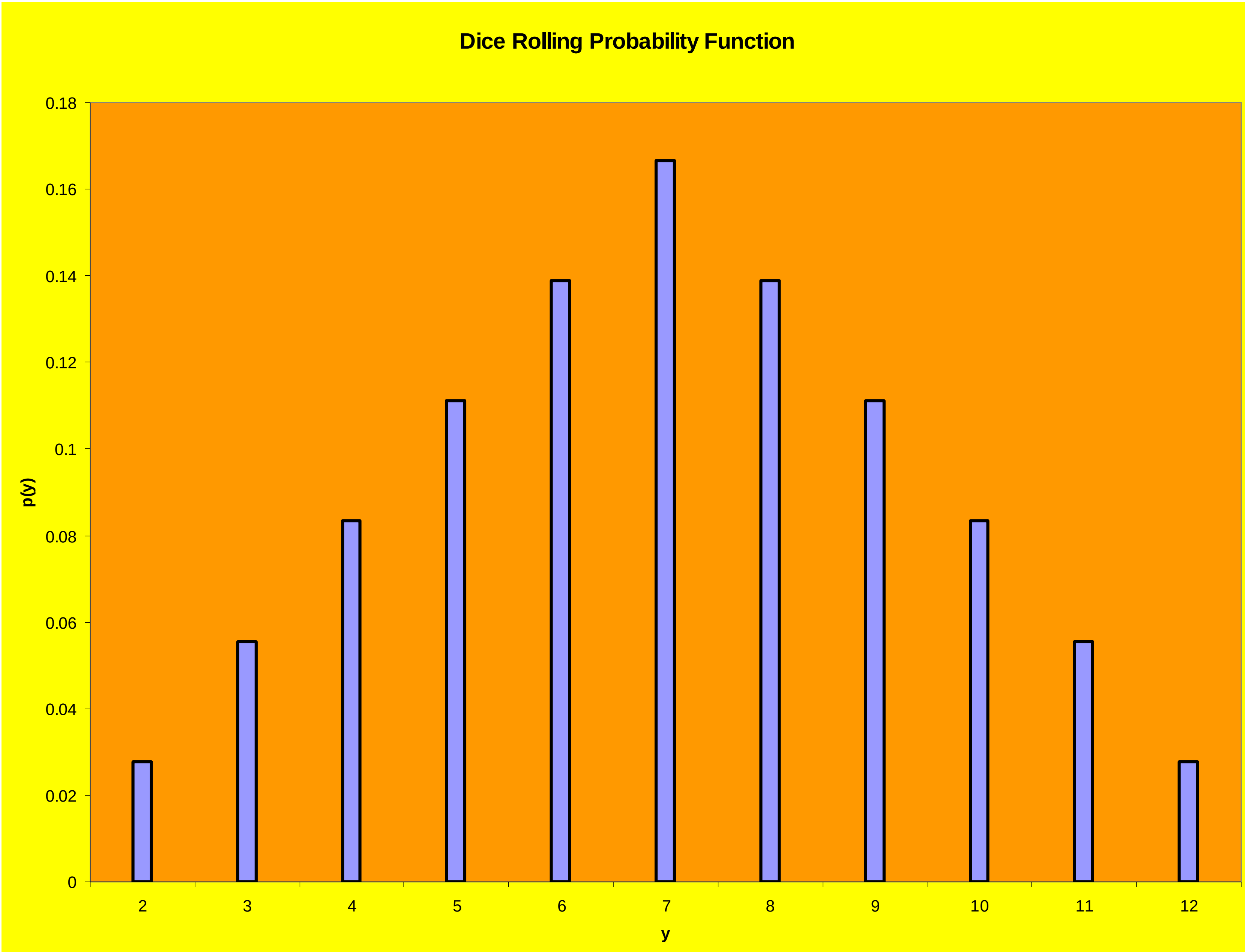
Rolling 2 Dice – Probability Mass Function & CDF

y	p(y)	F(y)
2	1/36	1/36
3	2/36	3/36
4	3/36	6/36
5	4/36	10/36
6	5/36	15/36
7	6/36	21/36
8	5/36	26/36
9	4/36	30/36
10	3/36	33/36
11	2/36	35/36
12	1/36	36/36

