

Time response of second order system

UNIT 3

TIME RESPONSE OF SECOND ORDER SYSTEM

$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$C(s) = R(s) \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

stepi / p — — $r(t) = d$

$$R(s) = d / s$$

$$C(s) = \frac{d}{s} * \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = [(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)]$$

$$= d \left[\frac{1}{s} - \frac{s + 2\zeta\omega_n}{[(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)]} \right]$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

$$d \left[\frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \right]$$

$$d \left[\frac{1}{s} - \frac{s + \zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_d^2} - \frac{\zeta \omega_n}{\omega_d} * \frac{\omega_d}{(s + \zeta \omega_n)^2 + \omega_d^2} \right]$$

taking laplace inverse

$$\ell^{-1} C(s) = d \ell^{-1} \left[\frac{1}{s} - \frac{s + 2\zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_d^2} - \frac{\zeta \omega_n}{\omega_d} * \frac{\omega_d}{(s + \zeta \omega_n)^2 + \omega_d^2} \right]$$

$$c(t) = d \left[1 - e^{-\zeta \omega_n t} \cdot \cos \omega_d t - \frac{\zeta \omega_n}{\omega_d} \cdot e^{-\zeta \omega_n t} \cdot \sin \omega_d t \right]$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

$$c(t) = d \left[1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1 - \zeta^2}} \left[\sqrt{1 - \zeta^2} \cdot \cos \omega_d t + \zeta \cdot \sin \omega_d t \right] \right]$$

$$c(t) = d \left[1 - e^{-\zeta \omega_n t} \left[\cos \omega_n \sqrt{1 - \zeta^2} t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \cdot \sin \omega_n \sqrt{1 - \zeta^2} t \right] \right]$$

$$c(t) = d \left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \left[\sqrt{1-\zeta^2} \cdot \cos \omega_d t + \zeta \cdot \sin \omega_d t \right] \right]$$

$$\sin \phi = \sqrt{1-\zeta^2} \quad \therefore \cos \phi = \zeta$$

$$c(t) = d \left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \left[\sin \phi \cdot \cos \omega_d t + \cos \phi \cdot \sin \omega_d t \right] \right]$$

$$c(t) = d \left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \phi) \right]$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2} \text{ and } \phi = \tan^{-1} \left(\frac{\sqrt{1-\zeta^2}}{\zeta} \right)$$

RISE TIME FOR UNDERDAMPED SECOND ORDER SYSTEM

$$c(t) = d \left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \phi) \right]$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2} \text{ and } \phi = \tan^{-1} \left(\frac{\sqrt{1-\zeta^2}}{\zeta} \right)$$

$$C(t) = d$$

then

$$1 = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin \left[(\omega_n \sqrt{1-\zeta^2} \cdot) t_r + \tan^{-1} \left(\frac{\sqrt{1-\zeta^2}}{\zeta} \right) \right]$$

$$\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin \left[(\omega_n \sqrt{1-\zeta^2} \cdot) t_r + \tan^{-1} \left(\frac{\sqrt{1-\zeta^2}}{\zeta} \right) \right] = 0$$

Rise time

$\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$ is infinite

$$\sin \left[(\omega_n \sqrt{1-\zeta^2}) t_r + \tan^{-1} \left(\frac{\sqrt{1-\zeta^2}}{\zeta} \right) \right] = 0$$

$$(\omega_n \sqrt{1-\zeta^2}) t_r + \tan^{-1} \left(\frac{\sqrt{1-\zeta^2}}{\zeta} \right) = \pi$$

$$t_r = \frac{\pi - \phi}{\omega_n \sqrt{1-\zeta^2}}$$

$$\phi = \tan^{-1} \left(\frac{\sqrt{1-\zeta^2}}{\zeta} \right)$$

Maximum overshoot peak time

The maximum positive deviation of the output

With respect to its desired value M_p

$$M_p = C(t)_{\max} - 1$$

$$\%M_p = \frac{C(t)_{\max} - 1}{1} * 100$$

t_p

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_n \sqrt{1-\zeta^2} t + \phi)$$

$$\frac{dc(t)}{dt} = -\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \omega_n \sqrt{1-\zeta^2} t \cos(\omega_n \sqrt{1-\zeta^2} t + \phi) - \frac{-\zeta\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_n \sqrt{(1-\zeta^2)} t + \phi)$$

