Energy Spectral Density

The total signal energy in an energy signal is

$$E_{\mathrm{x}} = \int_{-\infty}^{\infty} |\mathrm{x}(t)|^2 \, dt = \int_{-\infty}^{\infty} |\mathrm{X}(f)|^2 \, df \quad \text{or} \quad E_{\mathrm{x}} = \sum_{n=-\infty}^{\infty} |\mathrm{x}[n]|^2 = \int_{1} |\mathrm{X}(F)|^2 \, dF$$

The quantity, $|\mathrm{X}(f)|^2$, or $|\mathrm{X}(F)|^2$, is called the *energy spectral density (ESD)* of the signal, x, and is conventionally given the symbol, Ψ. That is,

$$\Psi_{\mathrm{x}}(f) = |\mathrm{X}(f)|^2 \qquad \text{or} \qquad \Psi_{\mathrm{x}}(F) = |\mathrm{X}(F)|^2$$

It can be shown that if x is a real-valued signal that the ESD is even, non-negative and real.

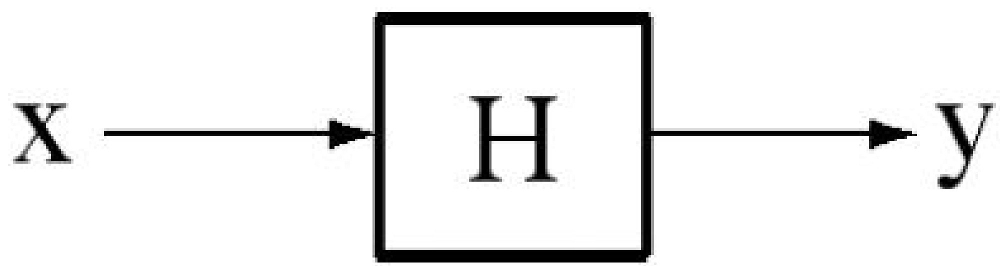
Energy Spectral Density

Probably the most important fact about ESD is the relationsh between the ESD of the excitation of an LTI system and the ESD of the response of the system. It can be shown (pp. 606 607) that they are related by

$$\Psi_y(f) = |H(f)|^2 \Psi_x(f) = H(f) H^*(f) \Psi_x(f)$$

or

$$\Psi_y(F) = |H(F)|^2 \Psi_x(F) = H(F) H^*(F) \Psi_x(F)$$



For an Energy Signal

ESD and autocorrelation form a Fourier transform pair.

$$R_x(t) \xleftrightarrow{\mathcal{F}} \Psi_x(f) \quad \text{or} \quad R_x[n] \xleftrightarrow{\mathcal{F}} \Psi_x(F)$$

Power Spectral Density

Power spectral density (PSD) applies to power signals in the same way that energy spectral density applies to energy signals. The PSD of a signal x is conventionally indicated by the notation, $G_x(f)$ or $G_x(F)$. In an LTI system,

$$G_y(f) = |H(f)|^2 G_x(f) = H(f)H^*(f)G_x(f)$$

or

$$G_y(F) = |H(F)|^2 G_x(F) = H(F)H^*(F)G_x(F)$$

Also, for a power signal, PSD and autocorrelation form a Fourier transform pair.

$$R(t) \xleftrightarrow{\mathcal{F}} [G(f)] \quad \text{or} \quad R[n] \xleftrightarrow{\mathcal{F}} G(F)$$

Correlation of Energy Signals

The correlation between two energy signals, x and y, is the area under (for CT signals) or the sum of (for DT signals) the product of x and y*.

$$\int_{-\infty}^{\infty} \mathrm{x}(t)\mathrm{y}^*(t)dt \qquad \text{or} \qquad \sum_{n=-\infty}^{\infty} \mathrm{x}[n]\mathrm{y}^*[n]$$

The correlation *function* between two energy signals, x and y, is the area under (CT) or the sum of (DT) that product *as a function of how much* y *is shifted* relative to x.

$$\mathrm{R}_{\mathrm{xy}}(\tau) = \int_{-\infty}^{\infty} \mathrm{x}(t)\mathrm{y}^*(t+\tau)dt \qquad \text{or} \qquad \mathrm{R}_{\mathrm{xy}}[m] = \sum_{n=-\infty}^{\infty} \mathrm{x}[n]\mathrm{y}^*[n+m]$$