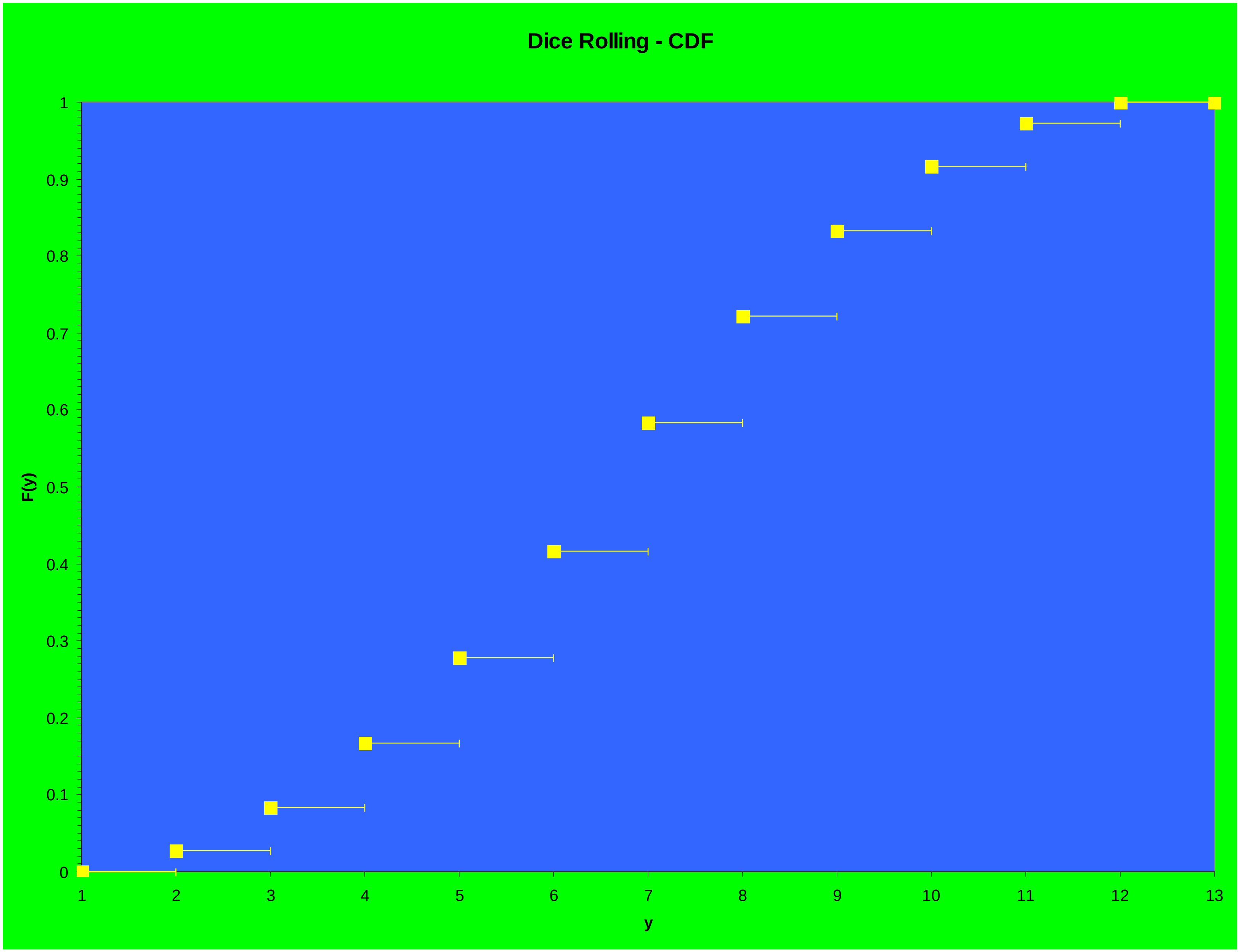
$$p(y) = \frac{\#\text{ of ways 2 die can sum to } y}{\#\text{ of ways 2 die can result in}}$$

