Performance of a second-order system

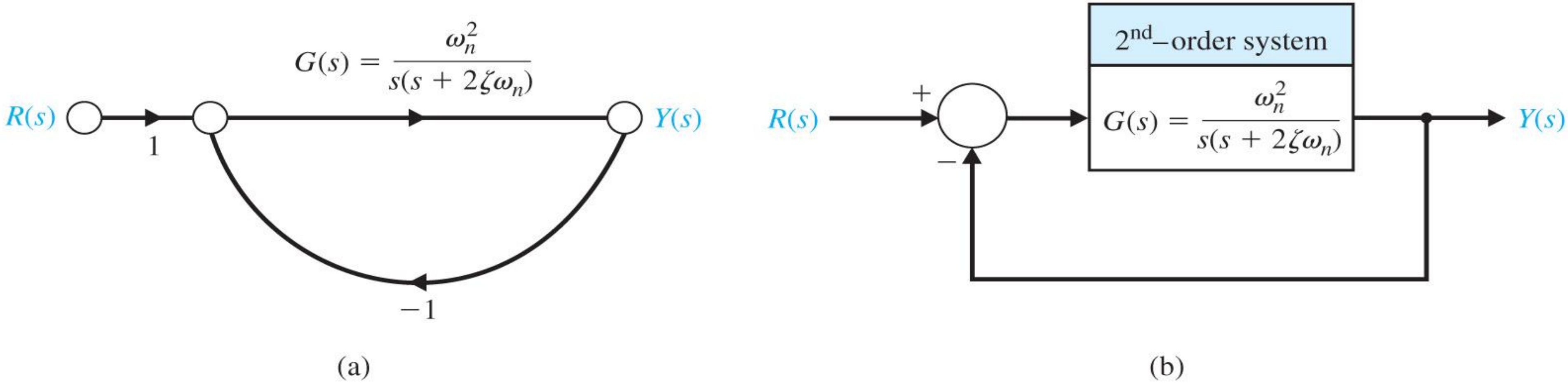


Figure: 05-04

Response to unit step input

$$Y(s) = \frac{G(s)}{1+G(s)} R(s)$$

$$Y(s) = \frac{\omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2} R(s)$$

$$Y(s) = \frac{\omega_n{}^2}{s(s^2 + 2\zeta\omega_n s + \omega_n{}^2)}$$

$$\beta = \sqrt{1-\zeta^2}$$

$$y(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t + \theta)$$

$$\theta = \cos^{-1}\zeta$$

- **Natural frequency ω_n - the frequency of natural oscillation that would occur for two complex poles if the damping were equal to zero**

- **Damping ratio ζ - a measure of damping for second-order characteristic equation**

