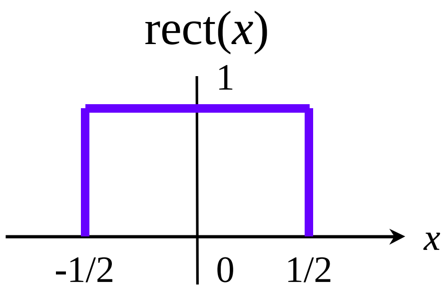
Some Fourier Transform Pairs

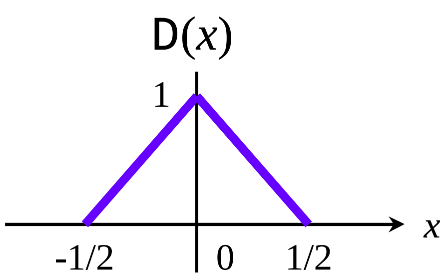
Useful Functions

- **Unit gate function (a.k.a. unit pulse function)**



$$\mathrm{rect}(x) = \begin{cases} 0 & |x| > \frac{1}{2} \\ \frac{1}{2} & |x| = \frac{1}{2} \\ 1 & |x| < \frac{1}{2} \end{cases}$$

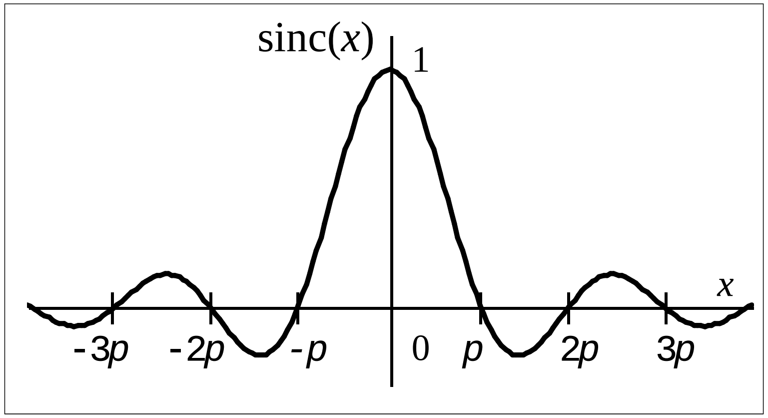
- **What does rect(x / a) look like?**
- **Unit triangle function**



$$\Delta(x) = \begin{cases} 0 & |x| > \frac{1}{2} \\ 1 - 2|x| & |x| < \frac{1}{2} \end{cases}$$

Useful Functions

- **Sinc function**



$$\mathrm{sinc}(x) = \frac{\sin(x)}{x}$$

How to compute sinc(0)?

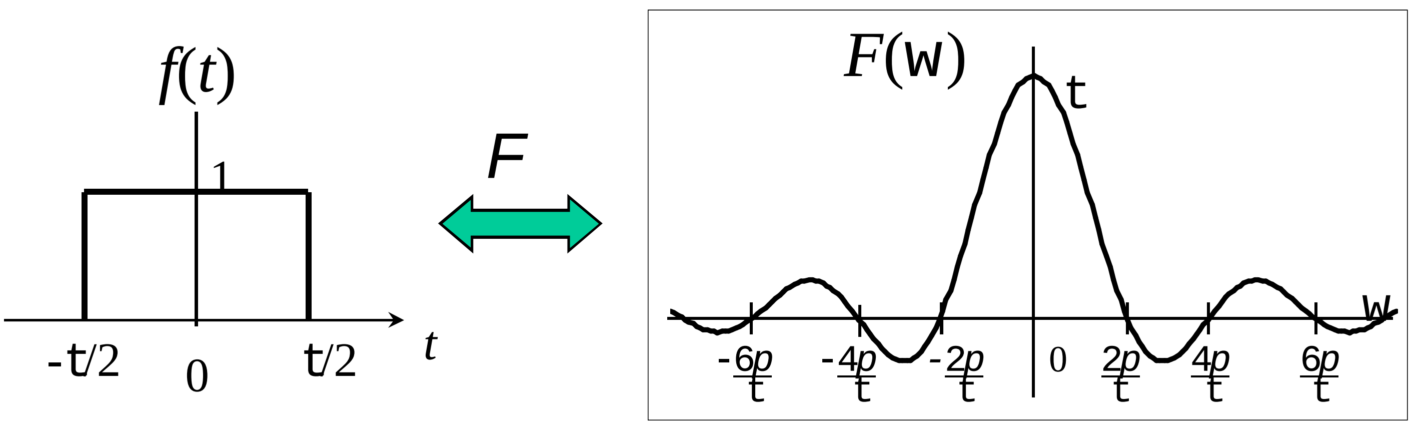
As $x \to 0$, numerator and denominator are both going to 0. How to handle it?