$$F(y) = \sum_{t=2}^{y} p(t)$$

Rolling 2 Dice – Probability Mass Function



Rolling 2 Dice – Cumulative Distribution Function



Expected Values of Discrete RV's

- Mean (aka Expected Value) – Long-Run average value an RV (or function of RV) will take on
- Variance – Average squared deviation between a realization of an RV (or function of RV) and its mean
- Standard Deviation – Positive Square Root of Variance (in same units as the data)
- Notation:
  - Mean: $E(Y) = m$
  - Variance: $V(Y) = s^2$
  - Standard Deviation: s

Expected Values of Discrete RV's

$$\text{Mean}: \ E(Y) = \mu = \sum_{\text{all } y} y p(y)$$

$$\text{Mean of a function } \ g(Y): \ E\left[g(Y)\right] = \sum_{\text{all } y} g(y) p(y)$$

$$\text{Variance}: V(Y) = \sigma^2 = E\left[(Y - E(Y))^2\right] = E\left[(Y - \mu)^2\right] =$$

$$= \sum_{\text{all } y} (y - \mu)^2 p(y) = \sum_{\text{all } y} \left(y^2 - 2y\mu + \mu^2\right) p(y) =$$

$$= \sum_{\text{all } y} y^2 p(y) - 2\mu \sum_{\text{all } y} y p(y) + \mu^2 \sum_{\text{all } y} p(y) =$$

$$= E\left[Y^2\right] - 2\mu(\mu) + \mu^2(1) = E\left[Y^2\right] - \mu^2$$

$$\text{Standard Deviation}: \sigma = +\sqrt{\sigma^2}$$

Expected Values of Linear Functions of Discrete RV's

$$\text{Linear Functions : } g(Y) = aY + b \quad (a, b \equiv \text{constants})$$

$$E[aY+b] = \sum_{\text{all } y} (ay+b)\, p(y) =$$

$$= a\sum_{\text{all } y} yp(y) + b\sum_{\text{all } y} p(y) = a\mu + b$$

$$V[aY+b] = \sum_{\text{all } y} \left((ay+b) - (a\mu+b)\right)^2 p(y) =$$

$$\sum_{\text{all } y} \left(ay - a\mu\right)^2 p(y) = \sum_{\text{all } y} \left[a^2 \left(y-\mu\right)^2\right] p(y) =$$

$$= a^2 \sum_{\text{all } y} (y-\mu)^2\, p(y) = a^2\sigma^2$$

$$\sigma_{aY+b} = |a|\sigma$$

Example – Rolling 2 Dice

y	p(y)	yp(y)	y^2p(y)
2	1/36	2/36	4/36
3	2/36	6/36	18/36
4	3/36	12/36	48/36
5	4/36	20/36	100/36
6	5/36	30/36	180/36
7	6/36	42/36	294/36
8	5/36	40/36	320/36
9	4/36	36/36	324/36
10	3/36	30/36	300/36
11	2/36	22/36	242/36
12	1/36	12/36	144/36
Sum	36/36 =1.00	252/36 =7.00	1974/36= 54.833

$$\mu = E(Y) = \sum_{y=2}^{12} yp(y) = 7.0$$

$$\sigma^2 = E\left[Y^2\right] - \mu^2 = \sum_{y=2}^{12} y^2 p(y) - \mu^2$$

$$= 54.8333 - (7.0)^2 = 5.8333$$

$$\sigma = \sqrt{5.8333} = 2.4152$$

Binomial Experiment

- Experiment consists of a series of n identical trials
- Each trial can end in one of 2 outcomes: Success (S) or Failure (F)
- Trials are independent (outcome of one has no bearing on outcomes of others)
- Probability of Success, p, is constant for all trials
- Random Variable Y, is the number of Successes in the n trials is said to follow Binomial Distribution with parameters n and p
- Y can take on the values $y=0,1,\ldots,n$
- Notation: $Y \sim Bin(n,p)$