Characteristic equation

$$s^2 + 2\zeta\omega_n s + {\omega_n}^2 = 0$$

$$s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

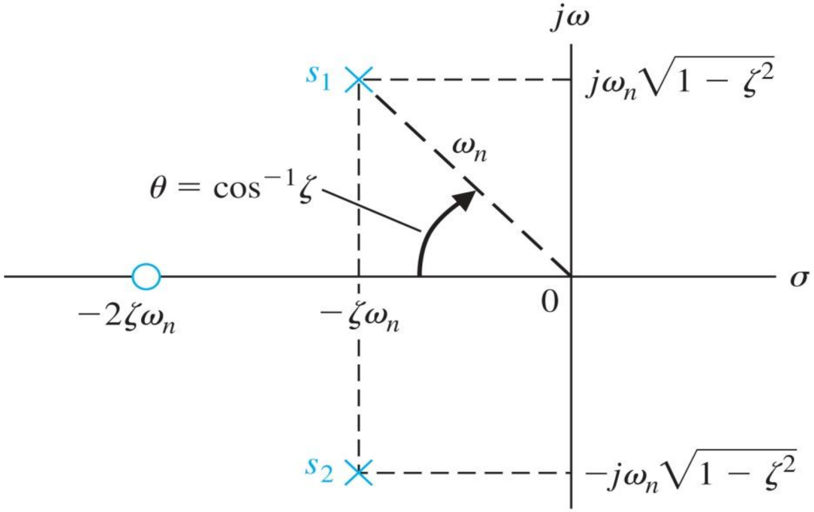


Figure: 02-09

Finding ω_n and ζ for a second-order system

$$G(s) = \frac{36}{s^2 + 4.2s + 36}$$

$$G(s) = \frac{\omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$$

$$s^2 + 2\zeta\omega_n s + \omega_n{}^2 = 0$$

$$s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

