Fundamental Limits on Performance

- Given an information source, and a noisy channel

  1) Limit on the minimum number of bits per symbol

  2) Limit on the maximum rate for reliable communication

➔ Shannon's 3 theorems

Uncertainty, Information and Entropy

- Let the source alphabet,

$$S = \{s_{0,}\ s_1,\ ..,\ s_{K\text{-}1}\}$$

with the prob. of occurrence

$$P(s = s_k) = p_k\ ,\ k = 0,1,\ ..,K\text{-}1\ \ and\ \sum_{k=0}^{K\text{-}1} p_k = 1$$

- Assume the discrete memoryless source (DMS)

What is the measure of information?

Uncertainty, Information, and Entropy (cont')

- **Interrelations between info., uncertainty or surprise**
  - **No surprise → no information**

$$(\approx Info. \ \propto \ \frac{1}{Prob.})$$

  - **If A is a surprise and B is another surprise, then what is the total info. of simultaneous A and B**

$$\longrightarrow Info.(A \cap B) \ \propto \ Info.(A) + Info.(B)$$

- **The amount of info may be related to the inverse of the prob. of occurrence.**

$$\therefore \ I(S_k) = \log(\frac{1}{p_k})$$

Property of Information

1) $I(\mathrm{s}_\mathrm{k}) = 0 \text{ for } \mathrm{p}_\mathrm{k} = 1$

2) $I(s_k) \geq 0 \quad \text{for} \quad 0 \leq \mathrm{p}_\mathrm{k} \leq 1$

3) $I(s_k) > I(s_i) \quad \text{for} \quad \mathrm{p}_\mathrm{k} < \mathrm{p}_\mathrm{i}$

4) $I(s_k s_i) = I(s_k) + I(s_i), \quad \text{if } \mathrm{s}_\mathrm{k} \text{ and } \mathrm{s}_\mathrm{i} \text{ statist. indep.}$

* Custom is to use logarithm of base 2

Entropy

- Def. : measure of average information contents per source symbol

➡ The mean value of $I(s_k)$ over S,

$$H(S) = \mathrm{E}[I(s_k)] = \sum_{k=0}^{K-1} p_k I(s_k) = \sum_{k=0}^{K-1} p_k \log_2\left(\frac{1}{p_k}\right)$$

➡ The property of H

$$0 \le H(S) \le \log_2 K, \; where\ K\ is\ radix\ (=\ \#\ of\ symbols)$$

1) H(S)=0,**iff** $p_k = 1$ **for some *k*, and all other** $p_i's = 0$

⟶ No Uncertainty

2) H(S)= $\log_2 K,\ iff\ p_k = \frac{1}{K}\ for\ all\ k$

⟶ Maximum Uncertainty

Entropy of Binary Memoryless Source

- **For binary source of symbol 0 with , p_0 and symbol 1 with** $p_1 (= 1 - p_0)$

$$H(S) = -p_0 \log_2 p_0 - p_1 \log_2 p_1$$

$$= -p_0 \log_2 p_0 - (1 - p_0) \log_2 (1 - p_0), \quad \text{(bits)}$$