Binomial Distribution

Consider outcomes of an experiment with 3 Trials :

$SSS \Rightarrow y = 3 \quad P(SSS) = P(Y = 3) = p(3) = p^3$

$SSF, SFS, FSS \Rightarrow y = 2 \quad P(SSF \cup SFS \cup FSS) = P(Y = 2) = p(2) = 3p^2(1 - p)$

$SFF, FSF, FFS \Rightarrow y = 1 \quad P(SFF \cup FSF \cup FFS) = P(Y = 1) = p(1) = 3p(1 - p)^2$

$FFF \Rightarrow y = 0 \quad P(FFF) = P(Y = 0) = p(0) = (1 - p)^3$

In General :

1) # of ways of arranging $y\ S^s$ (and $(n - y)\ F^s$) in a sequence of n positions $\equiv \binom{n}{y} = \dfrac{n!}{y!(n - y)!}$

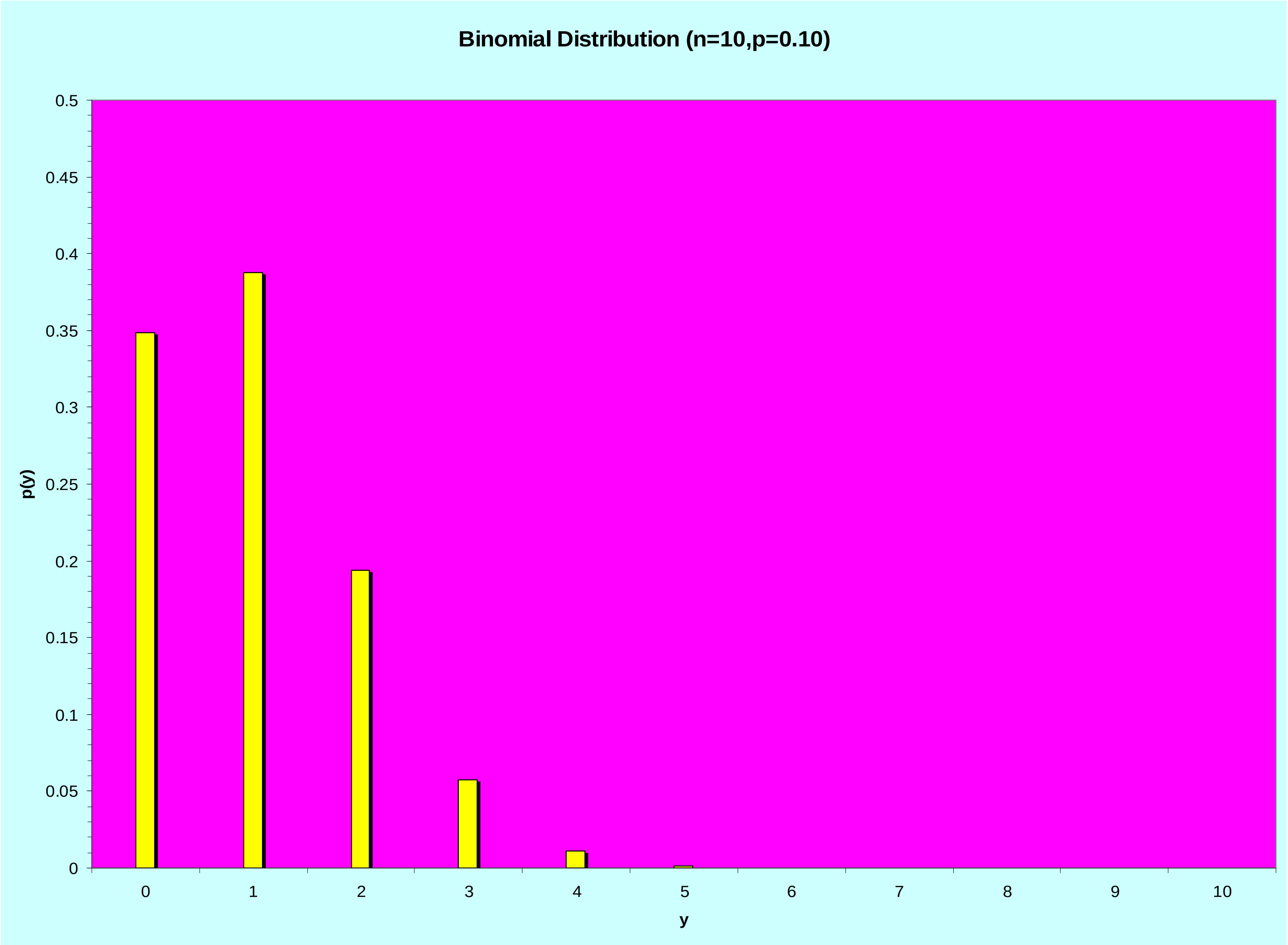
2) Probability of each arrangement of $y\ S^s$ (and $(n - y)\ F^s$) $\equiv p^y (1 - p)^{n-y}$