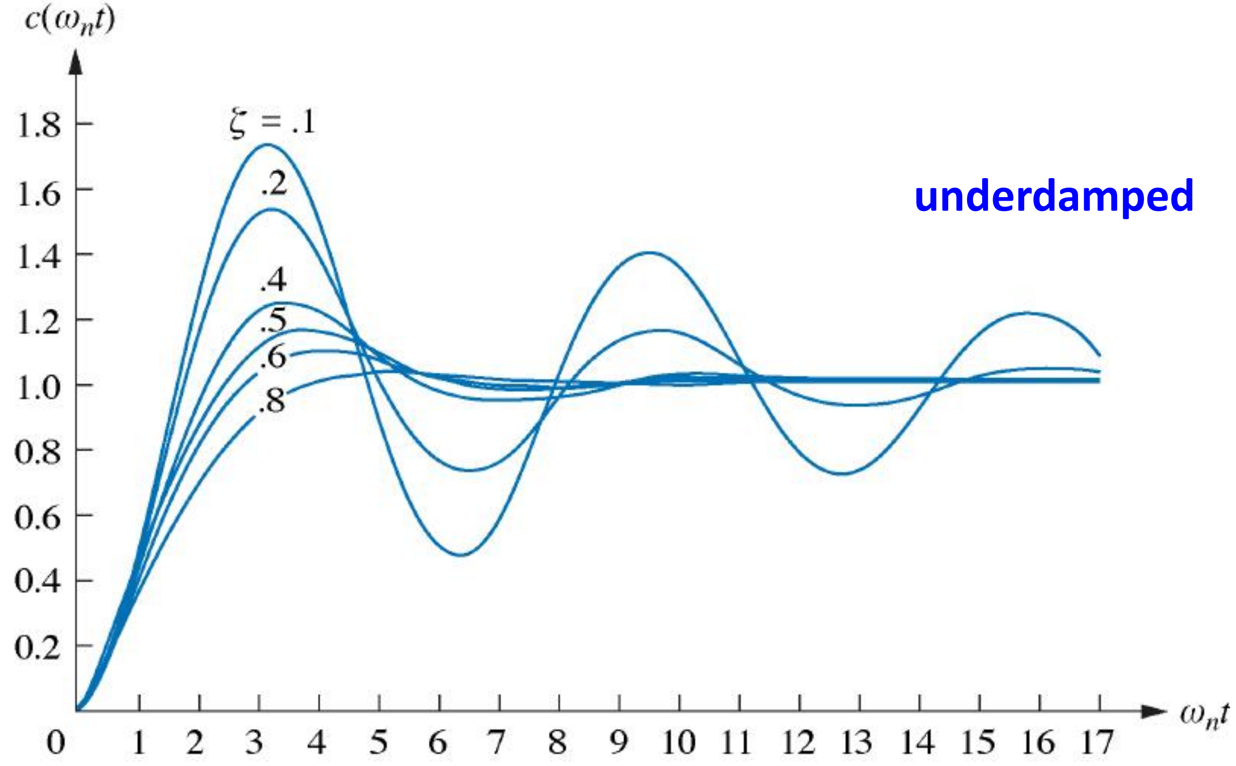
$$\omega_n{}^2 = 36$$

$$2\zeta\omega_n = 4.2$$

$$\omega_n = 6$$

$$\zeta = 0.35$$

Second-order responses for ζ



underdamped

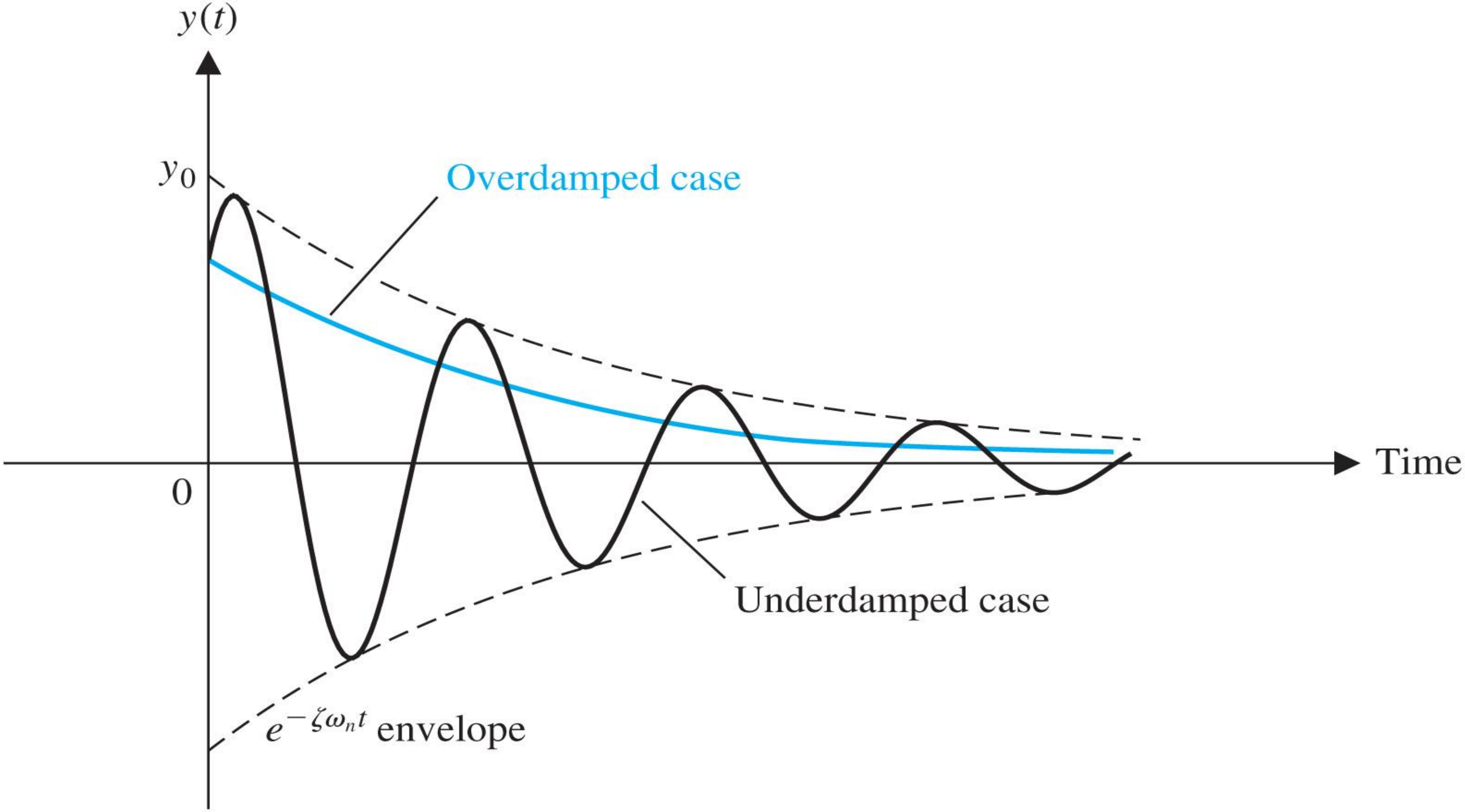
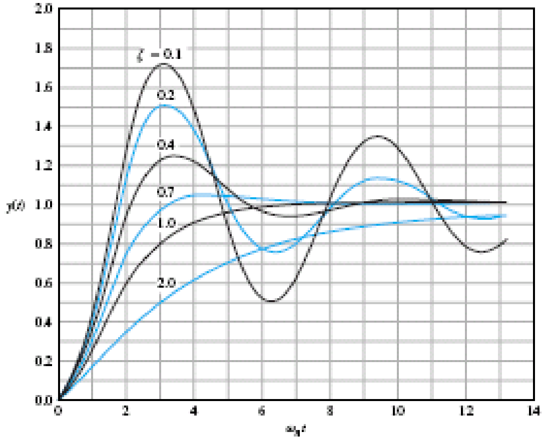

y(t)
y0
Overdamped case
Time
0
Underdamped case
$e^{-\zeta\omega_n t}$ envelope

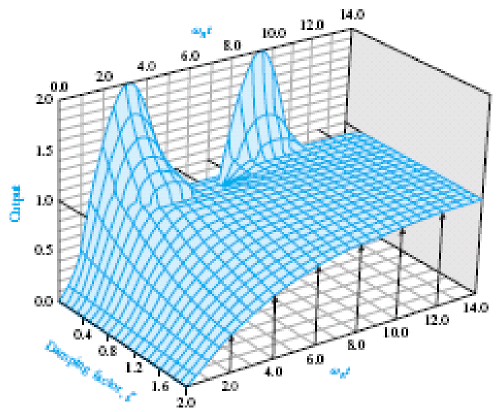
Figure: 02-12

Transit response



(a)

For step input as
a function of ζ



(b)

