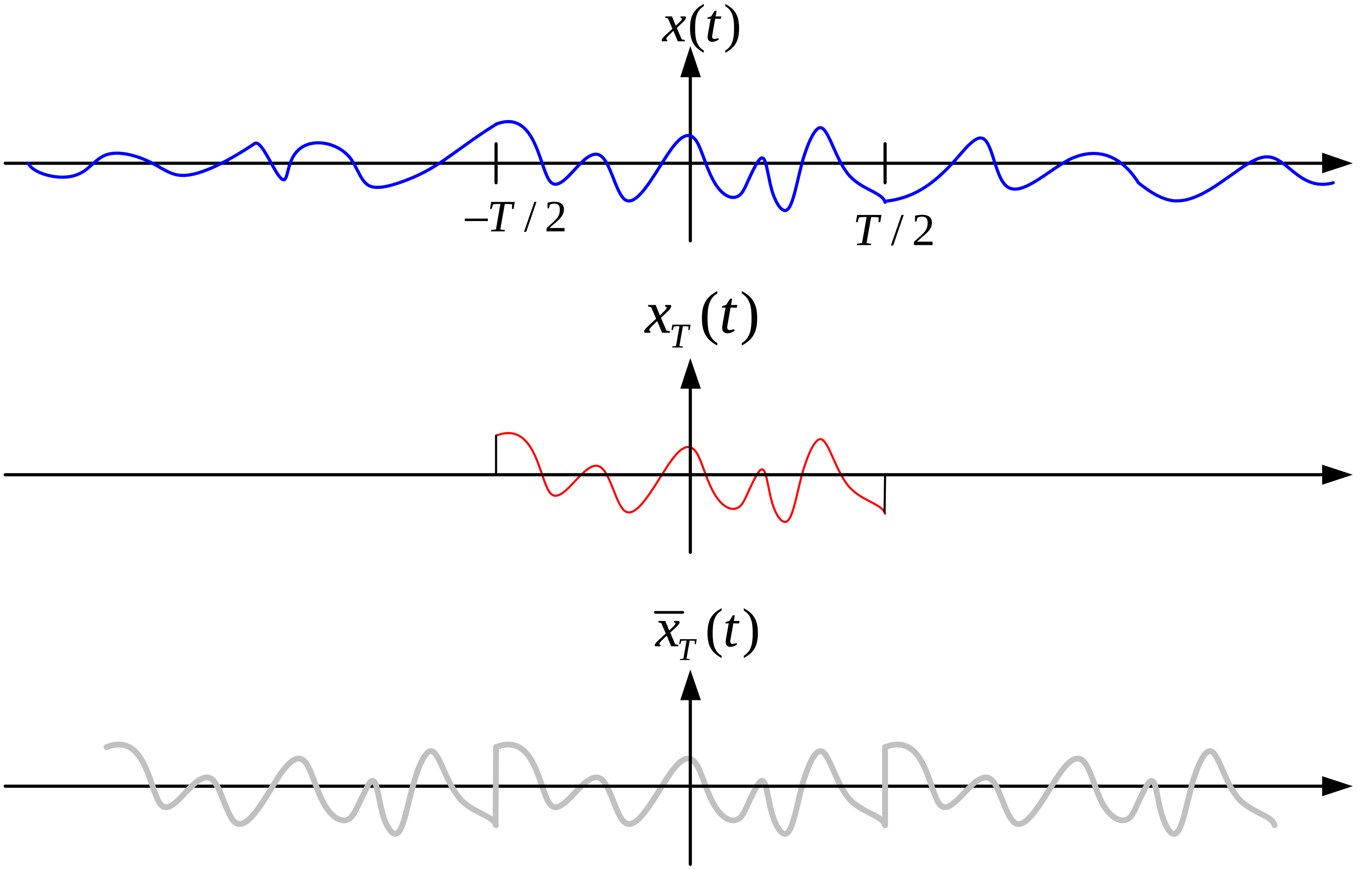
FOURIER TRANSFORMS

- Fourier transform is the extension of Fourier series to periodic and aperiodic signals.
- The signals are expressed in terms of complex exponentials of various frequencies, but these frequencies are not discrete.
- The extension of the Fourier series to aperiodic signals can be done by extending the period to infinity.
- The signal has a continuous spectrum as opposed to a discrete spectrum.