- Even function
- Zero crossings at $x = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$
- Amplitude decreases proportionally to 1/x

Fourier Transform Pairs

$$F(\omega) = \int_{-\infty}^{\infty} \text{rect}\left(\frac{t}{\tau}\right) e^{-j\omega t} dt = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt$$

$$= -\frac{1}{j\omega}\left(e^{-j\omega\tau/2} - e^{j\omega\tau/2}\right) = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega} = \tau\,\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)} = \tau\,\text{sinc}\left(\frac{\omega\tau}{2}\right)$$

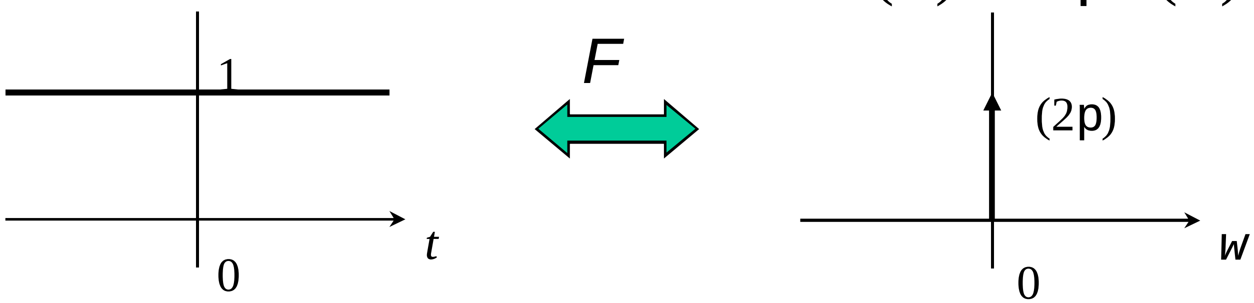


Fourier Transform Pairs

From the sampling property of the impulse,

$$F\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{-j0t} = 1$$

$f(t) = 1$ $\qquad\qquad$ $F(\omega) = 2\pi\delta(\omega)$



(2π) means that the area under the spike is (2π)