For step input as
a function of ζ and $\omega_n t$

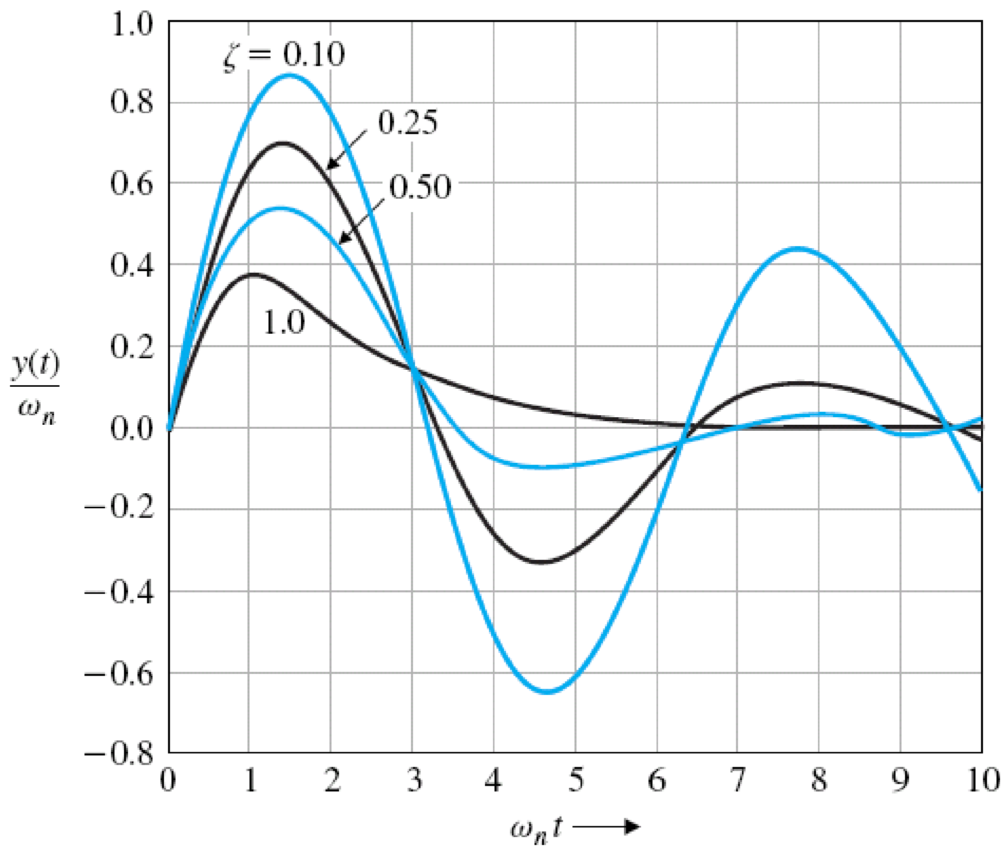
Unit impulse response

$$Y(s) = \frac{\omega_n{}^2}{(s^2 + 2\zeta\omega_n s + \omega_n{}^2)} R(s)$$

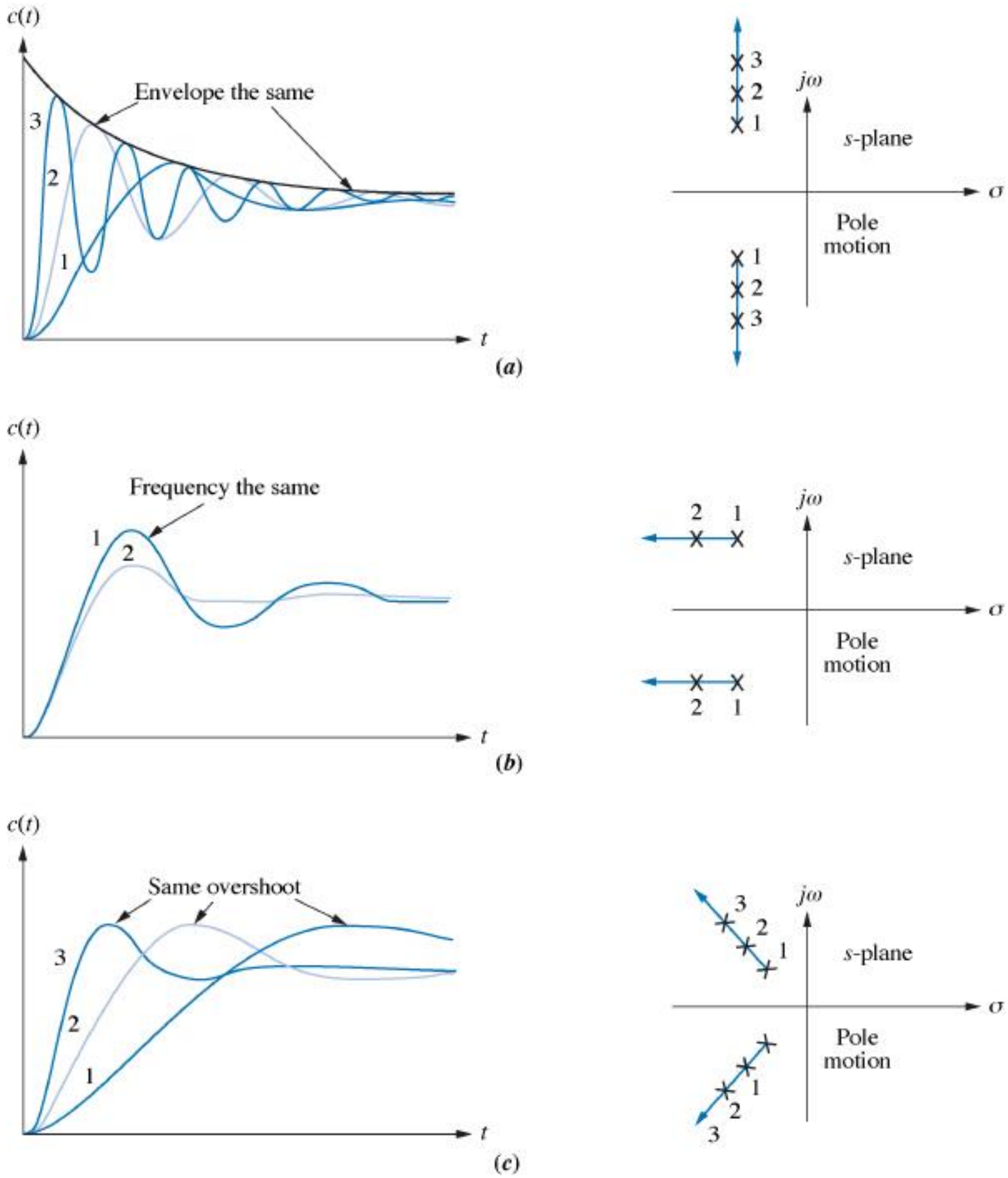
R(s)=1

T(s)=Y(s)