In the very common case in which x and y are both real,

$$\mathrm{R}_{\mathrm{xy}}(\tau) = \int_{-\infty}^{\infty} \mathrm{x}(t)\mathrm{y}(t+\tau)dt \qquad \text{or} \qquad \mathrm{R}_{\mathrm{xy}}[m] = \sum_{n=-\infty}^{\infty} \mathrm{x}[n]\mathrm{y}[n+m]$$

Correlation of Energy Signals

The correlation function for two real energy signals is very similar to the convolution of two real energy signals.

$$\mathrm{x}(t) * \mathrm{y}(t) = \int_{-\infty}^{\infty} \mathrm{x}(t-\tau)\mathrm{y}(\tau)d\tau \quad \text{or} \quad \mathrm{x}[n] * \mathrm{y}[n] = \sum_{m=-\infty}^{\infty} \mathrm{x}[n-m]\mathrm{y}[m]$$

Therefore it is possible to use convolution to find the correlation function.

$$\mathrm{R}_{\mathrm{xy}}(\tau) = \mathrm{x}(-\tau) * \mathrm{y}(\tau) \quad \text{or} \quad \mathrm{R}_{\mathrm{xy}}[m] = \mathrm{x}[-m] * \mathrm{y}[m]$$

It also follows that

$$\mathrm{R}_{\mathrm{xy}}(\tau) \xleftrightarrow{\mathcal{F}} \mathrm{X}^*(f)\mathrm{Y}(f) \quad \text{or} \quad \mathrm{R}_{\mathrm{xy}}[m] \xleftrightarrow{\mathcal{F}} \mathrm{X}^*(F)\mathrm{Y}(F)$$

Correlation of Power Signals

The correlation function between two power signals, x and y, is the average value of the product of x and y* as a function of how much y* is shifted relative to x.

$$R_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_T x(t) y^*(t+\tau) dt \quad \text{or} \quad R_{xy}[m] = \lim_{N \to \infty} \frac{1}{N} \sum_{n=\langle N \rangle} x[n] y^*[n+m]$$

If the two signals are both periodic and their fundamental periods have a finite least common period,

$$R_{xy}(\tau) = \frac{1}{T} \int_T x(t) y^*(t+\tau) dt \quad \text{or} \quad R_{xy}[m] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] y^*[n+m]$$

where T or N is any integer multiple of that least common period. For real periodic signals these become

$$R_{xy}(\tau) = \frac{1}{T} \int_T x(t) y(t+\tau) dt \quad \text{or} \quad R_{xy}[m] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] y[n+m]$$