- Assume that the Fourier series of periodic extension of the nonperiodic signal $x(t)$ exists.
- Define $x_T(t)$ as the truncation of $x(t)$ over $-\frac{T}{2} < t < \frac{T}{2}$, i.e.,

$$x_T(t) = \Pi(\frac{t}{T})x(t) = \begin{cases} x(t), & -\frac{T}{2} < t < \frac{T}{2} \\ 0, & \text{otherwise.} \end{cases}$$


$x(t)$
t
$-T/2$
$T/2$
$x_T(t)$
t
$\overline{x}_T(t)$
t

- Denote the periodic signal

$$\bar{x}_T(t) = \sum_{k=-\infty}^{\infty} x_T(t - kT).$$

- Conversely, we may express the truncated signal by

$$x_T(t) = \begin{cases} \bar{x}_T(t), & -\dfrac{T}{2} \le t \le \dfrac{T}{2} \\ 0, & \text{otherwise.} \end{cases}$$

- If we let the period T approach infinity, then in the limit, the periodic signal approximately becomes the aperiodic signal

$$x(t) = \lim_{T \to \infty} x_T(t) = \lim_{T \to \infty} \overline{x}_T(t).$$

- This periodic signal with fundamental period T has a complex exponential Fourier series that is given by

$$\overline{x}_T(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi n f_0 t},$$

$$x_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \overline{x}_T(t) e^{-j2\pi n f_0 t} dt.$$

- As far as the integration is concerned, the integrand on this integral can be rewritten as

$$x_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \overline{x}_T(t)\, e^{-j2\pi n f_0 t} dt = \frac{1}{T}\int_{-\infty}^{\infty} x_T(t) e^{-j2\pi n f_0 t} dt.$$

- Define

$$X_T(f) = \int_{-\infty}^{\infty} x_T(t) e^{-j2\pi f t} dt.$$

- We have

$$x_n = \frac{1}{T} X_T(n f_0).$$

$$\bar{x}_T(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} X_T(nf_0) e^{j2\pi n f_0 t} = \sum_{n=-\infty}^{\infty} X_T(nf_0) e^{j2\pi n f_0 t} f_0.$$

$$x(t) = \lim_{T\to\infty} \bar{x}_T(t) = \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} X_T(nf_0) e^{j2\pi n f_0 t} f_0$$

$$T \to \infty,\ f_0 \to 0.$$

- The summation turns to become an integral

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df.$$

- $x(t)$ is the inverse Fourier transform of $X(f)$.
- The Fourier transform of *x(t)* is

$$X(f) = \lim_{T \to \infty} X_T(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt.$$

- **Definition III.** Suppose that, $x(t), -\infty < t < \infty$ is a signal such that it is absolutely integrable, that is,

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty.$$

Then the ***Fourier transform*** of $x(t)$ is defined as

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt.$$

The inverse Fourier transform is given by

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df.$$

Fourier transform - Sufficient conditions

- The waveform $w(t)$ is Fourier transformable if it satisfies both Dirichlet conditions**:**
  1) Over any time interval of finite length, the function $w(t)$ is single valued with a finite number of maxima and minima, and the number of discontinuities (if any) is finite.
  2) $w(t)$ is absolutely integrable. That is,

$$\int_{-\infty}^{\infty} |w(t)| dt < \infty$$

- Above conditions are sufficient, but not necessary.
- A weaker sufficient condition for the existence of the Fourier transform is:

$$E = \int_{-\infty}^{\infty} |w(t)|^2 dt < \infty$$

Finite Energy

- where E is the normalized energy.
- This is the finite-energy condition that is satisfied by all physically realizable waveforms.
- Conclusion: Generally physical waveforms encountered in engineering practice are Fourier transformable**.**