$$y(t) = \frac{\omega_n}{\beta} e^{-\xi\omega_n t} \sin \omega_n \beta t$$



ζ = 0.10
0.25
0.50
1.0
y(t)/ωn
ωn t
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
0
1
2
3
4
5
6
7
8
9
10



Constant real part

Constant imaginary part

Constant damping ratio

Standard performance measures

$$T_s(s) = 4\tau = \frac{4}{\zeta\omega_n}$$

Settling time

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Peak time

$$M_{pt} = 1 + e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

Peak response

$$P.O. = 100e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

Percent overshoot

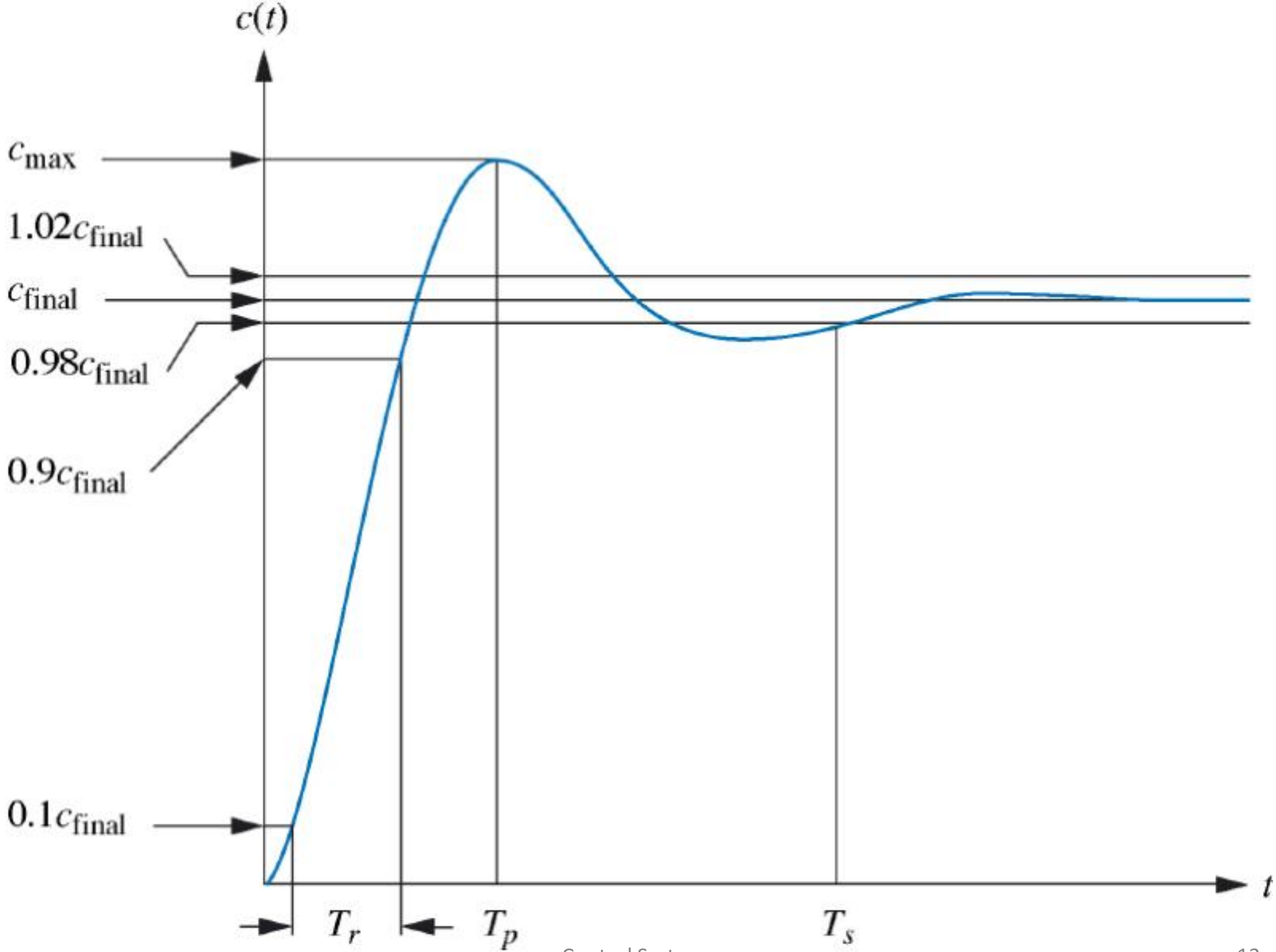

c(t)
c_max
1.02c_final
c_final
0.98c_final
0.9c_final
0.1c_final
t
T_r
T_p
T_s

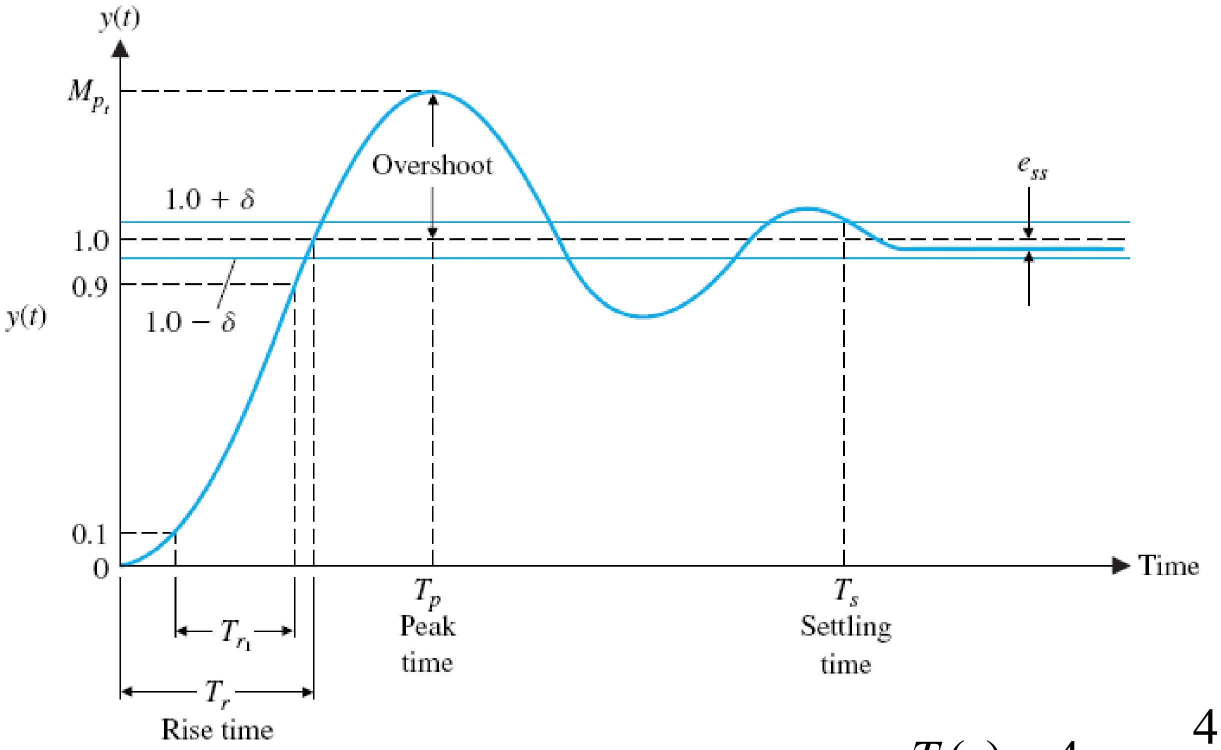
fig_04_14

Settling time

- **The settling time is defined as the time required for a system to settle within a certain percentage of the input amplitude.**