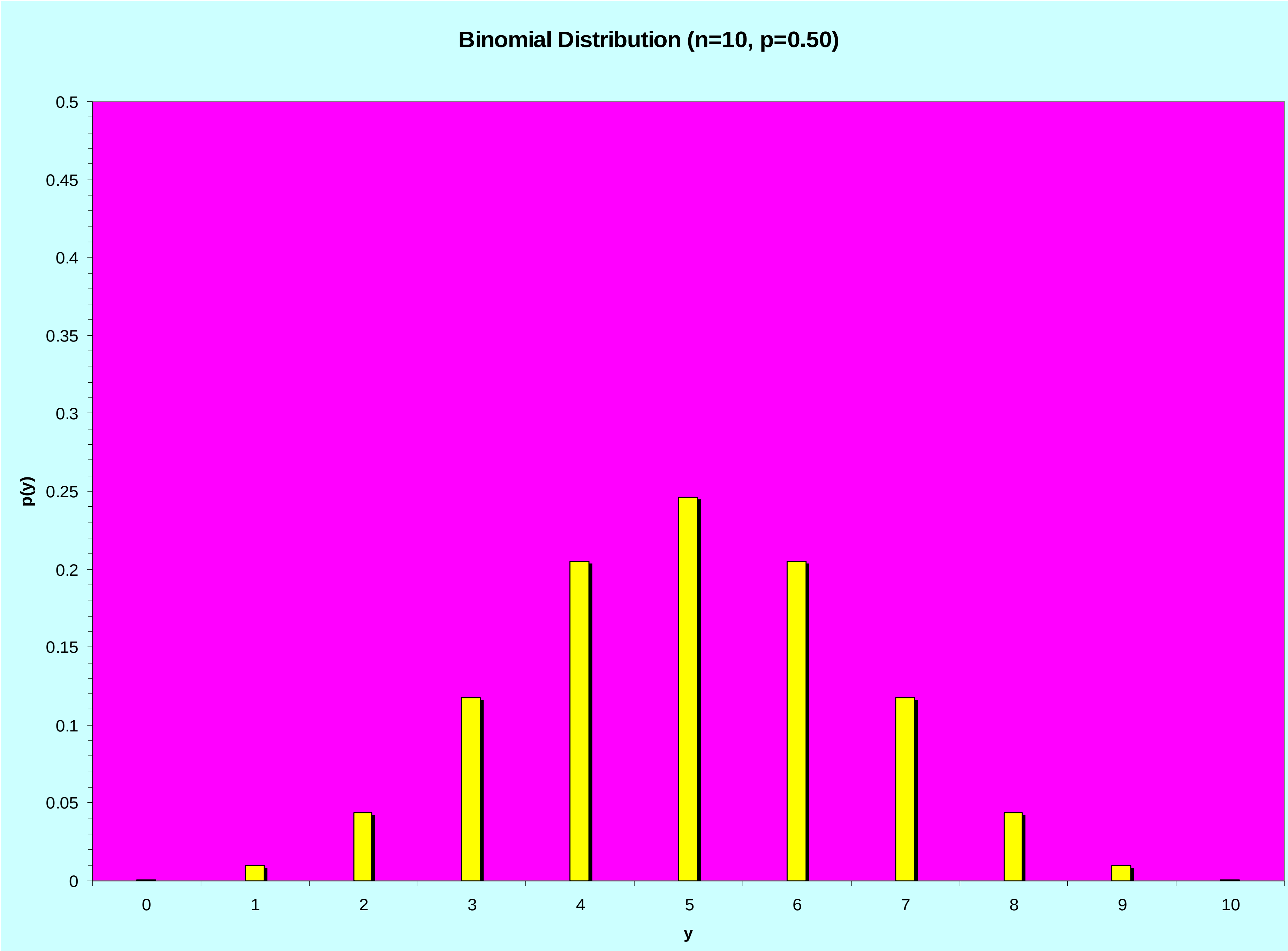
3) $\Rightarrow P(Y = y) = p(y) = \binom{n}{y} p^y (1 - p)^{n-y} \quad y = 0, 1, \ldots, n$