put

$$\frac{dc(t)}{dt} = 0$$

$$\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \{-\omega_n \sqrt{1-\zeta^2} \cos((\omega_n \sqrt{1-\zeta^2})t + \phi) + \zeta\omega_n \sin(\omega_n \sqrt{(1-\zeta^2)}t + \phi)\} = 0$$

$$\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \text{ is finite}$$

$$\omega_n \sqrt{1-\zeta^2} \cos[(\omega_n \sqrt{1-\zeta^2})t + \phi] = \zeta\omega_n \sin(\omega_n \sqrt{(1-\zeta^2)}t + \phi)$$

$$\tan[(\omega_n \sqrt{1-\zeta^2})t + \phi] = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\tan[\omega_n \sqrt{1 - \zeta^2} t + \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta}] = \frac{\sqrt{1 - \zeta^2}}{\zeta}$$

$$\omega_n \sqrt{1 - \zeta^2} t = n\pi \text{ --- } n = 0, 1, 2$$

$$t_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$$

$$c(t)_{\max} = 1 - \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1 - \zeta^2}} \sin(\omega_n \sqrt{1 - \zeta^2} t_p + \phi)$$

$$c(t)_{\max} = 1 - \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1 - \zeta^2}} \sin \left[\omega_n \sqrt{1 - \zeta^2} \left(\frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} \right) + \phi \right]$$

$$1 - \frac{e^{-\zeta\omega_n \left(\frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} \right)}}{\sqrt{1 - \zeta^2}} \sin(\pi + \phi)$$

$$1 - \frac{e^{-\zeta \frac{\pi}{\sqrt{1 - \zeta^2}}}}{\sqrt{1 - \zeta^2}} \sin(-\phi)$$

$$\phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta} \text{ and } \sin \phi = \sqrt{1-\zeta^2}$$

$$c(t)_{\max} = 1 + e^{-\zeta \frac{\pi}{\sqrt{1-\zeta^2}}}$$

$$M_P = C(t)_{\max} - 1$$

$$M_P = e^{-\zeta \frac{\pi}{\sqrt{1-\zeta^2}}}$$

Maximum overshoot: M_p

$$M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

$$\% M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} * 100$$

PEAK TIME

$$t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

Delay time & rise time

Time required to reach 50% of the final value in first attempt

Time required to reach its peak value

Settling time: the time needed to settle down afore said oscillations within 2% of the desired value

Time delay

$$td = \frac{1 + 0.7\zeta}{\omega_n}$$

SETTLING TIME T_s

$T_s = 4 * \text{time constant}$

$$= 4 * \frac{1}{\zeta\omega_n}$$

$$T_s = \frac{4}{\zeta\omega_n}$$

STEADY STATE ERROR

FINAL VALUE THEOREM

$$ess = \lim_{s \rightarrow 0} sE(s)$$

$$ess = \lim_{s \rightarrow 0} sR(s) \frac{1}{1 + G(S)H(S)}$$

STATIC POSITION ERROR COEFFICIENT

STEP 1 / P

$$ess = \lim_{s \rightarrow 0} sR(s) \frac{1}{1 + G(S)H(S)}$$

$$R(s) = \frac{1}{s}$$

$$ess = \lim_{s \rightarrow 0} s \cdot \frac{1}{s} \cdot \frac{1}{1 + G(S)H(S)} = \frac{1}{1 + \lim_{s \rightarrow 0} G(S)H(S)}$$

$$putKp = \lim_{s \rightarrow 0} G(S)H(S)$$

K_p is called positional error coefficient

$$ess = \frac{1}{1 + K_p}$$

STATIC VELOCITY ERROR COEFFICIENT

$$ess = \lim_{s \rightarrow 0} R(s) \frac{1}{1 + G(S)H(S)}$$

unit ramp i/p

$$R(s) = \frac{1}{s^2}$$

$$ess = \lim_{s \rightarrow 0} s \cdot \frac{1}{s^2} \cdot \frac{1}{1 + G(S)H(S)} = \lim_{s \rightarrow 0} \frac{1}{s + sG(S)H(S)}$$

$$= \lim_{s \rightarrow 0} \frac{1}{sG(S)H(S)}$$

$$put Kv = \lim_{s \rightarrow 0} sG(S)H(S)$$

K_v is called velocity error coefficient

$$ess = \frac{1}{K_v}$$

STATIC ACCELERATION ERROR COEFFICIENT

$$ess = \lim_{s \rightarrow 0} sR(s) \frac{1}{1 + G(S)H(S)}$$

unit PARABOLIC i/p

$$R(s) = \frac{1}{s^3}$$

$$ess = \lim_{s \rightarrow 0} s \cdot \frac{1}{s^3} \cdot \frac{1}{1 + G(S)H(S)} = \lim_{s \rightarrow 0} \frac{1}{s^2 + s^2 G(S)H(S)}$$