$$T_s(s) = 4\,\tau = \frac{4}{\zeta\omega_n}$$

Settling time



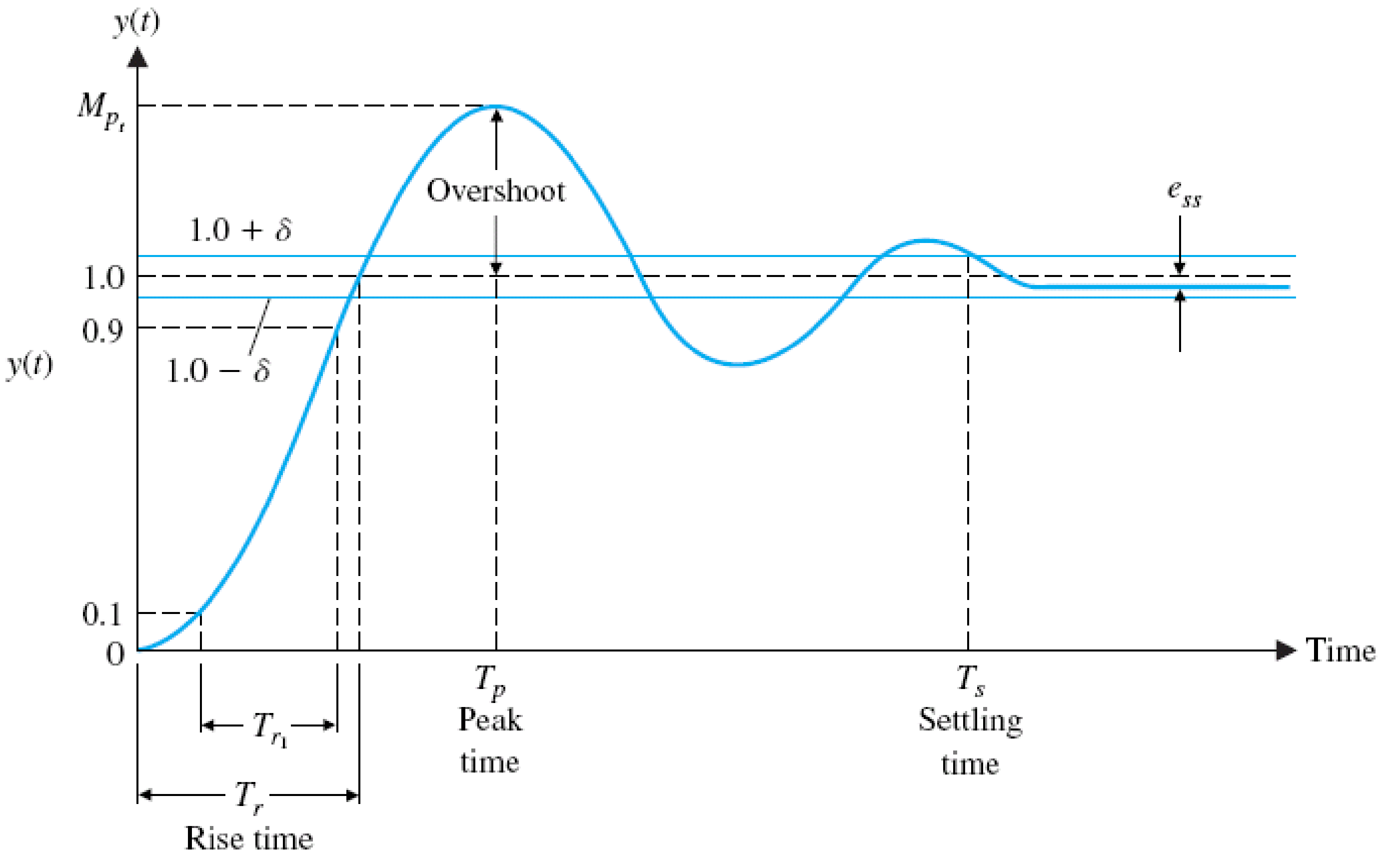
$$T_s(s) = 4\,\tau = \frac{4}{\zeta\omega_n}$$

Rise time

- **The time it takes for a signal to go from 10% of its value to 90% of its final value**

$$T_r(s) = \frac{2.16\zeta + 0.60}{\omega_n} \quad 0.3 \le \zeta \le 0.8$$

Rise time



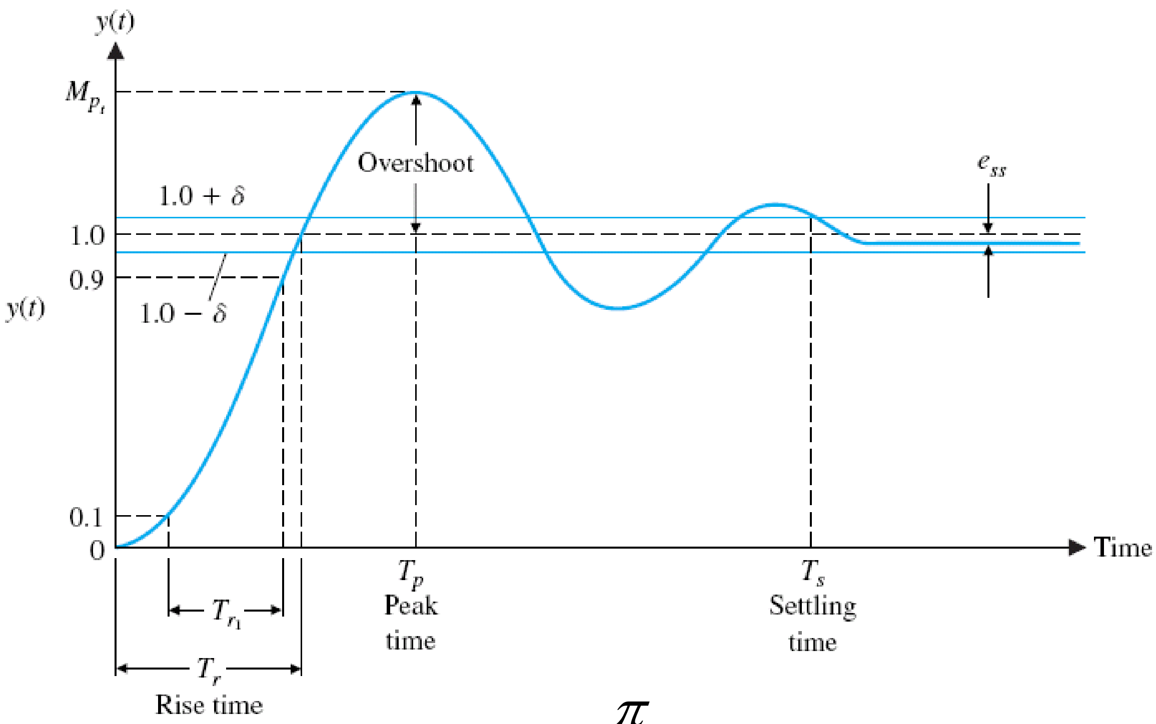
y(t)
M_{p_t}
Overshoot
e_{ss}
1.0 + δ
1.0
0.9
y(t)
1.0 − δ
0.1
0
Time
T_p
Peak
time
T_s
Settling
time
T_{r_1}
T_r
Rise time