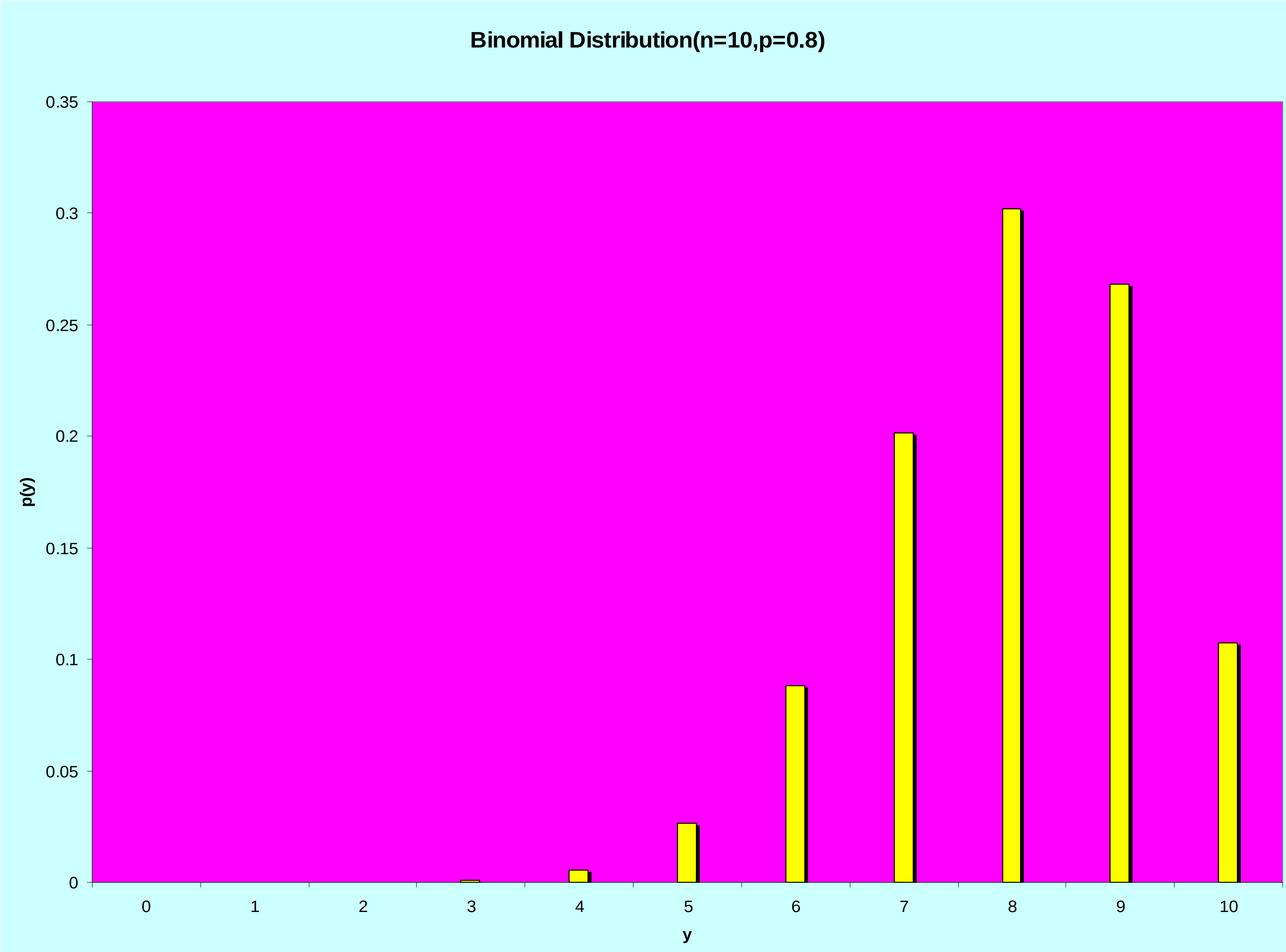
EXCEL Functions :