- Observations
  - *X(f)* is in general a complex function. The function *X(f)* is sometimes referred to as the *spectrum* of the signal *x(t)*.
  - To denote that *X(f)* is the Fourier transform of *x(t)*, the following notation is frequently employed

$$X(f) = \mathcal{F}[x(t)].$$

  - To denote that *x(t)* is the inverse Fourier transform of *X(f)* , the following notation is used

$$x(t) = \mathcal{F}^{-1}[X(f)].$$

  - Sometimes the following notation is used as a shorthand for both relations

$$x(t) \Leftrightarrow X(f).$$

– The Fourier transform and the inverse Fourier transform relations can be written as

$$x(t) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \tau} d\tau \right] e^{j2\pi ft} df$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{j2\pi f(t-\tau)} df \right] x(\tau) d\tau.$$

On the other hand,

$$x(t) = \int_{-\infty}^{\infty} \delta(t-\tau) x(\tau) d\tau,$$

where $\delta(t)$ is the unit impulse. From above equation, we may have

$$\delta(t-\tau) = \int_{-\infty}^{\infty} e^{j2\pi f(t-\tau)} df,$$

or, in general

$$\delta(t) = \int_{-\infty}^{\infty} e^{j2\pi ft} df.$$

Hence, the spectrum of $\delta(t)$ is equal to unity over all frequencies.

Example 2.2.1: Determine the Fourier transform of the signal $\Pi(t)$.

Solution: We have

$$
\begin{aligned}
\mathcal{F}\,[\Pi(t)] &= \int_{-\infty}^{\infty} \Pi(t) e^{-j2\pi ft} dt \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \Pi(t) e^{-j2\pi ft} dt \\
&= \frac{1}{-j2\pi f}\left[e^{-j\pi f} - e^{j\pi f}\right] \\
&= \frac{\sin(\pi f)}{\pi f} \\
&= \operatorname{sinc}(f).
\end{aligned}
$$

•The Fourier transform of $\Pi(t)$.