$$= \lim_{s \rightarrow 0} \frac{1}{s^2 G(S)H(S)}$$

$$put Ka = \lim_{s \rightarrow 0} s^2 G(S)H(S)$$

Ka is called acceleration error coefficient

$$ess = \frac{1}{Ka}$$

STEADY STATE ERROR

Type 0 system UNIT STEP

$$G(s)H(s) = \frac{K(1 + sT_a)(1 + sT_b)}{(1 + sT_1)(1 + sT_2)}$$

$$K_p = \lim_{s \rightarrow 0} G(s)H(s)$$

$$= \lim_{s \rightarrow 0} \frac{K(1 + sT_a)(1 + sT_b)}{(1 + sT_1)(1 + sT_2)}$$

$$\therefore K_p = K$$

$$e_{ss} = \frac{1}{1 + K_p}$$

$$e_{ss} = \frac{1}{1 + K}$$

STEADY STATE ERROR

Type 0 system UNIT RAMP

$$G(s)H(s) = \frac{K(1 + sT_a)(1 + sT_b)}{(1 + sT_1)(1 + sT_2)}$$

$$K_v = \lim_{s \rightarrow 0} sG(s)H(s)$$

$$= \lim_{s \rightarrow 0} s \frac{K(1 + sT_a)(1 + sT_b)}{(1 + sT_1)(1 + sT_2)}$$

$$\therefore K_v = 0$$

$$e_{ss} = \frac{1}{K_v}$$

$$e_{ss} = \frac{1}{0} = \infty$$

STEADY STATE ERROR

Type 0 system UNIT parabola

$$G(s)H(s) = \frac{K(1 + sT_a)(1 + sT_b)}{(1 + sT_1)(1 + sT_2)}$$

$$K_a = \lim_{s \rightarrow 0} s^2 G(s)H(s)$$

$$= \lim_{s \rightarrow 0} s^2 \frac{K(1 + sT_a)(1 + sT_b)}{(1 + sT_1)(1 + sT_2)}$$

$$\therefore K_a = 0$$

$$e_{ss} = \frac{1}{K_a}$$

$$e_{ss} = \frac{1}{0} = \infty$$

STEADY STATE ERROR

Type 1 system UNIT STEP

$$G(s)H(s) = \frac{K(1 + sT_a)(1 + sT_b)}{s(1 + sT_1)(1 + sT_2)}$$

$$K_p = \lim_{s \rightarrow 0} G(s)H(s)$$

$$= \lim_{s \rightarrow 0} \frac{K(1 + sT_a)(1 + sT_b)}{s(1 + sT_1)(1 + sT_2)}$$

$$K_p = \infty$$

$$e_{ss} = \frac{1}{1 + K_p}$$

$$e_{ss} = \frac{1}{1 + \infty}$$

$$e_{ss} = 0$$

STEADY STATE ERROR

Type 0 system UNIT RAMP

$$G(s)H(s) = \frac{K(1 + sT_a)(1 + sT_b)}{(1 + sT_1)(1 + sT_2)}$$

$$K_v = \lim_{s \rightarrow 0} sG(s)H(s)$$

$$= \lim_{s \rightarrow 0} s \frac{K(1 + sT_a)(1 + sT_b)}{(1 + sT_1)(1 + sT_2)}$$

$$\therefore K_v = 0$$

$$e_{ss} = \frac{1}{K_v}$$

$$e_{ss} = \frac{1}{0} = \infty$$

STEADY STATE ERROR

Type 0 system UNIT parabola

$$G(s)H(s) = \frac{K(1 + sT_a)(1 + sT_b)}{(1 + sT_1)(1 + sT_2)}$$

$$K_a = \lim_{s \rightarrow 0} s^2 G(s)H(s)$$

$$= \lim_{s \rightarrow 0} s^2 \frac{K(1 + sT_a)(1 + sT_b)}{(1 + sT_1)(1 + sT_2)}$$

$$\therefore K_a = 0$$

$$e_{ss} = \frac{1}{K_a}$$

$$e_{ss} = \frac{1}{0} = \infty$$