$p(y)$ is obtained by function : =BINOMDIST(y, n, p,0)

$F(y) = P(Y \le y)$ is obtained by function : =BINOMDIST(y, n, p,1)


Binomial Distribution (n=10,p=0.10)
p(y)
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
1
2
3
4
5
6
7
8
9
10
y


Binomial Distribution (n=10, p=0.50)
p(y)
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
1
2
3
4
5
6
7
8
9
10
y


Binomial Distribution(n=10,p=0.8)
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
p(y)
0
1
2
3
4
5
6
7
8
9
10
y

Poisson Distribution

- Distribution often used to model the number of incidences of some characteristic in time or space:
  - Arrivals of customers in a queue
  - Numbers of flaws in a roll of fabric
  - Number of typos per page of text.
- Distribution obtained as follows:
  - Break down the “area” into many small “pieces” (*n* pieces)
  - Each “piece” can have only 0 or 1 occurrences (p=P(1))
  - Let $\lambda = np \equiv$ Average number of occurrences over “area”
  - $Y \equiv$ # occurrences in “area” is sum of 0^s & 1^s over “pieces”
  - $Y \sim \text{Bin}(n,p)$ with $p = \lambda/n$
  - Take limit of Binomial Distribution as $n \rightarrow \infty$ with $p = \lambda/n$

$$p(y) = P(Y = y) = \frac{e^{-\lambda}\lambda^{y}}{y!} \quad \lambda > 0, \quad y = 0,1,2,\ldots$$

Negative Binomial Distribution

- Used to model the number of trials needed until the r^{th} Success (extension of Geometric distribution)
- Based on there being r-1 Successes in first y-1 trials, followed by a Success

$$p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r} = \frac{(y-1)!}{(r-1)!(y-r)!} p^r (1-p)^{y-r} =$$

$$= \frac{\Gamma(y)}{\Gamma(r)\Gamma(y-r+1)!} p^r (1-p)^{y-r} \qquad y = r, r+1, \ldots$$

$$E(Y) = \frac{r}{p} \qquad V(Y) = \frac{r(1-p)}{p^2}$$

$$\Gamma(a) = \int_0^\infty z^{a-1} e^{-z/a} dz \qquad \text{Note}: \Gamma(a) = (a-1)\Gamma(a)$$

Negative Binomial Distribution (II)

$$\text{Generalization to "domain" of } y^* = 0,1,\ldots$$

$$p(y^*) = \frac{\Gamma(y^* + k)}{\Gamma(k)\Gamma(y^* + 1)} \left(\frac{k}{\mu + k} \right)^k \left(\frac{\mu}{\mu + k} \right)^{y^*} \qquad y^* = 0,1,\ldots$$

where:

$$k = r \qquad y^* = y - r \qquad \frac{k}{\mu + k} = p \quad \frac{\mu}{\mu + k} = 1 - p$$

$$E(Y^*) = \mu \qquad V(Y^*) = \mu + \frac{\mu^2}{k}$$

This model is widely used to model count data when the Poisson model does not fit well due to over-dispersion: V(Y) > E(Y).

In this model, k is not assumed to be integer-valued and must be estimated via maximum likelihood (or method of moments)