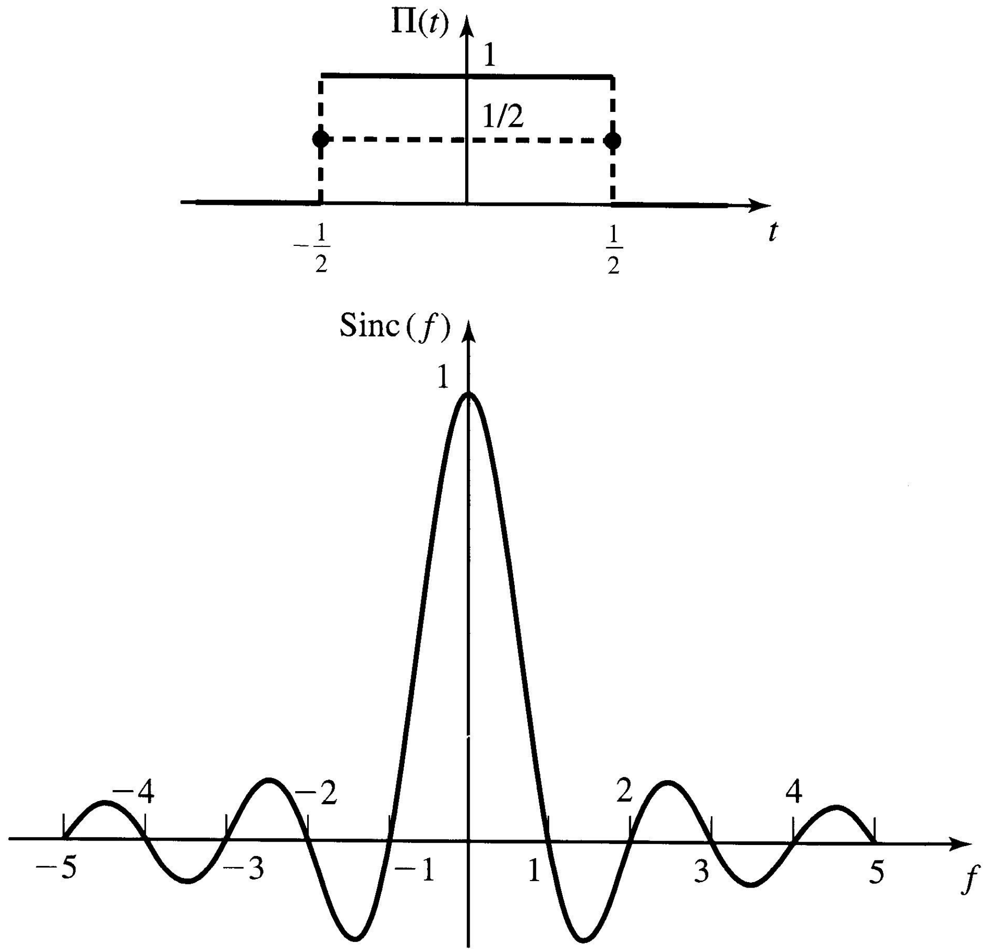


Figure 2.6 $\Pi(t)$ and its Fourier transform.

Example 2.2.2: Find the Fourier transform of the impulse signal $x(t) = \delta(t)$.

Solution: The Fourier transform can be obtained by

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt$$
$$= 1.$$

Similarly, from the relation

$$\int_{-\infty}^{\infty} \delta(f) e^{j2\pi ft} df = 1.$$

We conclude that

$$\mathcal{F}[1] = \delta(f).$$

•The Fourier transform of $\delta(t)$.

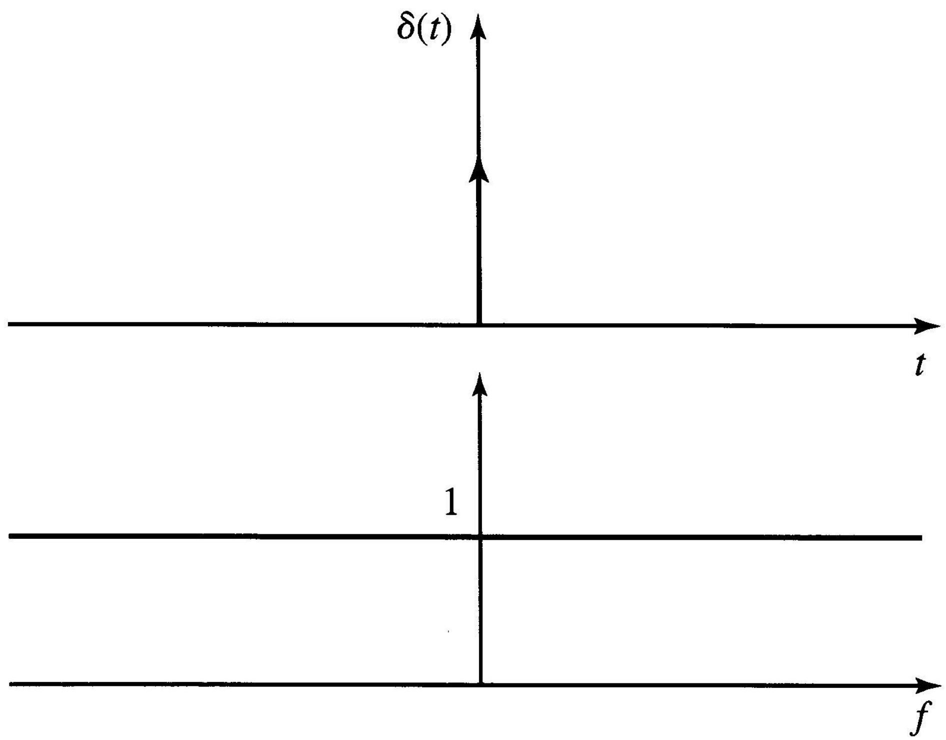


Figure 2.7 Impulse signal and its spectrum.