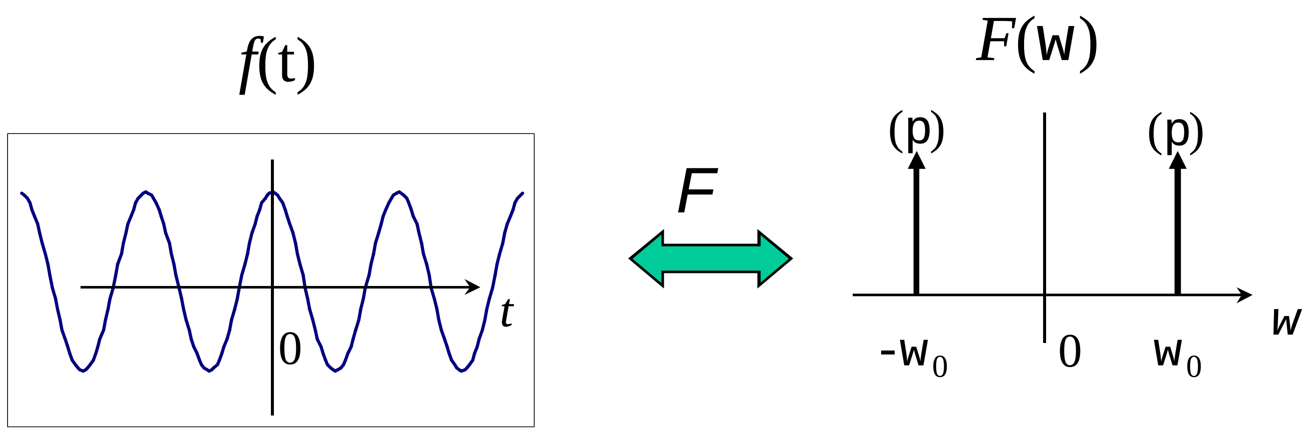
Fourier Transform Pairs

$$F^{-1}\{\delta(\omega-\omega_0)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)e^{j\omega t}d\omega=\frac{1}{2\pi}e^{j\omega_0 t}$$

$$\frac{1}{2\pi}e^{j\omega_0 t}\Leftrightarrow\delta(\omega-\omega_0)\quad\text{or}\quad e^{j\omega_0 t}\Leftrightarrow\delta(\omega-\omega_0)$$

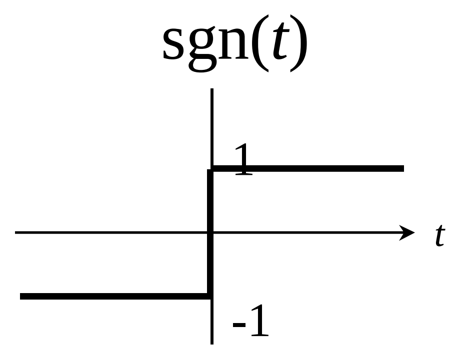
$$\text{Since }\cos(\omega_0 t)=\frac{1}{2}\left(e^{j\omega_0 t}+e^{-j\omega_0 t}\right)$$

$$\cos(\omega_0 t)\Leftrightarrow\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$$



Fourier Transform Pairs

$$\operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases} = \lim_{a \to 0}\left[e^{-at}u(t) - e^{at}u(-t)\right]$$



$$F\{\operatorname{sgn}(t)\} = \lim_{a \to 0}\left[F\left[e^{-at}u(t)\right] - F\left[e^{at}u(-t)\right]\right]$$

$$= \lim_{a \to 0}\left[\frac{1}{a + j\omega} - \frac{1}{a - j\omega}\right]$$

$$= \lim_{a \to 0}\left[\frac{-2j\omega}{a^2 + \omega^2}\right] = \frac{2}{j\omega}$$

Comparison with Laplace Transform

Fourier vs. Laplace Transform Pairs

$f(t)$	$F(s)$	Region of Convergence	$F(\omega)$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\} > -\mathrm{Re}\{a\}$	$\frac{1}{j\omega + a}$
$e^{-a\|t\|}$	$\frac{2a}{a^2 - s^2}$	$-\mathrm{Re}\{a\} < \mathrm{Re}\{s\} < \mathrm{Re}\{a\}$	$\frac{2a}{\omega^2 + a^2}$
$\delta(t)$	1	complex plane	1
1	$2\pi\delta(s)$	complex plane	$2\pi\delta(\omega)$
$u(t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\} > 0$	$\pi\delta(\omega) + 1/(j\omega)$
$\cos(\omega_0 t)$			$\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$
$\sin(\omega_0 t)$			$j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$
$e^{at}u(t)$	$\frac{1}{s-a}$	$\mathrm{Re}\{s\} > \mathrm{Re}\{a\}$	

Assuming that $\mathrm{Re}\{a\} > 0$