Correlation of Power Signals

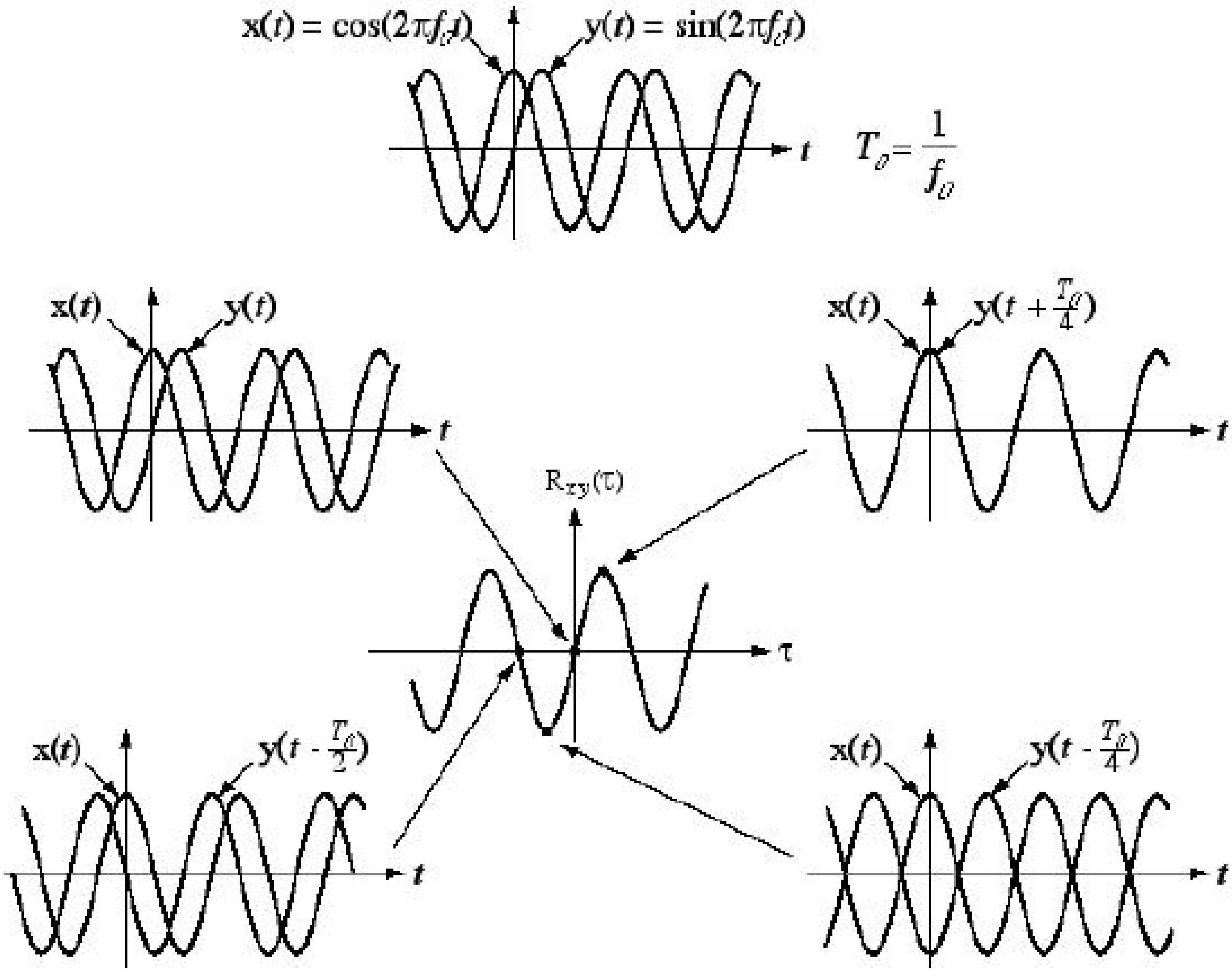
Correlation of real periodic signals is very similar to periodic convolution

$$R_{xy}(\tau) = \frac{x(-\tau) \circledast y(\tau)}{T} \quad \text{or} \quad R_{xy}[m] = \frac{x[-m] \circledast y[m]}{N}$$

$$R_{xy}(\tau) \xleftrightarrow{\mathcal{FS}} X^*[k]Y[k] \quad \text{or} \quad R_{xy}[m] \xleftrightarrow{\mathcal{FS}} X^*[k]Y[k]$$

where it is understood that the period of the periodic convolution is any integer multiple of the least common period of the two fundamental periods of x and y.

Correlation of Power Signals



Correlation of Sinusoids

- The correlation function for two sinusoids of different frequencies is always zero. (pp.

Autocorrelation

A very important special case of correlation is *autocorrelation*. Autocorrelation is the correlation of a function with a shifted version of *itself*. For energy signals,

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t+\tau)\,dt \quad \text{or} \quad R_{xx}[m] = \sum_{n=-\infty}^{\infty} x[n]x^*[n+m]$$

At a shift, τ or m, of zero,

$$R_{xx}(0) = \int_{-\infty}^{\infty} |x(t)|^2\,dt \quad \text{or} \quad R_{xx}[0] = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

which is the signal energy of the signal. For power signals,

$$R_{xx}(0) = \lim_{T\to\infty} \frac{1}{T}\int_T |x(t)|^2\,dt \quad \text{or} \quad R_{xx}[0] = \lim_{N\to\infty} \frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2$$

which is the average signal power of the signal.

Properties of Autocorrelation

For real signals, autocorrelation is an even function.

$$R_{xx}(\tau) = R_{xx}(-\tau) \qquad \text{or} \qquad R_{xx}[m] = R_{xx}[-m]$$

Autocorrelation magnitude can never be larger than it is at zero shift.

$$R_{xx}(0) \geq |R_{xx}(\tau)| \qquad \text{or} \qquad R_{xx}[0] \geq |R_{xx}[m]|$$

If a signal is time shifted its autocorrelation does not change.

The autocorrelation of a sum of sinusoids of different frequencies is the sum of the autocorrelations of the individual sinusoids.