Peak time

- **Peak time is the time required by a signal to reach its maximum value.**

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

Peak time



$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

Percent overshoot

- **Percent Overshoot is defined as:**

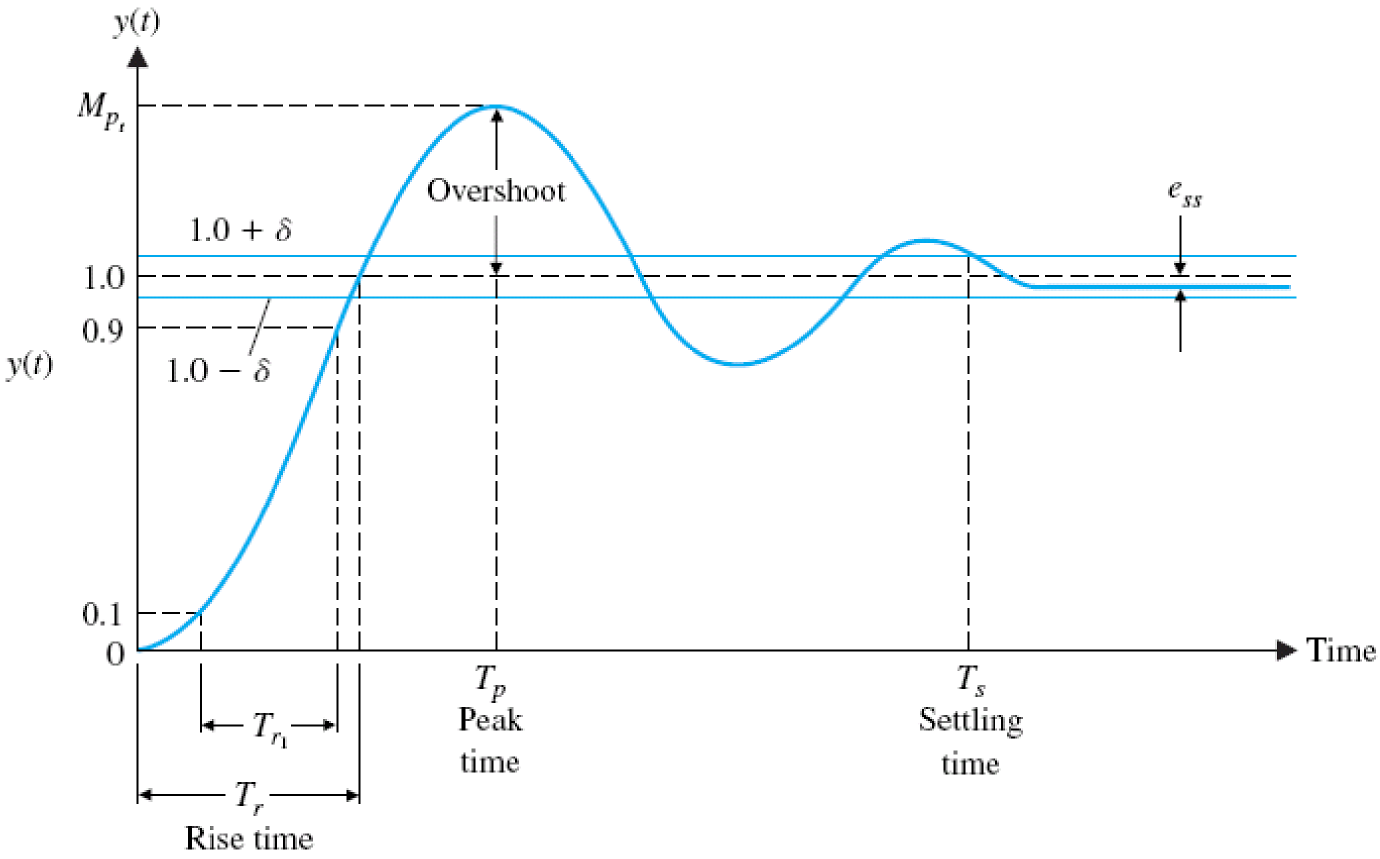
  P.O. = [(M_{pt} – fv) / fv] * 100%

  M_{pt} = The peak value of the time response

  fv = Final value of the response

$$P.O. = 100e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

Percent overshoot



Percent overshoot and normalized peak time versus ζ